# Groupe de travail
# “Théorie du potentiel et dynamique complexe”
# Orsay, 2023-2024

# Contents


**1 Subharmonic functions on open subsets of $\mathbb{C}$ (December 4th, Thomas Gauthier) 3**

**2 Fekete tuples and equidistribution (December 18th, Thomas Gauthier) 10**

**3 Entropy of complex polynomials (January 15th, Thomas Morand) 15**

**4 Equidistribution of equilibrium measures (January 29th, Mateo Ghezal) 20**

**5 Currents and plurisubharmonic functions (March 4th, Thomas Gauthier) 23**

**6 Maximal plurisubharmonic functions and a complex Monge-Ampère equation (March 18th, Damien Thomine) 28**

**7 Conjugation of automorphisms of $\mathbb{C}^2$ (April 8th, Thomas Gauthier) 35**

# 1 Subharmonic functions on open subsets of $\mathbb{C}$ (December 4th, Thomas Gauthier)

The main reference for this part is [6].

## 1.1 Definitions

**Definition 1.1** (Subharmonic function).
*Let $\Omega \subset \mathbb{C}$ be open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. We say that $u$ is* ***subharmonic*** *if $u \not\equiv -\infty$ and:*

- *$u$ is upper semi-continuous (USC): for all $\alpha \in \mathbb{R}$, the set $\{u < \alpha\}$ is open.*
- *$u$ satisfies the local submean inequality: $\forall z \in \Omega$, $\exists \varepsilon_0 > 0$, $\forall r \in [0, \varepsilon_0]$,*

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, \mathrm{d}\theta. \tag{1.1}$$

*We denote by $\mathcal{SH}(\Omega)$ the space of subharmonic functions on $\Omega$.*

**Example 1.2.**
*Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f \not\equiv 0$. Then $\ln|f|$ is subharmonic:*

- *We have $\{\ln|f| < \alpha\} = \{|f| < e^\alpha\} = f^{-1}(B(0, e^\alpha))$ for $\alpha \in \mathbb{R}$.*
- *For $z \in \mathbb{C}$ and all small enough $r > 0$,*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, \mathrm{d}\theta$$
$$\ln|f|(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \ln|f|(z + re^{i\theta})\, \mathrm{d}\theta.$$

**Example 1.3.**
*$\mathcal{SH}(\Omega)s$ is a cone : if $u$, $v$ are subharmonic on $\Omega$ and $\alpha$, $\beta \geq 0$, then $\alpha u + \beta v$ is subharmonic on $\Omega$.*

*If $u$, $v$ are subharmonic on $\Omega$, then $\max\{u, v\}$ is also subharmonic on $\Omega$.*

**Theorem 1.4** (Maximum principle).
*Let $\Omega$ be open and connected and $u$ subharmonic on $\Omega$.*

- *If $u$ attains its maximum in $\Omega$, then $u$ is constant.*
- *If $\limsup_{z \to \zeta} u(z) \leq 0$ for all $\zeta \in \partial\Omega$, then $u \leq 0$ on $\Omega$.*

**Theorem 1.5** (Criterions for subharmonicity).
*Let $\Omega$ be open and $u : \Omega \to \mathbb{R} \cup \{-\infty\}$. The following are equivalent:*

- *$u$ is subharmonic on $\Omega$.*
- *For all $h : \Omega \to \mathbb{R}$ harmonic, if[1] $\limsup_{z \to \zeta}(u(z) - h(z)) \leq 0$ for all $\zeta \in \partial\Omega$, then $u \leq h$ on $\Omega$.*
- *If furthermore $u$ is $\mathcal{C}^2$: $\Delta u \geq 0$ on $\Omega$.*

[1] More simply, but less rigorously: if $u \leq h$ on $\partial\Omega$ and $h$ is harmonic, then $u \leq h$ everywhere.

## 1.2 Regularization

We assume that $\Omega$ is connected.

Let $\chi \in \mathcal{C}^\infty(\mathbb{C}, \mathbb{R}_+)$ be supported on $\overline{D}(0,1)$ with integral 1. Write for $\varepsilon > 0$ and $z \in \mathbb{C}$:

$$\chi_\varepsilon(z) := \frac{1}{\varepsilon^2} \chi\left(\frac{z}{\varepsilon}\right).$$

For $\epsilon > 0$, write $\Omega_\varepsilon := \{z \in \Omega : \ B(z,\varepsilon) \subset \Omega\}$.

**Proposition 1.6.**
*Let $u \in \mathcal{SH}(\Omega)$. For all small enough $\varepsilon$, we have $u * \chi_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$.*

The key fact is:

**Lemma 1.7.**

$$\mathcal{SH}(\Omega) \subset \mathbb{L}^1_{loc}(\Omega).$$

*Proof.*
Let $u$ be a subharmonic function on $\Omega$ which is not locally integrable. Then there exists $z \in \Omega$ and $r > 0$ such that

$$\int_{\overline{D}(z,r)} |u|(w) \,\mathrm{d}\,\mathrm{Leb}(w) = +\infty.$$

Since $u$ is upper semi-continuous, it is bounded from above on compact subsets of $\Omega$, and in particular on $\overline{D}(z,r)$. Hence

$$\int_{\overline{D}(z,r)} u(w) \,\mathrm{d}\,\mathrm{Leb}(w) = -\infty.$$

By the submean inequality, $u \equiv -\infty$ on $\overline{D}(z,r)$. But then, by connectedness, $u \equiv -\infty$ on $\Omega$, which is impossible. □

## 1.3 Generalized Laplacian

For $f \in \mathcal{C}^2(\Omega, \mathbb{R})$, let

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Instead of using the real coordinates $(x,y)$, we use complex coordinates $(z, \overline{z})$:

$$\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z},$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. More succintly, we may write

$$\mathrm{d} = \partial + \overline{\partial},$$

where $\partial f = \frac{\partial f}{\partial z}\mathrm{d}z$ and $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$.

Then, the Laplace operator may be written

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta,$$

or again more succintly

$$i\partial\overline{\partial} f = \frac{1}{2}(\Delta f)\mathrm{d}x \wedge \mathrm{d}y.$$

We are going to investigate the operator $\mathrm{dd}^c = \frac{i}{\pi}\partial\overline{\partial}$, defined by duality for $u \in \mathbb{L}^1_{\mathrm{loc}}$ by:

$$\left\langle \frac{i}{\pi}\partial\overline{\partial} u, \varphi \right\rangle := \int u \cdot \frac{i}{\pi}\partial\overline{\partial}\varphi \,\mathrm{d}\,\mathrm{Leb} \quad \forall \varphi \in \mathcal{C}^\infty_c(\Omega, \mathbb{R}).$$

**Theorem 1.8.**

*If $u \in \mathcal{SH}(\omega)$, then $\mathrm{dd}^c u \geq 0$ in the sense of distributions. In particular, $\mathrm{dd}^c u$ extends as a positive continuous linear from on $\mathcal{C}^0_c(\Omega)$, i.e. as a Radon measure.*

*Proof.*

We write $L(\varphi) := \langle \mathrm{dd}^c u, \varphi \rangle$.

Let $\varphi \in \mathcal{C}^\infty_c(\Omega, \mathbb{R})$ be nonnegative. Let $(u_n)$ be a sequence of subharmonic smooth functions on $\Omega$ decreasing to $u$. Then, by monotonic convergence and integration by parts,

$$\begin{aligned} L(\varphi) &= \int_\Omega u \cdot \mathrm{dd}^c\varphi \,\mathrm{d}\,\mathrm{Leb} \\ &= \int_\Omega \lim_{n\to+\infty} u_n \cdot \mathrm{dd}^c\varphi \,\mathrm{d}\,\mathrm{Leb} \\ &= \lim_{n\to+\infty} \int_\Omega u_n \cdot \mathrm{dd}^c\varphi \,\mathrm{d}\,\mathrm{Leb} \\ &= \lim_{n\to+\infty} \frac{1}{2\pi} \int_\Omega u_n \Delta\varphi \,\mathrm{d}\,\mathrm{Leb} \\ &= \lim_{n\to+\infty} \frac{1}{2\pi} \int_\Omega (\Delta u_n)\varphi \,\mathrm{d}\,\mathrm{Leb} \\ &\geq 0. \end{aligned}$$

Let $K \subset \Omega$ be compact and $V$ be a neighborhood of $K$ compactly contained in $\Omega$. Let $\psi \in \mathcal{C}^\infty_c(\Omega, [0,1])$ with $\Psi \equiv 1$ on $V$. If $\varphi$ is smooth and supported on $V$, then $|\varphi| \leq \|\varphi\|_\infty \psi$, so that $|L(\varphi)| \leq \|\varphi\|_\infty |L(\psi)|$.

Let $\varphi \in \mathcal{C}^0_c(\Omega, \mathbb{R})$ be supported on $K$. For all small enough $\varepsilon$, the function $\varphi * \chi_\varepsilon$ is supported on $V$. We thus extend $L$ to $\varphi$, with $\|L\| \leq |L(\psi)|$. □

## 1.4 Potential of a measure

Let $\mu$ be a Borel, finite, nonnegative measure with compact support in $\mathbb{C}$. The **potential** of $\mu$ is

$$p_\mu : \left\{ \begin{array}{ccc} \mathbb{C} & \to & \mathbb{R} \cup \{-\infty\} \\ z & \mapsto & \int_{\mathbb{C}} \ln|z-w| \,\mathrm{d}\mu(w) \end{array} \right. .$$

**Theorem 1.9.**

*$p_\mu$ is subharmonic on $\mathbb{C}$, and harmonic on $\mathbb{C} \setminus \mathrm{Supp}(\mu)$;*

$$p_\mu(z) =_\infty \mu(\mathbb{C}) \ln|z| + O\left(z^{-1}\right) \;;$$

*and $\mathrm{dd}^c p_\mu = \mu$.*

*Proof.*

The first point follows from Fatou's inequality and Fubini's theorem. Let $(z_n)_{n\in\mathbb{N}}$ be converging to $z \in \mathbb{C}$. Then

$$\limsup_{n\to+\infty} p_\mu(z_n) \leq \int_{\mathbb{C}} \limsup_{n\to+\infty} \ln|z_n - w| \,\mathrm{d}\mu(w) \leq \int_{\mathbb{C}} \ln|z-w| \,\mathrm{d}\mu(w) = p_\mu(z),$$

so that $p_\mu$ is upper semi-continuous.

Let $z \in \mathbb{C}$ and $r > 0$. Then

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} p_\mu(z+re^{i\theta}) \,\mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} \ln|z+re^{i\theta}-w| \,\mathrm{d}\mu(w)\,\mathrm{d}\theta \\
&= \int_{\mathbb{C}} \frac{1}{2\pi}\int_0^{2\pi} \ln|z+re^{i\theta}-w| \,\mathrm{d}\theta\,\mathrm{d}\mu(w) \\
&= \int_{\mathbb{C}} \ln\max\{|z-w|, r\} \,\mathrm{d}\mu(w) \\
&\geq p_\mu(z),
\end{aligned}$$

so that $p_\mu$ is subharmonic. In addition, the last inequality is an equality for all small enough $r$ if $z \in \mathbb{C} \setminus \mathrm{Supp}(\mu)$, so that $p_\mu$ is harmonic on $\mathbb{C} \setminus \mathrm{Supp}(\mu)$.

Let $R$ be such that $\mathrm{Supp}(\mu) \subset \overline{D}(0,R)$. For all $z$ such that $|z| > R$,

$$p_\mu(z) = \mu(\mathbb{C})\ln(z) + \int_{\overline{D}(0,R)} \ln\left|1 - \frac{w}{z}\right| \,\mathrm{d}\mu(w).$$

If $|z| > 2R$, then for all $w \in \mathrm{Supp}(\mu)$,

$$\left|\ln\left|1-\frac{w}{z}\right|\right| \leq \frac{2R}{|z|},$$

whence $p_\mu(z) =_\infty \mu(\mathbb{C})\ln|z| + O\left(z^{-1}\right)$.

Let $\varphi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R})$. Then

$$\begin{aligned}
\langle \mathrm{dd}^c p_\mu, \varphi\rangle &= \int_{\mathbb{C}} p_\mu \cdot \mathrm{dd}^c\varphi \,\mathrm{d}\,\mathrm{Leb} \\
&= \int_{\mathbb{C}}\int_{\mathbb{C}} \ln|z-w| \,\mathrm{d}\mu(w) \frac{1}{2\pi}\Delta\varphi(z) \,\mathrm{d}\,\mathrm{Leb}(z) \\
&= \int_{\mathbb{C}} \frac{1}{2\pi}\int_{\mathbb{C}} \ln|z-w|\Delta\varphi(z) \,\mathrm{d}\,\mathrm{Leb}(z)\,\mathrm{d}\mu(w).
\end{aligned}$$

But:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{C}} \ln|z-w| \cdot \Delta\varphi(z) \,\mathrm{d}\,\mathrm{Leb}(z) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\{|z-w|>\varepsilon\}} \ln|z-w| \cdot \Delta\varphi(z) \,\mathrm{d}\,\mathrm{Leb}(z) \\
&= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} \left(\varphi(w+\varepsilon e^{i\theta}) - \varepsilon\ln(\varepsilon)\frac{\partial\varphi}{\partial r}(w+\varepsilon e^{i\theta})\right) \,\mathrm{d}\theta \\
&= \varphi(w).
\end{aligned}$$

□

## 1.5 Continuity principle and minimum principle

**Proposition 1.10.**
*Let $K \subset \mathbb{C}$ be compact and $\mu$ be supported on $K$. For all $z_0 \in \operatorname{Supp}(\mu)$,*

$$\liminf_{z \to z_0} p_\mu(z) = \limsup_{\substack{z \to z_0 \\ z \in K}} p_\mu(z).$$

*In addition, $\inf_{\mathbb{C}} p_\mu = \inf_{\operatorname{Supp}(\mu)} p_\mu$.*

*Proof.*
If $p_\mu(z_0) = -\infty$, the result follows by upper semi-continuity of $p_\mu$.

Assume that $p_\mu(z_0) > -\infty$. Then $\mu(\{z_0\}) = 0$. Hence, for all $\varepsilon > 0$, there exists $r > 0$ such that $\mu(\overline{D}(z_0, r)) \leq \varepsilon$. For $z \in \mathbb{C}$, let $\zeta \in K$ be a point which minimizes $|\cdot - z|$ on $K$. Then, for all $w \in K$,

$$\frac{|\zeta - w|}{|z - w|} \leq \frac{|\zeta - z| + |z - w|}{|z - w|} \leq 2.$$

Hence,

$$p_\mu(z) = p_\mu(\zeta) - \int_{\mathbb{C}} \ln \left| \frac{\zeta - w}{z - w} \right| \, \mathrm{d}\mu(w) \geq p_\mu(z) - \varepsilon \ln(2) - \int_{\mathbb{C} \setminus D(\zeta_0, r)} \ln \left| \frac{\zeta - w}{z - w} \right| \, \mathrm{d}\mu(w).$$

If $z$ converges to $z_0 \in K$, then $\zeta$ also converges to $z_0$. Hence,

$$\liminf_{z \to \zeta_0} p_\mu(z) \geq \limsup_{\substack{\zeta \to z_0 \\ \zeta \in K}} p_\mu(\zeta) - \varepsilon \ln(2).$$

As $\varepsilon$ is arbitrary, we get the claim. The last statement follows by upper semi-continuity. □

## 1.6 Energy of a measure

Let $K \subset \mathbb{C}$ be compact and $\mu$ be nonnegative, finite and supported on $K$.

**Definition 1.11** (Energy of a measure)**.**
*We define the **energy** of $\mu$ as*

$$I(\mu) := \int_{\mathbb{C}} p_\mu \, \mathrm{d}\mu = \int_{\mathbb{C}} \ln |z - w| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(w). \tag{1.2}$$

If $\mu(\{z_0\}) > 0$, then $I(\mu) = -\infty$.

**Definition 1.12** (Borel-polar subsets)**.**
*A subset $E \subset \mathbb{C}$ is **Borel-polar** if $I(\nu) = -\infty$ for all probability measures $\nu$ supported on $E$.*

Borel-polar subsets are stable under countable unions. In particular, countable subsets are Borel-polar. Conversely, if $E$ has positive Lebesgue measure, then a uniform measure supported on $E$ has finite energy, so that $E$ is not Borel-polar: Borel-polar subsets have Lebesgue measure zero.

**Definition 1.13** (Equilibrium measures)**.**
*A probability measure $\nu \in \mathcal{P}(K)$ is an **equilibrium measure of** $K$ if*

$$I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu).$$

**Theorem 1.14.**

*A nonempty compact set $K$ has an equilibrium measure. It is unique if $K$ is non-Borel polar.*

*Proof.*

Let $\mu \in \mathcal{P}(K)$, and $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{P}(K)$ converging in distribution to $\mu$. Note that $\mu_n \otimes \mu_n$ converges to $\mu \otimes \mu$ in distribution.

Let $m \in \mathbb{R}$. Set $g_m(z,w) := \max\{\ln|z-w|, m\}$. Then $g_m$ is continuous, and $\ln|z-w| \leq g_m(z,w)$, so that

$$\limsup_{n\to+\infty} I(\mu_n) \leq \limsup_{n\to+\infty} \int_{\mathbb{C}^2} g_m \, \mathrm{d}\mu_n \otimes \mu_n = \int_{\mathbb{C}^2} g_m \, \mathrm{d}\mu \otimes \mu.$$

By monotonic convergence, $\lim_{m\to-\infty} \int_{\mathbb{C}^2} g_m \, \mathrm{d}\mu \otimes \mu = I(\mu)$. Hence, $I$ is upper semi-continuous on $\mathcal{P}(K)$.

To conclude, let $(\mu_n)_{n\in\mathbb{N}}$ be such that $I(\mu_n)$ converges to $\sup_{\mu\in\mathcal{P}(K)} I(\mu)$. Up to extraction of a subsequence, $(\mu_n)_{n\in\mathbb{N}}$ converges in distribution to $\nu \in \mathcal{P}(K)$, and $\nu$ is an equilibrium measure.

The uniqueness relies on **Frostman's lemma**:

**Lemma 1.15** (Frostman)**.**

*If $\nu$ is an equilibrium measure of $K$, then $p_\nu \geq I(\nu)$ on $\mathbb{C}$. In addition, there exists a Borel-polar subset $E \subset K$ such that $p_\nu \equiv I(\nu)$ on $K \setminus E$.*

We ignore polar sets (they have Lebesgue measure 0, so we can work outside of a polar set, and conclude by upper semi-continuity). Let $\nu$ be an equilibrium measure on $K$, and $\nu'$ an equilibrium measure on $\partial K$. Then $p_\nu \geq I(\nu)$, and $p_\nu \equiv I(\nu)$ on $K$. In addition, $p_{\nu'} \geq I(\nu')$, and $p_{\nu'} \equiv I(\nu')$ on $\partial K$. By the maximum principle, $p_{\nu'} \equiv I(\nu')$ on $K$.

$p_\nu - p_{\nu'}$ is harmonic on $\mathbb{C} \setminus K$, and constant (equal to $I(\nu) - I(\nu')$) on $K$. Hence $p_\nu - p_{\nu'} =_\infty o(1)$.

By the maximum principle for harmonic functions, $p_\nu - p_{\nu'} \leq I(\nu) - I(\nu')$ on $\mathbb{C} \setminus K$. Taking the limit at $\infty$, we get $0 \leq I(\nu) - I(\nu')$. The same reasoning with $p_{\nu'} - p_\nu$ yields $0 \leq I(\nu') - I(\nu)$. Hence, $I(\nu) = I(\nu')$ and $p_\nu = p_{\nu'}$.

All that remains is to prove Frostman's lemma.

*Proof of Frostman's lemma.*

By the minimum principle, we need to show that $p_\nu \geq I(\nu)$ on $\mathrm{Supp}(\nu)$. Let $K_n := K \cap \{p_\nu \geq I(\nu) + \frac{1}{n}\}$.

We shall show that $K_n$ is polar for all $n$, so that $\bigcup_{n\geq 1} K_n = K \cap \{p_\nu > I(\nu)\}$ is also polar. If not, there exists a probability measure $\mu$ supported on $K_n$ with $I(\mu) > -\infty$. The main thrust of the argument is that, if we exchange a little mass of $\nu$ where $p_\nu$ is small (i.e. where the measure $\nu$ is too concentrated) with $\mu$, which is supported where $p_\nu$ is large (i.e. where the measure $\nu$ is less concentrated), we can improve the energy, which contradicts the hypothesis that $\nu$ be an equilibrium measure.

As $I(\nu) = \int_{\mathbb{C}} p_\nu \, \mathrm{d}\nu$, there exists $z_0 \in \mathrm{Supp}(\nu)$ such that $p_\nu(z_0) \leq I(\nu)$. Since $p_\nu$ is upper semi-continuous, there exists $r > 0$ such that $p_\nu < I(\nu) + \frac{1}{2n}$ on $\overline{D}(z_0, r)$.

Let $\nu_1 := \nu_{|\overline{D}(z_0,r)^c} + \nu(\overline{D}(z_0,r))\mu$. For all $t \in [0,1]$, let $\nu_t := (1-t)\nu + t\nu_1$, which is supported on $K$. We shall prove that $I(\nu_t) > I(\nu)$ for all small enough non-zero $t$, which provides the desired contradiction as $\nu$ maximizes the energy on $K$. Since $I(\mu) > -\infty$, the function $t \mapsto I(\nu_t)$ is quadratic.

We compute

$$\begin{aligned}\frac{\mathrm{d}I(\nu_t)}{\mathrm{d}t}(0) &= 2\int\int \ln|z-w|\,\mathrm{d}\nu(z)\,\mathrm{d}(\nu_1-\nu)(w)\\ &= 2\int_{\mathbb{C}} p_\nu\,\mathrm{d}(\nu_1-\nu)\\ &= 2\left(\nu(\overline{D}(z_0,r))\int_{\mathbb{C}} p_\nu\,\mathrm{d}\mu - \int_{\overline{D}(z_0,r)} p_\nu\,\mathrm{d}\nu\right).\end{aligned}$$

But $\mu$ is supported on $K_n$ and $p_\nu \geq I(\nu)+\frac{1}{n}$ on $K_n$, so that

$$\nu(\overline{D}(z_0,r))\int_{\mathbb{C}} p_\nu\,\mathrm{d}\mu \geq \nu(\overline{D}(z_0,r))\left(I(\nu)+\frac{1}{n}\right).$$

In addition, $p_\nu \leq I(\nu)+\frac{1}{2n}$ on $\overline{D}(z_0,r)$, so that

$$\int_{\overline{D}(z_0,r)} p_\nu\,\mathrm{d}\nu \leq \nu(\overline{D}(z_0,r))\left(I(\nu)+\frac{1}{2n}\right).$$

Together, these two inequalities yield $\frac{\mathrm{d}I(\nu_t)}{\mathrm{d}t}(0) \geq \frac{\nu(\overline{D}(z_0,r))}{n} > 0$, which is what we wanted. □

This finishes the proof of the uniqueness of the equilibrium measure. □

# 2 Fekete tuples and equidistribution (December 18th, Thomas Gauthier)

## 2.1 Fekete tuples and capacity

Let $K \subset \mathbb{C}$ be a non-polar compact subset.

**Definition 2.1** ($n$th-diameter)**.**
*The $n$th diamter of $K$ is*

$$\delta_n(K) := \sup \left\{ \prod_{1 \leq i < j \leq n} |w_i - w_j|^{\frac{2}{n(n-1)}} \; : \; w_1, \dots, w_n \in K \right\}.$$

*A $n$-tuple $(w_1, \dots, w_n) \in K^n$ is a **Fekete $n$-tuple** if it realizes the supremum.*

A Fekete $n$-tuple always exists: the map

$$\begin{cases} \mathbb{C}^n & \to \quad \mathbb{R} \\ (w_1, \dots, w_n) & \mapsto \quad \prod_{1 \leq i < j \leq n} |w_i - w_j|^{\frac{2}{n(n-1)}} \end{cases}$$

is continuous, and $K^n$ is compact.

There may not be a unique Fekete $n$-tuple : the image of a Fekete $n$-tuple by an isometry is a Fekete $n$-tuple, so a circle or a disk has infinitely many Fekete $n$-tuples.

**Theorem 2.2** (Fekete, Szëgo)**.**
*Let $K$ be a non-polar compact subset of $\mathbb{C}$. Then $(\delta_n(K))_{n \in \mathbb{N}}$ is non-increasing, and*

$$\lim_{n \to +\infty} \delta_n(K) = c(K) := \sup_{\mu \in \mathcal{P}(K)} e^{I(\mu)}.$$

*Proof.*
**The $n$th diameter in nonincreasing:** Let $n \geq 1$ and $(w_1, \dots, w_{n+1})$ be a Fekete $(n+1)$-tuple. Fix $1 \leq i_0 \leq n+1$. Then

$$\prod_{\substack{1 \leq i < j \leq n+1 \\ i,j \neq i_0}} |w_i - w_j| \leq \delta_n(K)^{\frac{n(n-1)}{2}}.$$

Taking the product over $1 \leq i_0 \leq n+1$,

$$\delta_{n+1}(K)^{\frac{(n-1)n(n+1)}{2}}, = \prod_{1 \leq i < j \leq n+1} |w_i - w_j|^{n-1} \leq \left( \delta_n(K)^{\frac{n(n-1)}{2}} \right)^{n+1} = \delta_n(K)^{\frac{(n-1)n(n+1)}{2}},$$

which gives the claim. In particular, the sequence $(\delta_n(K))_{n \in \mathbb{N}}$ converges.

**$\delta_n(K) \geq c(K)$ for all $n$:** Let $n \geq 1$ and $(w_1, \dots, w_n) \in K^n$. Then

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n+1} \ln(|w_i - w_j|) \leq \ln(\delta_n(K)).$$

Let us integrate each term with respect to $\mu_K \otimes \mu_K$, where $\mu_K$ is the equilibrium measure of $K$. Then

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n+1} \int_{\mathbb{C}^2} \ln(|w_i - w_j|) \, \mathrm{d}\mu_K(w_i) \, \mathrm{d}\mu_K(w_j) \leq \ln(\delta_n(K))$$

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n+1} I(\mu_K) \leq \ln(\delta_n(K))$$

$$I(\mu_K) \leq \ln(\delta_n(K)).$$

But $I(\mu_K) = \ln(c(K))$, which gives the claim.

$\lim_{n \to +\infty} \delta_n(K) \leq c(K)$ **for all** $n$**:** Let $n \geq 1$ and $(w_1, \ldots, w_n) \in K^n$ be a Fekete $n$-tuple. Fix $\varepsilon > 0$. Let $\mu_{i,\varepsilon}$ be the uniform measure on the circle $S(w_i, \varepsilon)$, and $\nu_{n,\varepsilon}$ be the average of $(\mu_{i,\varepsilon})_{1 \leq i \leq n}$. Then $\nu_{n,\varepsilon}$ is a probability measure supported on $\overline{B}(K, \varepsilon)$, and

$$\begin{aligned}
I(\nu_{n,\varepsilon}) &= \int_{\mathbb{C}^2} \ln(|z-w|) \, \mathrm{d}\nu_{n,\varepsilon}(w) \, \mathrm{d}\nu_{n,\varepsilon}(z) \\
&= \frac{1}{n^2} \sum_{i=1}^n \int_{\mathbb{C}^2} \ln(|z-w|) \, \mathrm{d}\mu_{i,\varepsilon}(w) \, \mathrm{d}\mu_{i,\varepsilon}(z) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{C}^2} \ln(|z-w|) \, \mathrm{d}\mu_{i,\varepsilon}(w) \, \mathrm{d}\mu_{j,\varepsilon}(z) \\
&= \frac{1}{n^2} \sum_{i=1}^n I(\mu_{i,\varepsilon}) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{C}} p_{\mu_{i,\varepsilon}} \, \mathrm{d}\mu_{j,\varepsilon}(z) \\
&\geq \frac{1}{n} \ln(\varepsilon) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} p_{\mu_{i,\varepsilon}}(w_j)
\end{aligned}$$

where we used the submean inequality at the last step. Finally, by the submean inequality again,

$$p_{\mu_{i,\varepsilon}}(w_j) = \int_{\mathbb{C}} \ln(|z - z_j|] \, \mathrm{d}\mu_{i,\varepsilon} \geq \ln(|z_i - z_j|).$$

Hence,

$$I(\nu_{n,\varepsilon}) \geq \frac{1}{n} \ln(\varepsilon) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \ln(|z_i - z_j|) = \frac{1}{n} \ln(\varepsilon) + \frac{n-1}{n} \ln(\delta_n(K)).$$

In addition, $\nu_{n,\varepsilon}$ is supported on $\overline{B}(K, \varepsilon)$, so that

$$c(K_\varepsilon) \geq e^{I(\nu_{n,\varepsilon})} \geq \varepsilon^{\frac{1}{n}} (\delta_n(K))^{1-\frac{1}{n}}.$$

Taking the limit as $n \to +\infty$, we get $c(K_\varepsilon) \geq \lim_{n \to +\infty} \delta_n(K)$.

Since $K \subset K_\varepsilon$, we have $c(K) \leq c(K_\varepsilon)$. Now, take a sequence $(\varepsilon_n)_{n \geq 1}$ which converges to 0, and $\mu_{K_{\varepsilon_n}}$ a sequence of equilibrium measures of $K_{\varepsilon_n}$. Let $\nu$ be a limit point in distribution, which exists since all measure are supported on the compact $K_1$. Then $\nu$ is supported on $\bigcap_{n \geq 1} K_{\varepsilon_n} = K$, and in particular $c(K) \geq e^{I(\nu)}$.

The semi-continuity of the energy yields

$$\limsup_{n \to +\infty} c(K_{\varepsilon_n}) = \limsup_{n \to +\infty} e^{I(\mu_{K_{\varepsilon_n}})} \leq e^{I(\nu)} \leq c(K).$$

Hence, $\lim_{\varepsilon \to 0} c(K_\varepsilon) = c(K)$, which finishes the proof. □

## 2.2 Equidistribution of Fekete tuples

**Theorem 2.3** (Fekete)**.**

*Let $K$ be a non-polar compact subset of $\mathbb{C}$. Let $(w_{1,n}, \dots, w_{n,n})_{n\geq 2}$ be a sequence of Fekete $n$-tuples. Then the sequence of probability measures $\left(\frac{1}{n}\sum_{k=1}^n \delta_{w_{i,n}}\right)_{n\geq 2}$ converges in distribution to $\mu_K$.*

*Proof.*

Let $\nu \in \mathcal{P}(K)$ be a limit point of $\left(\frac{1}{n}\sum_{k=1}^n \delta_{w_{i,n}}\right)_{n\geq 2}$ for the weak-$*$ convergence. Then, up to extraction of a subsequence,

$$\lim_{n\to+\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n}^{n} \delta_{(w_{i,n}, w_{j,n})} = \nu \otimes \nu.$$

Let $\varepsilon > 0$. Take $\varphi_\varepsilon(z, w) := \ln(\max\{|z - w|, \varepsilon\})$. Then

$$\lim_{n\to+\infty} \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n}^{n} \int_{\mathbb{C}^2} \varphi_\varepsilon \, \mathrm{d}\delta_{(w_{i,n}, w_{j,n})} = \int_{\mathbb{C}^2} \varphi_\varepsilon \, \mathrm{d}\nu \otimes \nu.$$

In addition,

$$\ln(c(K)) = \lim_{n\to+\infty} \ln(\delta_n(K)) \leq \int_{\mathbb{C}^2} \varphi_\varepsilon \, \mathrm{d}\nu \otimes \nu \quad ;$$

by monotonic convergence when $\varepsilon$ vanishes, $\ln(c(K)) \leq I(\nu)$. But $\nu$ is supported on $K$, so that Hence, $\ln(c(K)) = I(\nu)$. In addition, $K$ is non-polar, so its equilibrium measure is unique, and $\nu = \mu_K$. □

## 2.3 Quantitative equidistribution

**Definition 2.4** (Green function)**.**

*Let $K$ be a non-polar compact subset of $\mathbb{C}$. Let $\mu_K$ be its equilibrium measure. The* ***Green function*** *of $K$ is*

$$G_K := p_{\mu_K} - I(\mu_K).$$

**Definition 2.5.**

*Let $\alpha \in (0, 1]$. A compact $K$ is* ***$\alpha$-Hölder*** *if there exists $C \geq 0$ such that $|G_K(x) - G_K(y)| \leq C|x - y|^\alpha$ for all $x$, $y \in K$, i.e. if its Green function is $\alpha$-Hölder.*

**Theorem 2.6** (Favre, Rivera, Letelier, 2004; Pritsker, 2011)**.**

*Let $\alpha \in (0, 1]$ and $C \geq 0$. Let $K$ be a $(C, \alpha)$-Hölder compact subset of $\mathbb{C}$. There exists $A(C, \alpha) \geq 0$ such that, for all $\varphi \in W^{1,2} \cap \mathrm{Lip}(\mathbb{C}, \mathbb{R})$, for all $n \geq 2$ and any Fekete $n$-tuple $(w_1, \dots, w_n)$,*

$$\left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \frac{1}{n} \sum_{k=1}^n \varphi(w_i) \right| \leq A \sqrt{\frac{\ln(n)}{n}} \left\|\varphi'\right\|_{\mathbb{L}^2} + \frac{1}{n^{\frac{1}{\alpha}}} \left\|\varphi\right\|_{\mathrm{Lip}}.$$

*Proof.*

Let $n \geq 2$ and $\varepsilon > 0$. Set $\mu_n := \frac{1}{n}\sum_{i=1}^n \delta_{w_i}$ and $\nu_{n,\varepsilon} := \frac{1}{n}\sum_{i=1}^n \lambda_{S(w_i,\varepsilon)}$, where $\lambda_{S(w_i,\varepsilon)}$ is the uniform measure on $S(w_i, \varepsilon)$. Then

$$\left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_n \right| \leq \left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} \right| + \left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} - \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_n \right|.$$

The latter term of the right hand-side is at most $\varepsilon \|\varphi\|_{\text{Lip}}$. In addition,

$$\int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} = \int_{\mathbb{C}} \varphi \, \mathrm{dd}^c(p_{\mu_K} - p_{\nu_{n,\varepsilon}}) = -\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c(p_{\mu_K} - p_{\nu_{n,\varepsilon}}).$$

We use:

**Lemma 2.7** (Key lemma)**.**
*Let $\rho = \mathrm{dd}^c g$ be a signed measure, where $g \in \mathcal{C}^0(\mathbb{P}_1(\mathbb{C}), \mathbb{R})$. Then*

$$(\rho, \rho) := -\int_{\mathbb{C}^2} \ln(|z - w|) \, \mathrm{d}\rho(z) \, \mathrm{d}\rho(w) = \int_{\mathbb{C}} \mathrm{d}g \wedge \mathrm{d}^c g \geq 0.$$

*In addition, $(\rho, \rho) = 0$ if and only if $\rho = 0$.*

For now, let us assume this lemma. Then

$$\begin{aligned}\left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} \right| &= \left| \int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c(p_{\mu_K} - p_{\nu_{n,\varepsilon}}) \right| \\ &\leq \sqrt{\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c\varphi} \times \sqrt{\int_{\mathbb{C}} \mathrm{d}(p_{\mu_K} - p_{\nu_{n,\varepsilon}}) \wedge \mathrm{d}^c(p_{\mu_K} - p_{\nu_{n,\varepsilon}})}.\end{aligned}$$

But

$$\begin{aligned}\int_{\mathbb{C}} \mathrm{d}(p_{\mu_K} - p_{\nu_{n,\varepsilon}}) \wedge \mathrm{d}^c(p_{\mu_K} - p_{\nu_{n,\varepsilon}}) &= -\int_{\mathbb{C}^2} \ln(|z - w|) \, \mathrm{d}(p_{\mu_K} - p_{\nu_{n,\varepsilon}})(z) \, \mathrm{d}(p_{\mu_K} - p_{\nu_{n,\varepsilon}})(w) \\ &= -\int_{\mathbb{C}^2} \ln(|z - w|) \, \mathrm{d}\mu_K(z) \, \mathrm{d}\mu_K(w) \\ &\quad - \int_{\mathbb{C}^2} \ln(|z - w|) \, \mathrm{d}\nu_{n,\varepsilon}(z) \, \mathrm{d}\nu_{n,\varepsilon}(w) \\ &\quad + 2\int_{\mathbb{C}^2} \ln(|z - w|) \, \mathrm{d}\mu_K(z) \, \mathrm{d}\nu_{n,\varepsilon}(w) \\ &= -\ln(c(K)) - I(\nu_{n,\varepsilon}) + 2\int_{\mathbb{C}} p_{\mu_K} \, \mathrm{d}\nu_{n,\varepsilon}.\end{aligned}$$

Thus,

$$\begin{aligned}\left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} \right| &\leq \sqrt{\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c\varphi} \times \sqrt{\ln(c(K)) - I(\nu_{n,\varepsilon}) + 2\int_{\mathbb{C}} G_K \, \mathrm{d}\nu_{n,\varepsilon}} \\ &= \sqrt{\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c\varphi} \times \sqrt{\frac{\ln(c(K)) - \ln(\varepsilon)}{n} + 2\int_{\mathbb{C}} G_K \, \mathrm{d}\nu_{n,\varepsilon}}.\end{aligned}$$

Since $G_K \equiv 0$ on $K$,

$$\int_{\mathbb{C}} G_K \, \mathrm{d}\nu_{n,\varepsilon} = \frac{1}{n}\sum_{i=1}^{n} \int_{\mathbb{C}} G_K \, \mathrm{d}\lambda_{S(w_i,\varepsilon)} = \frac{1}{n}\sum_{i=1}^{n} \int_{\mathbb{C}} (G_K - G_K(w_i)) \, \mathrm{d}\lambda_{S(w_i,\varepsilon)} \leq C\varepsilon^{\alpha}.$$

We finally get

$$\left| \int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon} \right| \leq \sqrt{\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c\varphi} \times \sqrt{\frac{\ln(c(K)) - \ln(\varepsilon)}{n} + 2C\varepsilon^{\alpha}}$$

We choose $\varepsilon := \frac{1}{n^{\frac{1}{\alpha}}}$:

$$\left|\int_{\mathbb{C}} \varphi \, \mathrm{d}\mu_K - \int_{\mathbb{C}} \varphi \, \mathrm{d}\nu_{n,\varepsilon}\right| \leq \sqrt{\int_{\mathbb{C}} \mathrm{d}\varphi \wedge \mathrm{d}^c\varphi} \times \sqrt{\frac{C + \ln(c(K))}{n} + \frac{1}{\alpha}\frac{\ln(n)}{n}}.$$

This yields the theorem with $A = \sqrt{C + \ln(c(K)) + \frac{1}{\alpha}}$. All that remains is to prove the lemma.

*Proof of the key lemma.*

Let $\rho = \rho_+ - \rho_-$ be a signed measure, and assume that $\rho_+$, $\rho_-$ are probability measures and neither give mass to singletons. Let:

$$g_\rho(w) := \int_{\mathbb{C}} \ln(|z-w|) \, \mathrm{d}\rho(z) = p_{\rho_+} - p_{\rho_-}.$$

Then $\mathrm{dd}^c g_\rho = \mathrm{dd}^c p_{\rho_+} - \mathrm{dd}^c p_{\rho_-} = \rho = \mathrm{dd}^c g$. Hence, $g - g_\rho$ is harmonic on $\mathbb{P}_1(\mathbb{C})$, and thus constant, and

$$(\rho, \rho) = -\int_{\mathbb{C}^2} \ln(|z-w|) \, \mathrm{d}\rho(z) \, \mathrm{d}\rho(w) = -\int_{\mathbb{C}} g_\rho \, \mathrm{d}\rho = \int_{\mathbb{C}} \mathrm{d}g_\rho \wedge \mathrm{d}^c g_\rho = \int_{\mathbb{C}} \mathrm{d}g \wedge \mathrm{d}^c g.$$

If $g$ is $\mathcal{C}^1$, then

$$\mathrm{d}g \wedge \mathrm{d}^c g = \frac{1}{2\pi}\left(\left|\frac{\partial g}{\partial x}\right|^2 + \left|\frac{\partial g}{\partial y}\right|^2\right) \mathrm{d}\,\mathrm{Leb}\,.$$

In particular, $\int_{\mathbb{C}} \mathrm{d}g \wedge \mathrm{d}^c g = \|g'\|^2_{\mathbb{L}^2} \geq 0$.

If $g$ is only continuous, we approximate it by convolution. □

□

# 3 Entropy of complex polynomials (January 15th, Thomas Morand)

The references for this talk are [1, 2, 4, 6].

## 3.1 The Riemann sphere

Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. It is a compact space, which we may identify with $\mathbb{S}_2 \subset \mathbb{R}^3$. Let $z \mapsto z^*$ be the inverse of the stereographic projection. We define a distance, and its associated topology, on $\widehat{\mathbb{C}}$:

$$\sigma(z, w) = \|z^* - w^*\|_{\mathbb{R}^3} = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} \quad \forall z, w \in \mathbb{C},$$

$$\sigma(z, \infty) = \lim_{w \to \infty} \|z^* - w^*\|_{\mathbb{R}^3} = \frac{|z|}{\sqrt{1 + |z|^2}} \quad \forall z \in \mathbb{C}.$$

The open subsets of $\widehat{\mathbb{C}}$ are:

- the open subsets of $\mathbb{C}$;
- the sets $\{\infty\} \cup (\mathbb{C} \setminus K)$, with $K \subset \mathbb{C}$ compact.

The space $\widehat{\mathbb{C}}$ is a dimension 1 complex manifold, when endowed with the charts $z \mapsto z$ on $\mathbb{C}$ and $z \mapsto 1/z$ on $\widehat{\mathbb{C}} \setminus \{0\}$.

The **spherical derivative** of $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is

$$|f'(z)|_\sigma := \lim_{w \to z} \frac{\sigma(f(w), f(z))}{\sigma(w, z)}.$$

**Definition 3.1.**
*A function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is* ***holomorphic*** *if $f$ (or $1/f$) is analytical at each $z_0 \in \mathbb{C}$, and $z \mapsto f(1/z)$ (or $z \mapsto 1/f(1/z)$ if $f(\infty) = \infty$) is analytical on a neighborhood of $0$.*

*We denote by $\mathcal{O}(\widehat{\mathbb{C}})$ the space of holomorphic functions on $\widehat{\mathbb{C}}$.*

**Proposition 3.2.**
*The set of holomorphic functions on $\widehat{\mathbb{C}}$ is the set of rational functions on $\mathbb{C}$.*

*Proof.*
Let $f \in \mathcal{O}(\widehat{\mathbb{C}})$ be nonconstant. Let $\{z_i,\ i \in I\}$ be the set of its poles. By removing theses poles, we get a function $h = f - \sum_{i \in I} g_i - g$ which is continuous on $\widehat{\mathbb{C}}$ and thus continuous and bounded on $\mathbb{C}$. By Liouville's theorem, $h$ is constant, so $f = h + g + \sum_{i \in I} g_i$ is rational. □

## 3.2 Entropy

Let $X$ be compact and $f \in \mathcal{C}(X, X)$. We denote by $h_{\mathrm{top}}(f)$ its **topological entropy**, and by $h_\mu(f)$ its **metric entropy**.

Let $d$ be a distance on $X$. For all $n \in \mathbb{N}$ and $x$, $y \in X$, set

$$d_n(x, y) := \max_{0 \le k \le n-1} d(f^k(x), f^k(y)).$$

Then $d_n$ is a distance on $X$. We denote by $B_n(x, r)$ the associated open ball.

**Theorem 3.3** (Brin-Katok)**.**
*Let $\mu \in \mathcal{P}(X)$ be ergodic. For $\mu$-almost every $x \in X$,*

$$h_\mu(f) = \sup_{\varepsilon>0} \liminf_{n\to+\infty} -\frac{1}{n}\ln(\mu(B_n(x,\varepsilon))). \tag{3.1}$$

A subset $F \subset X$ is said to be $(n,\varepsilon)$-**separated** if, for all $x \neq y \in F$, we have $d_n(x,y) \geq \varepsilon$; or, in other words, if the balls $(B_n(x,\varepsilon/2))_{x\in F}$ are pairwise disjoint.

Let $N_d(f,\varepsilon,n)$ be the maximal cardinal of a $(n,\varepsilon)$-separated subset, and

$$h_d(f,\varepsilon) := \limsup_{n\to+\infty} \frac{1}{n}\ln(N_d(f,\varepsilon,n)).$$

Then

$$h_{\text{top}}(f) = \lim_{\varepsilon\to 0} h_d(f,\varepsilon) = \sup_{\varepsilon\to 0} h_d(f,\varepsilon).$$

We recall:

**Proposition 3.4** (Variational principle)**.**

$$h_{\text{top}}(f) = \sup_{\mu\in\mathcal{P}(X)} h_\mu(f). \tag{3.2}$$

## 3.3 Dynamics

We now study specifically the dynamics of a polynomial $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$.

### 3.3.1 Entropy of $P$

**Theorem 3.5** (Misurewicz, Przytycki)**.**
*For any $P \in \mathbb{C}[X]$ of degree $d$,*

$$h_{\text{top}}(P) \geq \ln(d). \tag{3.3}$$

*Proof.*
We write $L := \sup_{z\in\widehat{\mathbb{C}}} |P'(z)|_\sigma$. Since $\widehat{\mathbb{C}}$ is compact and $z \mapsto |P'(z)|_\sigma$ is continuous, $L < +\infty$. Let $\alpha \in (0,1)$; set $\varepsilon := L^{\frac{\alpha}{\alpha-1}}$ and $B := \{z \in \widehat{\mathbb{C}} :\ |P'(z)|_\sigma \geq \varepsilon\}$.

**A subset of good preimages.** Fix $\delta > 0$ such that, whenever $z \neq w \in B$ and $\sigma(z,w) < \delta$, we have $P(z) \neq P(w)$.

Let $n \in \mathbb{N}$ and $x$ be a regular value of $P^n$. Then $P^{-1}(\{x\})$ contains $d$ preimages. If $d$ of these preimages are in the good set $B$, we set $Q_1(x)$ to be those $d$ preimages. Otherwise, we choose one of these preimages $y \in B^c$, and set $Q_1(x) = \{y\}$.

Since all the points in $Q_1(x)$ are regular values of $P^{n-1}$, we may iterate this process, getting $Q_2(x) \subset P^{-2}(\{x\})$, …, $Q_n(x) \subset P^{-n}(\{x\})$. By construction, $Q_n(x)$ is $(n,\delta)$-separated.

**A point whose preimages are mostly good.** Set

$$A := \{y \in \mathbb{C} :\ \text{Card}(B \cap \{y, P(y), \dots, P^{n-1}(y)\}) \leq \alpha n\}.$$

Let us show that, for all $n \in \mathbb{N}$, there exists $x \in \widehat{\mathbb{C}} \setminus P^n(A)$ which is a regular value of $P^n$. Let $y \in A$. Then

$$|(P^n)'(y)|_\sigma \leq |P'(P^{n-1}(y))|_\sigma \dots |P'(P(y))|_\sigma |P'(y)|_\sigma.$$

There are at least $n - \rfloor \alpha n \lfloor$ instances where $|P'(P^{n-k}(y))|_\sigma < \varepsilon$, whence

$$|(P^n)'(y)|_\sigma < \varepsilon^{n-\rfloor \alpha n \lfloor} L^{\lfloor \alpha n \rfloor} \leq \varepsilon^{n-\alpha n} L^{\alpha n} = 1.$$

By Sard's lemma, there exists $x \in \widehat{\mathbb{C}} \setminus P^n(A)$ which is a regular value of $P^n$.

**Lower bound on the entropy.** Let $x \in \widehat{\mathbb{C}} \setminus P^n(A)$ be a regular value of $P^n$. Since $x = P^n(Q_n(x)) \notin A$, we have in particular $Q_n(x) \cap A = \emptyset$.

Let $y \in Q_n(x)$. Since $y \notin A$, we have $\mathrm{Card}(B \cap \{y, P(y), \ldots, P^{n-1}(y)\}) > \alpha n$. In other words, to go from $y$ to $x$, we have at least $\lfloor \alpha n \rfloor + 1$ good transitions, so that $\mathrm{Card}(Q_n(x)) \geq d^{\alpha n}$.

Hence, $Q_n(x)$ is a $(n, \delta)$-separated subset whose cardinal is at least $d^{\alpha n}$. Hence, $h_d(P, \delta) = \limsup_{n \to +\infty} \frac{1}{n} \ln(d^{\alpha n}) \geq \alpha \ln(d)$. Taking the limit as $\delta \to 0$, we get $h_{\mathrm{top}}(P) \geq \alpha \ln(d)$. Since this holds for all $\alpha \in (0,1)$, finally, $h_{\mathrm{top}}(P) \geq \ln(d)$. □

### 3.3.2 Julia set

Let $P(X) = \sum_{k=0}^d a_k X^k$ be a polynomial of degree $d \geq 2$. Then

$$|P(z)| = |z||z^{d-1}| \left| \sum_{k=0}^d a_k z^{k-d} \right|.$$

The rightmost term is bounded on $\mathbb{C} \setminus B(0,1)$. Since $d \geq 2$, there exists $M \geq 0$ such that $|P(z)| \geq 2|z|$ whenever $|z| \geq M$. Hence, either $(P^n(z))_{n \in \mathbb{N}}$ is bounded, or it converges to $\infty$.

Set $A(\infty) := \{z \in \widehat{\mathbb{C}} : \lim_{n \to +\infty} P^n(z) = \infty\}$ and $K_P := A(\infty)^c = \{z \in \widehat{\mathbb{C}} : (P^n(z))_{n \in \mathbb{N}} \text{ is bounded}\}$ the **filled-in Julia set**. Set also $J_P := \partial K_P$ the **Julia set**, and $F_P := J_P^c$ the **Fatou set**.

**Lemma 3.6.**
*$A(\infty)$ is open.*

*Proof.*
$A(\infty) = \bigcup_{n=0}^{+\infty} P^{-n}(\overline{B}(0,M)^c)$. □

**Theorem 3.7.**
$c(P^{-1}(K)) = \left(\frac{c(K)}{|a_d|}\right)^{\frac{1}{d}}$ *whenever $K$ is compact [6, Theorem 5.2.5].*
*In particular, $c(K_P) = |a_d|^{-\frac{1}{d-1}}$.*

### 3.3.3 Green function of $P$

Let us define

$$g_P := \left\{ \begin{array}{ccl} \mathbb{C} & \to & \mathbb{R} \\ z & \mapsto & \lim_{n \to +\infty} \frac{1}{d^n} \ln_+ |P^n(z)|. \end{array} \right.$$

Set $u(z) := \ln_+ |P(z)| - d \ln_+ |z|$. Whenever $|z| > \max\{1, M\}$, we have $u(z) = \ln|P(z)| - d \ln|z|$, so that $u$ is continuous and bounded on $\widehat{\mathbb{C}}$. In addition,

$$\begin{aligned} \ln_+ |z| + \sum_{n=0}^{+\infty} \frac{1}{d^{n+1}} u(P^n(z)) &= \ln_+ |z| + \lim_{N \to +\infty} \sum_{n=0}^{N} \left[ \frac{1}{d^{n+1}} \ln_+ |P^{n+1}(z)| - \frac{1}{d^n} \ln_+ |P^n(z)| \right] \\ &= \lim_{N \to +\infty} \frac{1}{d^{N+1}} \ln_+ |P^{N+1}(z)| \\ &= g_P(z). \end{aligned}$$

Hence, $g_P$ is well-defined, continuous, and

$$g_P \circ P = d g_P. \tag{3.4}$$

The function $g_P$ is subharmonic as a uniform limit of continuous and subharmonic functions. In addition, $g_P \equiv 0$ on $K_P$, and $\lim_{n\to+\infty} \frac{1}{d^n} \ln(|P^n(z)|)$ for $z \notin K_P$. In particular, $g_P$ is harmonic on $A(\infty)$.

### 3.3.4 Equilibrium measure

We know that $g_P : \mathbb{C} \to \mathbb{R}$ is continuous, subharmonic, harmonic on $A(\infty)$, zero on $K_P$, and such that $g_P(z) - \ln_+ |z| = \sum_{n=0}^{+\infty} \frac{1}{d^{n+1}} u(P^n(z))$ is bounded.

Hence, $\mathrm{dd}^c g_P =: \mu_P$ is the equilibrium measure of $K_P$, and

$$I(\mu_P) = \ln(c(K_P)) = -\frac{1}{d-1} \ln(|a_d|).$$

**Definition 3.8.**
*Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a proper function and $\varphi \in \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R})$. Let $\mu$ be a measure on $\widehat{\mathbb{C}}$. We set*

$$f^*\varphi := \varphi \circ f,$$

*and, by duality,*

$$\langle f_*\mu, \varphi \rangle := \langle \mu, f^*\varphi \rangle = \langle \mu, \varphi \circ f \rangle.$$

Since $f$ is proper, it has a topological degree $d$, and thus

$$f_*\varphi(x) = \sum_{y \in f^{-1}(\{x\})} \varphi(y).$$

By duality,

$$\langle f^*\mu, \varphi \rangle := \langle \mu, f_*\varphi \rangle = \int_{\widehat{\mathbb{C}}} \sum_{y \in f^{-1}(\{x\})} \varphi(y) \, \mathrm{d}\mu(x).$$

**Proposition 3.9.**

- $P^*\mathrm{dd}^c g_P = \mathrm{dd}^c(g_P \circ P)$.
- $P^*\mu_P = d\mu_P$.
- $P_*P^*\mu_P = d\mu_P$.
- $P_*\mu_P = \mu_P$.

*Proof.*
Assume the first point. Then

$$P^*\mu_P = P^*\mathrm{dd}^c g_P = \mathrm{dd}^c(g_P \circ P) = d\mathrm{dd}^c g_P = d\mu_P,$$

which proves the second point.

Let $\varphi \in \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R})$. Then

$$\langle P_*P^*\mu_P, \varphi \rangle = \langle P^*\mu_P, P^*\varphi \rangle = \langle \mu_P, P_*P^*\varphi \rangle.$$

But

$$\begin{aligned}
P_*P^*\varphi(x) &= \sum_{y\in P^{-1}(\{x\})} (P^*\varphi)(y) \\
&= \sum_{y\in P^{-1}(\{x\})} \varphi(P(y)) \\
&= \sum_{y\in P^{-1}(\{x\})} \varphi(x) \\
&= d\varphi(x),
\end{aligned}$$

so that $\langle P_*P^*\mu_P, \varphi\rangle = \langle \mu_P, d\varphi\rangle = d\langle \mu_P, \varphi\rangle$. As this holds for all continuous $\varphi$, we get $P_*P^*\mu_P = d\mu_P$, which is the third point.

Finally, $P_*\mu_P = P_*(d^{-1}P^*\mu_P) = d^{-1}P_*P^*\mu_P = d^{-1}d\mu_P = \mu_P$, which proves the fourth and last point.‘ □

Hence, the Jacobian of $P$ with respect to $\mu_P$ is constant and equal to $d$: if $P$ is injective on $B$, then $\mu_P(P(B)) = d\mu_P(B)$. The measure $\mu_P$ is mixing, and thus ergodic.

**Theorem 3.10.**

$$h\ \mu_P(P) = \ln(d).$$

*Proof.*

By the variational principle,

$$h\ \mu_P(P) \leq h\ \mathrm{top}(P) \leq \ln(d).$$

By Brin-Katok's theorem, for all $\alpha > 0$, there exists $\varepsilon > 0$ and $X_\alpha$ a Borel subset of positive measure (for $\mu_P$) such that, for all large enough $n$ and all $x \in X_\alpha$,

$$\mu_P(B_n(x,\varepsilon)) \leq d^{-(1-\alpha)n}.$$

Hence, $h\ \mathrm{top}(P) \geq \ln(d)$.

Let $\alpha > 0$, and $U$ a neighborhood of the set $V$ of critical values of $P$ such that $\mu_P(U) \leq \alpha/2$. Set

$$X_\alpha(n) := \left\{x \in \widehat{\mathbb{C}} :\ \mathrm{Card}\{0 \leq k \leq n-1 :\ P^k(x) \in U\} \leq n\alpha\right\},$$

and $X_\alpha := \liminf_{n\to+\infty} X_\alpha(n)$. By Birkhoff's ergodic theorem, for $\mu_P$-almost every $x \in \widehat{\mathbb{C}}$,

$$\lim_{n\to+\infty} \frac{\mathrm{Card}\{0 \leq k \leq n-1 :\ P^k(x) \in U\}}{n} = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_U(P^k(x)) = \mu_P(U) \leq \frac{\alpha}{2}.$$

In particular, $\mu_P$-almost surely, the point $x$ belongs to $X_\alpha(n)$ for all large enough $n$, whence $\mu_P(X_\alpha) = 1$.

Let $x \in X_\alpha$ and $\varepsilon < d(V, \partial U)$ be small enough. Note that $P$ sends $B_{n-k}(P^k(x), \varepsilon)$ into $B_{n-k-1}(P^{k+1}(x), \varepsilon)$. If $P^{k+1}(x) \in U$, then $\mu_P(B_{n-k}(P^k(x), \varepsilon)) \leq \mu_P(B_{n-k-1}(P^{k+1}(x), \varepsilon))$, since $\mu_P$ is $P$-invariant. Otherwise, by taking $\varepsilon$ small enough, we can ensure that $P$ is injective on $B_{n-k}(P^k(x), \varepsilon)$, and

$$\mu_P(B_{n-k}(P^k(x),\varepsilon)) = d\mu_P(P(B_{n-k}(P^k(x),\varepsilon))) \leq \mu_P(B_{n-k-1}(P^{k+1}(x),\varepsilon)).$$

Hence, for all $x \in X_\alpha$ and all large enough $n$,

$$\mu_P(B_n(x,\varepsilon)) \leq d^{-(1-\alpha)n}.$$ □

# 4 Equidistribution of equilibrium measures (January 29th, Mateo Ghezal)

The references for this talk are [3, 6].

## 4.1 Recalls

Let $P$ be a monic polynomial of degree $d \geq 2$.

**Definition 4.1.**
*Let $\mu \in \mathcal{P}(\mathbb{C})$ with compact support. We set*

$$p_\mu(z) := \int_{\mathbb{C}} \ln|z - w| \, \mathrm{d}\mu(w).$$

*The **energy** of $\mu$ is then*

$$I(\mu) := \int_{\mathbb{C}} p_\mu(z) \, \mathrm{d}\mu(z) = \int_{\mathbb{C}} \int_{\mathbb{C}} \ln|z - w| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(w).$$

*If $K$ is compact, then the **capacity** of $K$ is*

$$c(K) := \sup_{\mu \in \mathcal{P}(K)} e^{I(\mu)}.$$

**Definition 4.2.**
*We set*

$$A(\infty) := \left\{ z \in \mathbb{C} : \lim_{n \to +\infty} P^n(z) = +\infty \right\}.$$

*The **filled-in Julia set** is $K_P := A(\infty)^c$, and $J_P := \partial K_P$*

If $\mu_P$ is the equilibrium measure of $K_P$, then $\mu_P := \mathrm{dd}^c G_P$.

**Theorem 4.3.**

$$c(K_P) = 1 \text{ and } I(\mu_P) = 0. \tag{4.1}$$

## 4.2 Asymptotic distribution of preimages

For $z_0 \in J_P$, we set $\mu_n := d^{-n} \sum_{x : P^n(x) = z_0} \delta_x$.

**Theorem 4.4.**
*For all $z_0 \in J_P$, the sequence $(\mu_n)_{n \geq 0}$ converges in distribution to $\mu_P$.*

*Proof.*
We know that $\limsup_{n \to +\infty} p_{\mu_n} \leq 0$ on $J_P$.
Let $z \in K_P$. Then

$$p_{\mu_n}(z) = d^{-n} \sum_{k=1}^{d^n} \ln|z - z_k|,$$

were $(z_k)_{1 \leq k \leq d^n}$ is an enumeration of $P^{-n}(\{z_0\})$. Hence,

$$p_{\mu_n}(z) = d^{-n} \ln|P^n(z) - z_0|,$$

and $\limsup_{n \to +\infty} \ln|P^n(z) - z_0| \leq 0$. In addition, we recall

**Lemma 4.5** (Frostman)**.**

$$p_{\mu_P} \geq I(\mu_P).$$

Hence,

$$\int_{J_P} \limsup_{n\to+\infty} p_{\mu_n} \,\mathrm{d}\mu_P \geq \limsup_{n\to+\infty} \int_{J_P} p_{\mu_n} \,\mathrm{d}\mu_P = \limsup_{n\to+\infty} \int_{J_P} p_{\mu_n} \,\mathrm{d}\mu_n \geq 0.$$

Finally, $\limsup_{n\to+\infty} p_{\mu_n}(z_0) \geq 0$ for $\mu_P$-almost every $z_0 \in J_P$.

Let $\mu$ be a limit point of $(\mu_n)_{n\geq 0}$. Then $\limsup_{n\to+\infty} p_{\mu_n}(z) \leq p_\mu(z)$ on $J_P$, so that $p_\mu(z) \geq 0$ for $\mu$-almost every $z \in J_P$. Denote by $\Omega$ this set. Then $\mu_P(\overline{\Omega}) = 1$, and $\mathrm{Supp}(\mu_P) \subset \overline{\Omega}$.

But $p_\mu$ is upper semi-continuous. Let $(z_n)$ be a sequence of $\overline{\Omega}$ converging to $z$. Then $p_\mu(z) \geq \limsup_{n\to+\infty} p_\mu(z_n) \geq 0$. Hence, $I(\mu) = \int_{J_P} p_\mu \,\mathrm{d}\mu \geq 0 = I(\mu_P)$. By uniqueness of the equilibrium measure, $\mu = \mu_P$. □

**Remark 4.6.**

*We can count points without multiplicity. We have* $\mu_n \mathrm{dd}^c d^{-n} \ln|P^n(\cdot) - z_0| + o(1)$*. Writing* $z_1$, …, $z_{m_n}$ *the roots of* $P^n - z_0$*, we can use instead* $\mu_n' := m_n^{-1} \sum_{k=1}^{m_n} \delta_{z_k}$*. Indeed,*

$$\left|p_{\mu_n} - d^{-n} \ln|P^n - z_0|\right| = \frac{1}{d^n} \left| \sum_{k=m_n+1}^{d^n} \ln|z_k - \cdot| \right|.$$

*If this error term is not in* $o(1)$*, then there exists a point* $z$ *such that* $P^{-1}(\{z\}) = \{w\}$*. But then* $z = w$ *is a super-attracting critical point.*

**Remark 4.7.**

*We can choose* $z_0 \in K_P$ *such that* $P^{-1}(\{z_0\}) \neq \{z_0\}$*.*

## 4.3 Asymptotic distribution of periodic points

Let $\mathrm{Per}_n := \{z \in \mathbb{C} : \ P^n(z) = z\}$, and

$$\mu_n := \frac{1}{\mathrm{Card}(\mathrm{Per}_n)} \sum_{z : P^n(z) = z} \delta_z.$$

We want to show that any accumulation point of $(\mu_n)_{n\geq 1}$ is supported on $J_P$, and that $\mu_n = \mathrm{dd}^c d^{-n} \ln|P^n - z| + o(1)$.

Let us prove the first point. Let $\mu$ be a limit point of $(\mu_n)_{n\geq 1}$ and $U$ be a connected component of the interior of $K_P$. Then $U$ contains at most one periodic point $z$. If $z$ has multiplicity 2 or higher, then it is parabolic, which is absurd. Hence, $z$ in simple, and thus $\mu_n(U) \leq d^{-n}$ and $\mu(U) = 0$. Finally, $\mu$ is supported on $J_P$.

The proof of the second point is very similar to the proof for the preimages. Actually, Brolin has shown a more general statement:

**Lemma 4.8** (Brolin)**.**

*Let* $E \subset H$ *be two compact subsets such that* $c(E) = e^V > 0$*. Let* $(\mu_n)_{n\geq 1}$ *be a sequence of probability measures supported on* $H$*, converging to* $\mu \in \mathcal{P}(E)$*.*

*If* $\limsup_{n\to+\infty} p_{\mu_n} \leq V$*, then* $\mu = \mu_E$*.*

## 4.4 Application

We can draw $J_P$.

**Corollary 4.9.**

*If* $z_0 \in J_P$*, then*

$$J_P = \overline{\bigcup_{n \geq 0} P^{-n}(z_0)}.$$

*Proof.*

Write $X := \overline{\bigcup_{n \geq 0} P^{-n}(z_0)}$. Since $J_P$ is $P$-invariant, we have $X \subset J_P$. In addition, $J_P = \mathrm{Supp}(\mu_P) \subset X$. □

# 5 Currents and plurisubharmonic functions (March 4th, Thomas Gauthier)

## 5.1 Forms and currents on manifolds

### 5.1.1 Differential forms

**Definition 5.1** (Differential forms).
*Let $U \subset \mathbb{R}^n$ be open. A $p$-**form** on $U$ is a section of $\Lambda^p(T^*U)$. In coordinates,*

$$\varphi(x) = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=p}} \varphi_I(x) \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p},$$

*where, if $f_1, \ldots, f_p$ are linear forms,*

$$(f_1 \wedge \ldots \wedge f_p)(x_1, \ldots, x_p) = \det(f_j(x_i)).$$

**Example 5.2.**
*A 0-form is a smooth function.*

*Let $f : U \to \mathbb{R}^n$ be a smooth function. Then*

$$\mathrm{d}f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \mathrm{d}x_i$$

*is a 1-form.*

**Proposition 5.3.**
*Let $\Omega^p(U)$ be the space of $\mathcal{C}^\infty$ $p$-forms on $U$. Then $\Omega^p(U)$ is a $\mathbb{R}$-vector space and a $\mathcal{C}^\infty(U)$-modulo of finite type: whenever $\varphi$, $\psi \in \Omega^p(U)$ and $f$, $g \in \mathcal{C}^\infty(U)$, we have $f\varphi + g\psi \in \Omega^p(U)$.*

### 5.1.2 Exterior product and differential

**Definition 5.4** (Exterior product).
*Let $p$, $q$ be such that $p + q \leq n$. Let $\varphi \in \Omega^p(U)$ and $\psi \in \Omega^q(U)$. We define*

$$\varphi \wedge \psi(x) := \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=p}} \sum_{\substack{J \subset \{1,\dots,n\} \\ |J|=q}} \varphi_I(x)\psi_J(x) \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p} \wedge \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_q}. \tag{5.1}$$

**Remark 5.5.**
*The exterior product in noncommutative: $\varphi \wedge \psi = (-1)^p \psi \wedge \varphi$.*

*In Equation (5.1), whenever $I \cap J \neq \emptyset$, the term in the sum vanishes.*

**Definition 5.6** (Derivation).
*Let $\varphi \in \Omega^p(U)$. We define*

$$\mathrm{d}\varphi(x) := \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=p}} \mathrm{d}\varphi_I \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}. \tag{5.2}$$

**Remark 5.7.**

*We compute*

$$\mathrm{d}^2\varphi = \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=p}} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}.$$

*But, by Schartz's lemma,* $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$*, so that* $\frac{\partial^2 f}{\partial x_i \partial x_j}\mathrm{d}x_i \wedge \mathrm{d}x_j = -\frac{\partial^2 f}{\partial x_j \partial x_i}\mathrm{d}x_j \wedge \mathrm{d}x_i$*. Hence,* $\mathrm{d}^2\varphi = 0$*, or more summarily,* $\mathrm{d}^2 = 0$.

**Definition 5.8** (Compactly-supported $p$-forms)**.**

*Given a $p$-form $\varphi$, we define its* ***support*** *as* $\bigcup_{\substack{I\subset\{1,\dots,n\}\\|I|=p}} \mathrm{Supp}(\varphi_I)$.

*We denote by $\mathcal{D}^p(U)$ the space of smooth compactly supported $p$-forms on $U$. It is endowed with a generating family of semi-norms* $\|\varphi\|_{K,N} = \max \|\varphi_I\|_{\mathcal{C}^N(K)}$ *for any $N \geq 0$ and $K$ compactly embedded in $U$.*

### 5.1.3 Currents

**Definition 5.9** ($(n-p)$-current)**.**

*A $(n-p)$-**current** on $U$ is a continuous linear form $T : \mathcal{D}^p(U) \to \mathbb{R}$. We denote by $(\mathcal{D}^p(U))'$ the space of $(n-p)$-currents.*

*The space $(\mathcal{D}^p(U))'$ is endowed with the weak-$*$ topology: $(T_n)_{n\geq 0}$ converges to $T$ in $(\mathcal{D}^p(U))'$ if $(\langle T_n, \varphi\rangle)_{n\geq 0}$ converges to $\langle T, \varphi\rangle$ for all $\varphi \in \mathcal{D}^p(U)$.*

**Example 5.10.**

$\mathcal{D}'(U) = (\mathcal{D}^n(U))'$.

*Let $S \subset U$ be a submanifold of dimension $p$. Set*

$$\langle [S], \varphi\rangle := \int_S \varphi_S.$$

*Then $[S] \in (\mathcal{D}^p(U))'$ is the* ***integration current on*** $S$.

*Let $\psi \in \Omega^{n-p}(U)$. Then*

$$\langle \psi, \varphi\rangle := \int_U \psi \wedge \varphi.$$

**Definition 5.11** (Derivation and exterior product)**.**

*Let $T$ be a $(n-p)$-current. We define for $\varphi \in \mathcal{D}^{p-1}(U)$:*

$$\langle \mathrm{d}T, \varphi\rangle := (-1)^{p+1}\langle T, \mathrm{d}\varphi\rangle,$$

*which gives a derivation on $(\mathcal{D}^p(U))'$.*

*We define for $\alpha \in \Omega^q(U)$ and $\varphi \in \mathcal{D}^{p-q}(U)$:*

$$\langle T \wedge \alpha, \varphi\rangle := \langle T, \alpha \wedge \varphi\rangle,$$

*which gives a mixed exterior product between $(\mathcal{D}^p(U))'$ and $\Omega^q(U)$ whenever $q \leq p$.*

**Example 5.12.**

$\mathrm{d}[S] = (-1)^{p+1}[\partial S]$ *by Stokes' theorem.*

$\mathrm{d}\psi = \mathrm{d}\psi$*, where the left-hand side is understood as a current and the right-hand side as a differential form.*

**Remark 5.13.**

*In coordinates,* $T = \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=n-p}} T_I \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_{n-p}}$*, where* $T_I \in \mathcal{D}'(U)$*.*

*For a given* $I$*, we can recover* $T_I$ *in the following way. Write* $J := I^c$*. For any* $f \in \mathcal{D}(U)$*,*

$$\langle T_I, f\rangle = \langle T, f\mathrm{d}x_{j_1} \wedge \ldots \wedge x_{j_p}\rangle.$$

*If we are given a family* $(T_I)_{\substack{I\subset\{1,\dots,n\}\\|I|=n-p}}$ *of distributions, for any* $\varphi \in \mathcal{D}^p(U)$*, we may define*

$$\langle T, \varphi\rangle := \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=n-p}} \langle T_I, \varphi_{I^c}\rangle.$$

**Example 5.14.**

*Let* $(f_I)_{\substack{I\subset\{1,\dots,n\}\\|I|=n-p}}$ *be a family of functions in* $\mathbb{L}^1_{loc}(U)$*. Then*

$$f := \sum_{\substack{I\subset\{1,\dots,n\}\\|I|=n-p}} f_I \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_{n-p}} \in (\mathcal{D}^p(U))'.$$

### 5.1.4 de Rham cohomology via currents

**Definition 5.15** (Closed forms, exact forms)**.**

*A* $p$*-form* $\varphi$ *is* ***closed*** *if* $\mathrm{d}\varphi = 0$*.*

*A* $p$*-form* $\varphi$ *is* ***exact*** *if* $\varphi = \mathrm{d}\psi$ *for some* $(p-1)$*-form* $\psi$*.*

Since $\mathrm{d}^2 = 0$, any exact form is closed.

**Definition 5.16** (Cohomology groups)**.**

*Let* $M$ *be a* $n$*-dimensional manifold. The* $p$***th cohomology group*** *of* $M$ *is the* $\mathbb{R}$*-vector space*

$$H^p(M) := \{\text{closed } p \text{ forms}\}/\{\text{exact } p \text{ forms}\} = \mathrm{Ker}(\mathrm{d}:\ \Omega^p(M) \to \Omega^{p+1}(M))$$

**Example 5.17.** *By Poincaré's lemma, if* $U$ *is a star-shaped open set, then* $H^p(U) = 0$ *for all* $p \geq 1$*.*

**Definition 5.18** (Closed currents, exact currents)**.**

*A* $(n-p)$*-current* $T$ *is* ***closed*** *if* $\mathrm{d}T = 0$*.*

*A* $(n-p)$*-current* $T$ *is* ***exact*** *if* $T = \mathrm{d}S$ *for some* $(n-p-1)$*-current* $S$*.*

**Lemma 5.19** (Poincaré-like lemma)**.**

*Let* $U$ *be a star-shaped open set. If* $T$ *is a closed current on* $U$*, then* $T$ *is exact.*

**Corollary 5.20.**

$$H^p(M) := \{\text{closed } p \text{ currents}\}/\{\text{exact } p \text{ currents}\}$$

**Example 5.21.** *Let* $M = \mathbb{P}_n(\mathbb{C})$ *and* $H = \{P = 0\} \subset M$ *an hypersurface, where* $P \in \mathbb{C}[X_1, \dots, X_n]$ *is an homogeneous polynomial.*

*Then* $H$ *is closed (without boundary) and* $\{[H]\}_{H^2(\mathbb{P}_n(\mathbb{C}))} = \deg(P)$*.*

## 5.2 Positivity of currents on complex manifolds

### 5.2.1 Hodge theory

Let $U \subset \mathbb{C}^n$. The space $\mathbb{C}^n$ can be seen as $\mathbb{R}^{2n}$ together with a complex multiplication operator $J$.

The correspondance from $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ can be written, for instance, with $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (z_1, \ldots, z_n)$ with $z_j = x_j + iy_j$. We may then define

$$\begin{cases} \mathrm{d}z_j & := & \mathrm{d}x_j + i\mathrm{d}y_j \\ \mathrm{d}\overline{z_j} & := & \mathrm{d}x_j - i\mathrm{d}y_j. \end{cases}$$

Any real $r$-form $\alpha$ can be written as

$$\alpha = \sum_{p+q=r} \alpha_{p,q} \text{ with } \alpha_{p,q} := \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=p}} \sum_{\substack{J\subset\{1,\ldots,n\} \\ |J|=q}} i^{p-q}\alpha_{p,q,I,J}\mathrm{d}z_{i_1} \wedge \ldots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \ldots \wedge \mathrm{d}\overline{z_{j_q}}.$$

This yields a decomposition $\Omega^r(M) = \bigoplus_{p+q=r} \Omega^{p,q}(M)$.

Let $\mathrm{d} := \partial + \overline{\partial} := \sum_{j=1}^n \frac{\partial}{\partial x_j}\mathrm{d}x_j + \sum_{j=1}^n \frac{\partial}{\partial y_j}\mathrm{d}y_j$. Then

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \\ \frac{\partial}{\partial \overline{z_j}} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \\ \mathrm{d}z_j &= \left(\frac{\partial}{\partial z_j}\right)^*, \\ \mathrm{d}\overline{z_j} &= \left(\frac{\partial}{\partial \overline{z_j}}\right)^*. \end{aligned}$$

We get two operators

$$\begin{cases} \partial & : & \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \\ \overline{\partial} & : & \Omega^{p,q}(M) \to \Omega^{p,q+1}(M). \end{cases}$$

These two operators satisfy $\partial \circ \partial = \overline{\partial} \circ \overline{\partial} = 0$. We may then define the **Dolbeaut cohomology** $H^{p,q}_{\overline{\partial}} := \mathrm{Ker}(\overline{\partial}) / \mathrm{Im}(\overline{\partial})$.

**Definition 5.22** (Conjugation)**.**
*The conjugation from* $\Omega^{p,q}$ *to* $\Omega^{q,p}$ *is the operator which sends* $\alpha_{p,q,I,J}\mathrm{d}z_{i_1}\wedge\ldots\wedge\mathrm{d}z_{i_p}\wedge\mathrm{d}\overline{z_{j_1}}\wedge\ldots\wedge\mathrm{d}\overline{z_{j_q}}$ *to* $\overline{\alpha_{p,q,I,J}}\mathrm{d}\overline{z_{i_1}} \wedge \ldots \wedge \mathrm{d}\overline{z_{i_p}} \wedge \mathrm{d}z_{j_1} \wedge \ldots \wedge \mathrm{d}z_{j_q}$.

### 5.2.2 Positive forms, positive currents

Let $\beta := \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j = \frac{i}{2}\mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$. Then $\beta^n$ is the Lebesgue measure. It is positive in the sense that, whenever $f \in \mathcal{D}(\mathbb{C}^n)$ is positive, we have $\langle \beta^n, f \rangle \geq 0$.

Missing : strongly positive

**Example 5.23.**
*Let* $\omega := i\sum_{j,k=1}^n a_{k,l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$ *be a* $(1,1)$*-form. Then* $\omega \geq 0$ *if and only if* $(a_{j,k}(x))$ *is Hermitian positive for all* $x \in M$.

*Let* $\omega$ *be a* $(p,p)$*-form. Then* $\omega \geq 0$ *if and only if, for all* $p$*-dimensional complex submanifold* $S \subset M$*, we have* $\omega_{|S} \geq 0$ *as a measure.*

**Definition 5.24** (Strongly positive form)**.**

**Definition 5.25** (Positive current)**.**

*Let $T$ be a current on a complex manifold $M$. We say that $T$ has **bidegree** $(p,p)$ or **bidimension** $(n-p, n-p)$ if $T \in (\mathcal{D}^{(p,p)}(M))'$.*

*Let $T$ be a $(p,p)$-current. We say that $T \geq 0$ if $\langle T, \omega \rangle \geq 0$ for all strongly positive $\omega \in \mathcal{D}^{n-p,n-p}(M)$.*

**Example 5.26.**

*Let $f_1, \ldots, f_N : M \to \mathbb{C}$ be holomorphic, so that $V = \bigcap_{j=1}^N \{f_j = 0\}$ is a complex analytic subvariety. Set*

$$\langle [V], \omega \rangle := \int_{V \setminus V_{sing}} \omega_{|V \setminus V_{sing}}.$$

*Then $[V] \geq 0$ and $\mathrm{d}[V] = 0$.*

## 5.3 $\mathrm{dd}^c$ and plurisubharmonic functions

Let $\mathrm{d}^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$, so that $\mathrm{dd}^c = \frac{i}{\pi}\partial\overline{\partial} : \Omega^{p,q} \to \Omega^{p+1,q+1}$.

**Definition 5.27.**

*Let $\varphi : M \to [-\infty, +\infty)$ be non identically $-\infty$. We say that $\varphi$ is **plurisubharmonic** (PSH) if:*

- *$\varphi$ is upper semi-continuous;*
- *$\varphi$ is sub-harmonic in each coordinate (plurisubharmonic): for all $a \in M$ and all small enough $\varepsilon > 0$, for all $j$,*

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + \varepsilon e^{i\theta} e_j) \, \mathrm{d}\theta.$$

**Lemma 5.28.**

*Let $\varphi : M \to [-\infty, +\infty)$. There is equivalence between:*

- *$\varphi$ is PSH;*
- *$\varphi$ is upper semi-continuous, and subharmonic along all holomorphic discs: for all holmorphic $u : \mathbb{D} \to M$, the function $\varphi \circ u$ is subharmonic.*
- *$\varphi$ is upper semi-continuous, locally integrable, and $\mathrm{dd}^c\varphi \geq 0$ as a $(1,1)$-current.*

For the third point: if $\varphi$ is $\mathcal{C}^2$, then

$$\mathrm{dd}^c\varphi = \sum_{j,k=1}^n \frac{i}{2} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k},$$

whence $\mathrm{dd}^c\varphi \geq 0$. If $\varphi$ is not $\mathcal{C}^2$, then way can apply this reasoning to $\varphi * \chi_\varepsilon$ for some convolution kernel $\chi_\varepsilon$, and let $\varepsilon$ vanish.

**Example 5.29.**

*If $f$ is holomorphic, then $\ln|f|$ is PSH.*

*If $u$, $v$ are PSH, then so is $\max\{u, v\}$.*

*If $n = \dim(M) = 1$, then plurisubharmonic functions are the same as subharmonic functions.*

# 6 Maximal plurisubharmonic functions and a complex Monge-Ampère equation (March 18th, Damien Thomine)

The only reference for this talk is [5].

Unless specified otherwise, $\Omega$ is an open subset of $\mathbb{C}^n$. Let us recall the following inclusions:

$$\begin{array}{ccc}
\mathrm{PH}(\Omega) & \longrightarrow & \mathrm{H}(\Omega) \\
\downarrow & & \downarrow \\
\mathrm{MPSH}(\Omega) & & \\
\downarrow & & \downarrow \\
\mathrm{PSH}(\Omega) & \longrightarrow & \mathrm{SH}(\Omega) \\
 & & \downarrow \\
 & & \mathbb{L}^1_{\mathrm{loc}}(\Omega)
\end{array}$$

- $\mathrm{PH}(\Omega)$: PluriHarmonic (functions)
- $\mathrm{H}(\Omega)$: Harmonic
- $\mathrm{MPSH}(\Omega)$: Maximal PluriSubHarmonic
- $\mathrm{PSH}(\Omega)$: PluriSubHarmonic
- $\mathrm{SH}(\Omega)$: SubHarmonic
- $\mathbb{L}^1_{\mathrm{loc}}(\Omega)$: locally integrable

All these spaces, excluding $\mathrm{MPSH}(\Omega)$, have been previously defined. The space $\mathrm{MPSH}(\Omega)$ of Maximal PluriSubHarmonic functions will be the main focus of today's talk.

## 6.1 (Sub)harmonic and pluri(sub)harmonic functions

Harmonic and pluriharmonic behave very differently under transformations: the notion of (sub)harmonicity depends on a *conformal* structure, while the notion of pluri(sub)harmonicity depends on a *complex* structure.

**Proposition 6.1.**
*Let $\varphi \in \mathrm{PH}(\Omega)$ (resp. $\mathrm{PSH}(\Omega)$). Let $f : \Omega' \to \Omega$ be holomorphic. Then $\varphi \circ f \in \mathrm{PH}(\Omega')$ (resp. $\mathrm{PSH}(\Omega')$).*

*Proof.*
Let $u : \mathbb{D} \to \Omega'$ be holomorphic. Then $f \circ u : \mathbb{D} \to \Omega$ is holomorphic, so $\varphi \circ (f \circ u) = (\varphi \circ f) \circ u$ is harmonic (resp. subharmonic). As this holds for all such $u$, the function $\varphi \circ f$ itself is pluriharmonic (resp. plurisubharmonic). □

In particular, PH and PSH functions can be defined on any complex manifold. By contrast:

**Example 6.2.**
*The function $\varphi(z_1, z_2) = |z_1|^2 - |z_2|^2$ is harmonic on $\mathbb{C}^2$. Let $f(z_1, z_2) := (2z_1, z_2)$. Then $\varphi \circ f(z_1, z_2) = 4|z_1|^2 - |z_2|^2$ is neither harmonic nor subharmonic.*

The issue is that begin harmonic only give a control on the average of a function on balls, and holomorphic functions in dimension $\geq 2$ can distort these balls. Being pluriharmonic gives a control on all complex directions, and holomorphic functions are still conformal whe restricted to complex lines.

To justify further the naturality of the notion of pluri(sub)harmonicity, let us note that $\mathrm{PH}(\Omega)$ (resp. $\mathrm{PSH}(\Omega)$) is the largest subspace of $\mathrm{H}(\Omega)$ (resp. $\mathrm{PH}(\Omega)$) which is stable under holomorphic transformations.

**Theorem 6.3.**
*Let $\Omega \subset \mathbb{C}^n$ be open. A function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is PSH if and only if $\varphi \circ T$ is SH for all $T \in \mathrm{GL}_n(\mathbb{C})$.*

## 6.2 Polar and pluripolar sets

**Definition 6.4** (Polar and pluripolar sets)**.**

*A subset $E \subset \Omega \subset \mathbb{C}^n$ is **polar** if there exists $\varphi \in \mathrm{SH}(\Omega)$ such that $\varphi \equiv -\infty$ on $E$.*

*A subset $E \subset M$ is **pluripolar** if, locally, there exists $\varphi \in \mathrm{PSH}(\cdot)$ such that $\varphi \equiv -\infty$ on $E$.*

If $f$ is boholomorphic, then $E$ is pluripolar if and only if $f(E)$ is pluripolar. Indeed, if $E \subset \{\varphi = -\infty\}$ for some PSH function $\varphi$, then $f(E) \subset \{\varphi \circ f^{-1} = -\infty\}$ and $\varphi \circ f^{-1}$ is PSH.

In charts, any pluripolar set is polar, since any PSH function is SH.

A finite union of pluripolar sets is pluripolar, since a sum of PSH functions is PSH. This actually still holds for countable unions by Jakobson's theorem.

**Lemma 6.5.**

*Polar sets have Lebesgue measure zero.*

*Proof.*

Let $\varphi \in \mathrm{SH}(\Omega)$ and $E := \{\varphi = -\infty\}$. We want to prove that $\mathrm{Leb}(E) = 0$.

Let $E'$ be the closure of the Lebesgue density points of $E$. Then $E'$ is closed.

Let $x \in E'$ and $r > 0$ such that $B(x, 3r) \subset \Omega$. Let $y \in B(x, r)$. Then:

- $B(y, 2r) \subset B(x, 3r) \subset \Omega$, so that $\varphi$ is well-defined on $B(y, 2r)$.
- $B(x, r) \subset B(y, 2r)$.

By the definition of $E'$, the ball $B(x, r/2)$ contains a Lebesgue density point of $E$, so $\mathrm{Leb}(E \cap B(x, r)) > 0$. Hence, $\mathrm{Leb}(E \cap B(y, 2r)) > 0$. By the subharmonic property,

$$u(y) \leq \frac{1}{\mathrm{Leb}(B(y, 2r))} \int_{B(y,2r)} u(z) \, \mathrm{d}z = -\infty.$$

Since this holds for all $y \in B(x, r)$, we have $B(x, r) \subset E$. Hence all points of $B(x, r)$ are Lebesgue density points of $E$, so that $B(x, r) \subset E'$. In particular, $x \in E$, so that $E' \subset E$. In addition, $E'$ is open; since $E'$ is closed, $E'$ is an union of connected components of $\Omega$. But $\varphi \equiv -\infty$ on $E'$, and since $\varphi$ is SH, we get that $E'$ is empty. Hence, the set of Lebesgue density points of $E$ is empty. By the Lebsgue density theorem, $\mathrm{Leb}(E) = 0$. □

**Corollary 6.6.**

*Pluripolar sets have Lebesgue measure* 0.

**Example 6.7.**

- *In $\mathbb{C}^n$: $(z_1, \ldots, z_n) \mapsto \ln |z_n|$ is* PSH, *so $\{\ln |z_n| = -\infty\} = \{z_n = 0\}$ is pluripolar.*
- *Let $f : U \subset \mathbb{C}^{n-1} \to \mathbb{C}$ be analytic. The graph of $f$ is the image of $\{z_n = 0\} \subset U \times \mathbb{C}$ by the biholomorphism $(Z, z_n) \mapsto (Z, z_n + f(Z))$. Hence the graph of $f$ is pluripolar:*

$$\Gamma(f) := \{(Z, f(Z)) : \ Z \in U\} = \{\varphi(Z, z_n) = -\infty\} \text{ with } \varphi(Z, z_n) = \ln |z_n - f(Z)|.$$

- *Since being pluripolar is a local notion, complex submanifolds (of codimension at least* 1*) are pluripolar.*
- *Conversely, given $f \in \mathcal{C}(\mathbb{C}^{n-1}, \mathbb{C})$, its graph $\Gamma(f)$ is pluripolar if and only if $f$ is holomorphic (Shcherbina, 2005). For instance, $\{(z, z) : \ z \in \mathbb{C}\}$ is pluripolar, but $\{(z, \overline{z}) : \ z \in \mathbb{C}\}$ is not. Being pluripolar is not only about size, but also about orientation.*
- *Since points are pluripolar and countable unions of pluripolar sets are pluripolar, there are dense pluripolar sets.*

## 6.3 Maximal plurisubharmonic functions

### 6.3.1 Definition

Recall that:

**Proposition 6.8.**
*Let $\Omega \subset \mathbb{C}$ be open and bounded. Let $\varphi \in \mathrm{SH}(\Omega)$. Then, for all $\psi \in \mathrm{H}(\Omega)$,*

$$\text{if } \limsup_{\xi} (\varphi - \psi) \leq 0 \quad \forall \xi \in \partial\Omega, \text{ then } \varphi \leq \psi \text{ on } \Omega. \tag{6.1}$$

The condition "$\limsup_\xi (u - h) \leq 0$ for all $\xi \in \partial\Omega$" can loosely be understood as "$\varphi \leq \psi$ on $\partial\Omega$". This is a maximum principle: if an harmonic function dominates a subharmonic function on the boundary of a bounded domain, then it dominates it everywhere.

This proposition can be read as a characterisation of harmonic functions: $\psi$ is harmonic if and only if is is subharmonic and satisfies Equation (6.1) for all subharmonic $\varphi$, or in other (loose) words, if it is *maximal among subharmonic functions with the same boundary conditions*.

The higher dimensional analogue of this condition leads to:

**Definition 6.9** (Maximal plurisubharmonic functions)**.**
*Let $\Omega \subset \mathbb{C}^n$ be open and $\psi \in \mathrm{PSH}(\Omega)$. We say that $\psi$ is **maximal** if, for all $U$ open and compactly embedded in $\Omega$, for all $\varphi \in \mathrm{PSH}(U)$,*

$$\text{if } \limsup_{\xi} (\varphi - \psi) \leq 0 \quad \forall \xi \in \partial U, \text{ then } \varphi \leq \psi \text{ on } U. \tag{6.2}$$

*We denote by* $\mathrm{MPSH}(\Omega)$ *the space of maximal* PSH *functions on $\Omega$.*

The introduction of compact neighborhoods $U$ in this definition is to use the domination on bounded open sets, while $\Omega$ itself may not be bounded.

**Example 6.10.**
*The function $\psi(z_1, \ldots, z_n) := \ln(\max\{|z_1|, \ldots, |z_n|\})$ is maximal plurisubharmonic on $\mathbb{C}^n$. It is* PSH *as a maximum of* PSH *functions, and maximal because, for all $w \in \mathbb{C}^n \setminus \{0\}$, the map $z \mapsto \psi(zw)$ is harmonic, and one can use the maximum principle on each direction $\mathbb{C}w$.*

Following Klimek, the main ansatz is that MPSH (instead of PH) *is often a good higher-dimensional analogue of* H. The first reason comes from its definition; the second will soon be stated.

### 6.3.2 Characterization of $\mathcal{C}^2$ MPSH functions

To make the notion of MPSH function clearer, we will characterize it, first for $\mathcal{C}^2$ functions.

Let $\varphi \in \mathcal{C}^2(\Omega)$. Then $\varphi \in \mathrm{PSH}(\Omega)$ if and only if $\mathrm{dd}^c\varphi \geq 0$. But

$$\mathrm{dd}^c\varphi = \frac{i}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\mathrm{d}z_j \mathrm{d}\overline{z}_k} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$

is a $(1,1)$-current, and as such is positive if and only if the matrix $\left(\frac{\partial^2\varphi}{\mathrm{d}z_j \mathrm{d}\overline{z}_k}\right)_{1 \leq j,k \leq n}$ is Hermitian positive. We shall denote by $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ (the complex Hessian) this matrix. Then $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ is positive if and only if $\langle \vec{e}, \mathrm{Hess}_{\mathbb{C}}(\varphi)\vec{e}\rangle \geq 0$ for all $\vec{e} \in \mathbb{C}^n$. In particular, $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) \geq 0$.

**Proposition 6.11.**

*Let* $\varphi \in \mathcal{C}^2 \cap \mathrm{PSH}(\Omega)$.

$\varphi \in \mathrm{MPSH}(\Omega)$ *if and only if* $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) \equiv 0$.

*Proof.*

**Direct implication:** We already know that $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) \geq 0$. Assume that there exists $x \in \Omega$ such that $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) > 0$. Since $\varphi$ is $\mathcal{C}^2$, the matrix $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ is continuous. Hence there exists $\delta$, $r > 0$such that $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) \geq \delta$ on $B(x, r)$. Let $\rho \in \mathcal{C}_c(B(x, r), \mathbb{R}_+)$ be a non-zero bump function. Then $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi + t\rho)) \geq \delta/2 > 0$ for all small enough $t$. In particular, if $t$ is small enough and non-zero, $\varphi + t\rho$ is a PSH function on $B(x, r)$ which coincides with $\varphi$ on the boundary of $B(x, r)$ and dominates it strictly on $B(x, r)$. Hence $\varphi$ is not maximal.

**Indirect implication:** Assume that $\det(\mathrm{Hess}_{\mathbb{C}}(\varphi)) \equiv 0$ and that $\varphi$ is not maximal. Then we can find $U$ open and compactly embedded in $\Omega$, as well as $\phi \in \mathrm{PSH}(U)$, such that $\phi \leq \varphi$ on $\partial U$ but $\phi(x) > \varphi(x)$ for some $x \in U$.

We may replace $\phi$ by $(\phi - \delta) * \chi_\varepsilon$ for some small enough $\delta$ and some small enough $\varepsilon$ (depending on $\delta$). Doing so gives another PSH function, with the same relations with respect to $\varphi$, but which is in addition $\mathcal{C}^2$. Hence, without loss of generality, $\phi$ is $\mathcal{C}^2$.

We may further replace $\phi$ by $\phi - \delta + \varepsilon \|\cdot\|^2$ for some small enough $\delta$ and some small enough $\varepsilon$ (depending on $\delta$). Doing so gives another PSH function, with the same relations with respect to $\varphi$, but whose complex Hessian is in addition positive definite everywhere. Hence, without loss of generality, $\mathrm{Hess}_{\mathbb{C}}(\phi)$ is positive definite everywhere.

Let $x \in U$ be a point minimizing $\varphi - \phi$. Let $\vec{e} \in \mathrm{Ker}(\mathrm{Hess}_{\mathbb{C}}(\varphi)(x)) \setminus \{\vec{0}\}$. Finally, let $f(z) := (\varphi - \phi)(x + z\vec{e})$. Then 0 is a local minimum of $f$, so $\Delta f(0) \geq 0$. On the other hand, $\Delta f(0) = 4\langle \vec{e}, \mathrm{Hess}_{\mathbb{C}}(\varphi - \phi)(x)\vec{e}\rangle < 0$, which brings a contradiction. Out initial additional hypothesis cannot hold, and $\varphi$ is maximal. □

A $\mathcal{C}^2$ and MPSH function is thus not a function which is everywhere harmonic in *all* directions (that could be a PH function), but everywhere harmonic in *some* direction.

Given $\psi \in \mathcal{C}^2 \cap \mathrm{MPSH}(\Omega)$, the rank of $\mathrm{Hess}_{\mathbb{C}}(\psi)$ is at most $n - 1$ everywhere. This rank may not be constant. However, where this rank is maximal (equal to $n - 1$), we get a continuous field of lines $L(x) = \mathrm{Ker}(\mathrm{Hess}_{\mathbb{C}}(\psi)(x)) \subset T_x M$ with $\dim(L(\cdot)) \equiv 1$. It turns out that, if $\varphi$ is $\mathcal{C}^3$, then $L$ is $\mathcal{C}^1$ and integrable, and as such defines a foliaiton by Riemann surfaces; on each of these curves, $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ vanishes, so $\varphi$ is harmonic.

**Theorem 6.12** (Bedford, Kalka, 1977)**.**

*Let* $\psi \in \mathcal{C}^3 \cap \mathrm{MPSH}(\Omega)$. *Let* $\Omega' := \{x \in \Omega :\ \mathrm{rk}(\mathrm{Hess}_{\mathbb{C}}(\varphi)(x)) = n - 1\}$.

*Then* $\Omega'$ *is foliated by a family* $(S_\alpha)_{\alpha \in A}$ *of Riemann surfaces such that each* $\psi_{|S_\alpha}$ *is harmonic.*

Conversely, it is sometimes possible to construct a MPSH function from a foliation (e.g. a holomorphic foliation, or a codimension 1 foliation by complex hypersurfaces – Bedford, Kalka, 1977).

This finishes the characterisation of $\mathcal{C}^2$ MPSH functions. We want to extend it to a more general class of MPSH functions; however, in order to do this, we need to give a meaning to $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ for less regular PSH functions.

### 6.3.3 An homogeneous Monge-Ampère equation

**Definition 6.13** (Complex Monge-Ampèere operator)**.**

*The **complex Monge-Ampère operator** in $\mathbb{C}^n$ is*

$$(\mathrm{dd}^c)^n : \ \varphi \mapsto (\mathrm{dd}^c\varphi)^n = (\mathrm{dd}^c\varphi) \wedge \ldots \wedge (\mathrm{dd}^c\varphi),$$

*with $n$ terms in the exterior product.*

Beware that this functional is not linear!

**Example 6.14.**
*If $\varphi$ is $\mathcal{C}^2$, then*

$$(\mathrm{dd}^c\varphi)^n = 4^n n! \det(\mathrm{Hess}_{\mathbb{C}}(\varphi))\mathrm{d}V,$$

*where*

$$\mathrm{d}V = \left(\frac{i}{2}\right)^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$$

*is the usual volume form.*

In particular, if $\varphi \in \mathcal{C}^2 \cap \mathrm{PSH}$, then $\varphi$ is MPSH if and only if $(\mathrm{dd}^c\varphi)^n = 0$. This gives another justification for the analogy between MPSH functions in dimension $n$ and harmonic functions in dimension 1: then are both PSH solutions to the **complex Monge-Ampère equation** $(\mathrm{dd}^c\varphi)^n = 0$.

### 6.3.4 The Chern-Levine-Nirenberg inequality

The definition of $(\mathrm{dd}^c)^n$ involves second derivative. However, exploiting the form of convexity of PSH functions, it is possible to control it for PSH functions using only sup norms. This opens the possibility to extend the operator $(\mathrm{dd}^c)^n$ to less regular functions.

**Theorem 6.15** (Chern-Levine-Nirenberg estimate)**.**
*Let $\Omega \subset \mathbb{C}^n$ and $U$ be an open set compactly embedded in $\Omega$. Let $K \subset \Omega$ be a comapct neighborhood of $\overline{U}$. Then there exists a constant $C(U, K)$ such that, for all $(\varphi^{(j)})_{1\leq j\leq n}$ in $\mathcal{C}^2 \cap \mathrm{PSH}(\Omega)$,*

$$0 \leq \int_U (\mathrm{dd}^c\varphi^{(1)}) \wedge \ldots \wedge (\mathrm{dd}^c\varphi^{(n)}) \leq C \left\|\varphi^{(1)}\right\|_{K,\infty} \cdots \left\|\varphi^{(n)}\right\|_{K,\infty}. \tag{6.3}$$

*In particular, for all $\varphi \in \mathcal{C}^2 \cap \mathrm{PSH}(\Omega)$,*

$$0 \leq \int_U (\mathrm{dd}^c\varphi)^n \leq C \left\|\varphi\right\|^n_{K,\infty}. \tag{6.4}$$

*Proof.*
The goal is to replace all the $(1,1)$-forms $(\mathrm{dd}^c\varphi^{(j)})$ by constant $(1,1)$-form $\beta$ which dominates them. This is done by recursion on $j$.

Let $\beta := \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$ be the standard Kähler form on $\Omega$. It is strongly positive. Since the space of positive $(1,1)$-forms is a strictly convex cone, there exists $R > 0$ such that $\omega < R\left\|\omega\right\|_{\Omega,\infty}\beta$ for all positive $(1,1)$-form $\omega$, in the sense that $R\left\|\omega\right\|\beta - \omega$ is strongly positive. Then, for any positive $(n-1, n-1)$-form $\eta$ which is compactly supported with bounded coefficients,

$$\int_\Omega \eta \wedge \omega \leq R\left\|\omega\right\|_{\Omega,\infty} \int_\Omega \eta \wedge \beta.$$

Let $(\chi_j)_{1 \leq j \leq n}$ be a family of $\mathcal{C}_c^\infty$ functions such that $U \subset \{\chi_1 = 1\}$, $\mathrm{Supp}(\chi_j) \subset \{\chi_{j+1} = 1\}$ for all $j$ and $\mathrm{Supp}(\chi_n) \subset K$. Then

$$
\begin{aligned}
0 &\leq \int_U (\mathrm{dd}^c \varphi^{(1)}) \wedge \ldots \wedge (\mathrm{dd}^c \varphi^{(n)}) \\
&\leq \int_{\mathrm{Supp}(\chi_1)} \chi_1 (\mathrm{dd}^c \varphi^{(1)}) \wedge \ldots \wedge (\mathrm{dd}^c \varphi^{(n)}) \\
&= \int_{\mathrm{Supp}(\chi_1)} \varphi^{(1)} (\mathrm{dd}^c \chi_1) \wedge (\mathrm{dd}^c \varphi^{(2)}) \wedge \ldots \wedge (\mathrm{dd}^c \varphi^{(n)}) \\
&\leq R \left\|\mathrm{dd}^c \chi_1\right\|_{K,\infty} \left\|\varphi_1\right\|_{K,\infty} \int_{\mathrm{Supp}(\chi_1)} (\mathrm{dd}^c \varphi^{(2)}) \wedge \ldots \wedge (\mathrm{dd}^c \varphi^{(n)}) \wedge \beta,
\end{aligned}
$$

where the argument used at the second-to-third line is an integration by parts (an avatar of the computation used to show that the Laplace operator $\Delta$ is self-adjoint). Recursively, we get

$$
\int_U (\mathrm{dd}^c \varphi^{(1)}) \wedge \ldots \wedge (\mathrm{dd}^c \varphi^{(n)}) \leq R^n \left\|\mathrm{dd}^c \chi_1\right\|_{K,\infty} \ldots \left\|\mathrm{dd}^c \chi_n\right\|_{K,\infty} \left\|\varphi_1\right\|_{K,\infty} \ldots \left\|\varphi_n\right\|_{K,\infty} \int_{\mathrm{Supp}(\chi_n)} \beta^n.
$$

This is the claim, with

$$
C(U, K) = R^n \left\|\mathrm{dd}^c \chi_1\right\|_{K,\infty} \ldots \left\|\mathrm{dd}^c \chi_n\right\|_{K,\infty} \int_K \beta^n. \qquad \square
$$

Let $(\varphi_k)_{k \geq 0}$ be a sequence of $\mathcal{C}^2$ PSH functions on $\Omega$, converging locally uniformly (i.e. in $\mathbb{L}^\infty_{\mathrm{loc}}(\Omega)$) to $\varphi$. The function $\varphi$ belongs to $\mathrm{PSH}(\Omega)$. By the Chern-Levine-Nirenberg inequality, the sequence of functions $((\mathrm{dd}^c \varphi_k)^n)_{k \geq 0}$ is locally bounded in total variation. Hence, it is tight, and has limit points in distribution (i.e. for the weak $*$ topology in $\mathcal{C}_c(\Omega)^*$). If this limit can be characterized, then this gives a meaning to $(\mathrm{dd}^c \varphi)^n$ as a nonnegative Radon measure.

**Theorem 6.16.**
*Let $\Omega \subset \mathbb{C}^n$ be open and $(\varphi_k^{(j)})_{1 \leq j \leq n, k \geq 0}$ be $n$ decreasing sequences of $\mathcal{C}^2 \cap \mathrm{PSH}(\Omega)$. Assume that each $(\varphi_k^{(j)})_{k \geq 0}$ converges to $\varphi^{(j)} \in \mathbb{L}^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(\Omega)$. Then $(\mathrm{dd}^c \varphi_k^{(1)} \wedge \ldots \wedge (\mathrm{dd}^c \varphi_k^{(n)})_{k \geq 0}$ converges in distribution.*

Given $\varphi \in \mathbb{L}^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(\Omega)$ and $\varepsilon > 0$, the function $\varphi$ can be approximated on $\Omega_\varepsilon$ by a decreasing sequence of smooth PSH functions: $\varphi^{(k)} = \varphi * \chi_{1/k}$ with a convolution kernel $\chi$.

Hence, the operator $(\mathrm{dd}^c)^n$, initially defined from $\mathcal{C}^2 \cap \mathrm{PSH}(\Omega)$ to $\mathcal{C}(\Omega, \mathbb{R}_+)$ can be extended to an operator from $\mathbb{L}^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(\Omega)$ to nonnegative Radon measures[2].

In addition, Theorem 6.16 then stays true for $\varphi_k^{(j)} \in \mathbb{L}^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(\Omega)$.

**Remark 6.17.**
*This construction is parallel to the construction of Hessian measures for real convex functions.*

**Corollary 6.18.**
*The Chern-Levine-Nirenberg inequality stays true for functions in $\mathbb{L}^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(\Omega)$.*

[2]The hypothesis that the limit belongs to $\mathbb{L}^\infty_{\mathrm{loc}}(\Omega)$ can be somewhat weakened.

### 6.3.5 Comparison theorem

We have given a meaning to $(\mathrm{dd}^c\varphi)^n$ for $\varphi \in \mathbb{L}^\infty \cap \mathrm{PSH}(\Omega)$ as a nonnegative Radon measure. We no go back to the characterization of MPSH functions. We will use:

**Theorem 6.19** (Comparaison theorem; Bedford, Taylor, 1982)**.**
*Let $\Omega \subset \mathbb{C}^n$ be open and bounded. Let $\varphi_1$, $\varphi_2 \in \mathbb{L}^\infty \cap \mathrm{PSH}(\Omega)$.*
*Assume that $\limsup_\xi(\varphi_1 - \varphi_2) \leq 0$ for all $\xi \in \partial\Omega$ ("$\varphi_1 \leq \varphi_2$ on $\partial\Omega$"). Then*

$$\int_{\varphi_2<\varphi_1} (\mathrm{dd}^c\varphi_1)^n \leq \int_{\varphi_2<\varphi_1} (\mathrm{dd}^c\varphi_2)^n. \tag{6.5}$$

*Proof.*
We prove it when $\varphi_1$ and $\varphi_2$ are continuous; the general case follows by a (lenghty) approximation argument.

We replace $\Omega$ by $\{\varphi_2 < \varphi_1\}$, which is open by continuity. Without loss of generality, $\limsup_\xi(\varphi_1 - \varphi_2) = 0$ for all $\xi \in \partial\Omega$.

Let $\varepsilon > 0$. Set $\widetilde{\varphi}_\varepsilon := \max\{\varphi_2, \varphi_1 - \varepsilon\}$, so that $\widetilde{\varphi}_\varepsilon = \varphi_2$ on a neighborhood of $\partial\Omega$. We first show that we may replace $\varphi_2$ by $\widetilde{\varphi}_\varepsilon$ in the comparison theorem.

Let $\chi \in \mathcal{C}_c^\infty(\Omega, [0,1])$ such that $\chi \equiv 1$ on a neighborhood of $\{\widetilde{\varphi}_\varepsilon > \varphi_2\}$. This is possible since the latter subset does not intersect a neighborhood of $\partial\Omega$. Then $\mathrm{dd}^c\chi = 0$ whenever $\widetilde{\varphi}_\varepsilon > \varphi_2$, so

$$\begin{aligned}\int_\Omega \chi \cdot (\mathrm{dd}^c\widetilde{\varphi}_\varepsilon)^n &= \int_\Omega \widetilde{\varphi}_\varepsilon \cdot (\mathrm{dd}^c\chi) \wedge (\mathrm{dd}^c\widetilde{\varphi}_\varepsilon)^{n-1} \\ &= \int_\Omega \varphi_2 \cdot (\mathrm{dd}^c\chi) \wedge (\mathrm{dd}^c\varphi_2)^{n-1} \\ &= \int_\Omega \chi \cdot (\mathrm{dd}^c\varphi_2)^n.\end{aligned}$$

In addition, $\int_\Omega(1-\chi)\cdot(\mathrm{dd}^c\widetilde{\varphi}_\varepsilon)^n = \int_\Omega(1-\chi)\cdot(\mathrm{dd}^c\varphi_2)^n$ since $\widetilde{\varphi}_\varepsilon = \varphi_2$ whenever $\chi < 1$, so that $\int_\Omega(\mathrm{dd}^c\widetilde{\varphi}_\varepsilon)^n = \int_\Omega \chi \cdot (\mathrm{dd}^c\varphi_2)^n$.

Finally, $\varphi_1$ is the increasing limit of $(\widetilde{\varphi}_\varepsilon)_{\varepsilon>0}$. By another approximation principle for increasing sequences of PSH functions (Bedford, Taylor, 1982), $(\mathrm{dd}^c\varphi_1)^n$ is the limit in distribution of $((\mathrm{dd}^c\widetilde{\varphi}_\varepsilon)^n)_{\varepsilon>0}$, and convergence in distribution on an open set may only lessen the total mass of the measures. □

**Corollary 6.20.**
*Let $\Omega \subset \mathbb{C}^n$ be open and bounded. Let $\varphi_1$, $\varphi_2 \in \mathbb{L}^\infty \cap \mathrm{PSH}(\Omega)$.*
*Assume that $\limsup_\xi(\varphi_1 - \varphi_2) \leq 0$ for all $\xi \in \partial\Omega$ ("$\varphi_1 \leq \varphi_2$ on $\partial\Omega$") and $(\mathrm{dd}^c\varphi_1)^n \geq (\mathrm{dd}^c\varphi_2)^n$. Then $\varphi_1 \leq \varphi_2$.*

**Corollary 6.21.**
*Let $\Omega \subset \mathbb{C}^n$ be open. Let $\varphi \in \mathbb{L}^\infty \cap \mathrm{PSH}(\Omega)$.*
*$\varphi$ is MPSH if and only if $(\mathrm{dd}^c\varphi)^n = 0$.*

*Proof.*
We only prove the indirect direction using the comparison theorem. If $(\mathrm{dd}^c\varphi)^n = 0$, then for all $U$ compactly embedded in $\Omega$ and all $\widetilde{\varphi} \in \mathbb{L}^\infty \cap \mathrm{PSH}(U)$ such that $\widetilde{\varphi} \leq \varphi$ on $\partial U$, we have in addition $(\mathrm{dd}^c\widetilde{\varphi})^n \geq 0 = (\mathrm{dd}^c\varphi)^n$, so that $\widetilde{\varphi} \leq \varphi$ by the previous corollary. The case of general $\widetilde{\varphi} \in \mathrm{PSH}(U)$ follows by a truncation argument (apply is to $\max\{\widetilde{\varphi}, -R\}$ and take $R$ to $+\infty$). □

# 7 Conjugation of automorphisms of $\mathbb{C}^2$ (April 8th, Thomas Gauthier)

**Theorem 7.1** (Dujardin, Cantat, March 28th 2024)**.**

*Let $f,\ g : \mathbb{C}^2 \to \mathbb{C}^2$ be regular automorphisms conjugated by a biholomorphism $\varphi$. Then $\varphi$ is a polynomial automorphism.*

**Definition 7.2** (Automorphism)**.**

*We denote by* $\mathrm{Aut}(\mathbb{C}^2) := \{h = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2,\ P, Q \in \mathbb{C}[X]$ *and $h$ is bijective*$\}$*. For* $h \in \mathrm{Aut}(\mathbb{C}^2)$*, we write* $\deg(h) = \max\{\deg(P), \deg(Q)\}$ *its algebraic degree, and*

$$\lambda_1(h) := \lim_{n\to+\infty} \deg(h^n)^{\frac{1}{n}}$$

*its dynamical degree.*

There are two cases:

- Either $\lambda_1(h) = 1$. Then $h$ is elementary: up to conjugation, it is a composition of maps $(x, y) \mapsto (\alpha x + P(y), \beta y + \gamma)$.
- Or $\lambda_1(h) > 1$. Then $\lambda_1(h) = \deg(h)$ and, up to conjugation, $h$ is a composition of Hénon automorphisms $(x, y) \mapsto (\alpha y + P(x), x)$ with $\deg(P) \geq 2$.

In the latter case, we say that $h$ is *regular*.

**Lemma 7.3.**

*Let $f_1,\ f_2 : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ be meromorphic.*

$\deg(f_1 \circ f_2) < \deg(f_1)\deg(f_2)$ *if and only if there exists a curve $C \subset \mathbb{P}^2(\mathbb{C})$ such that $f_1(C) \subset$* $\mathrm{Ind}(f_2)$*, where* $\mathrm{Ind}(f_2)$ *is the (finite) set of points of indeterminacy of $f_2$.*

*Proof.*

We start with the direct part. Write $f_2 \circ f_1 = [P : Q : R]$ with $\deg(P) = \deg(Q) = \deg(R) < \deg(f_1)\deg(f_2)$, and $f_1 = [P_1 : Q_1 : R_1]$, $f_2 = [P_2 : Q_2 : R_2]$ with minimal degree. Then $P_2(P_1, Q_1, R_1),\ Q_2(P_1, Q_1, R_1),\ R_2(P_1, Q_1, R_1)$ have a common factor $H$. Then $C := \{H = 0\}$ satisfies the conclusion.

Let us prove the indirect part. Assume that $C = \{H = 0\}$ is such that $f_2 \circ f_{1|C} = [0 : 0 : 0]$. Then $f_2 \circ f_1 = [P : Q : R] = [H\widetilde{P} : H\widetilde{Q} : H\widetilde{R}] = [\widetilde{P} : \widetilde{Q} : \widetilde{R}]$ on $\{H \neq 0\}$. Hence, $\deg(f_2 \circ f_1) \leq \deg(f_1)\deg(f_2) - \deg(H)$. □

**Example 7.4.**

*Let $f(x, y) := (\alpha y + P(x), x)$ be a Hénon automorphism. Then $f([x : y : z]) = [\alpha y z^{d-1} + P(x/z)z^d : xz^{d-1} : z^d]$, so* $\mathrm{Ind}(f) \subset \{z = 0\}$*. By dividing coordinates by $z^{d-1}$, we find $x = 0$ and* $\mathrm{Ind}(f) = [0 : 1 : 0]$.

Let us write:

$$G_f^+ := \lim_{n\to+\infty} \frac{1}{d^n} \ln_+ \|f^n\|,$$

$$G_f^- := \lim_{n\to+\infty} \frac{1}{d^n} \ln_+ \|f^{-n}\|.$$

**Theorem 7.5** (Fornaess, Sibony).

*Let* $T \geq 0$ *be a closed form of bidegree* $(1,1)$. *Assume that* $c = \int T \wedge dd^c G_f^- < +\infty$ *and* $\mathrm{Supp}(T) \subset \{G_f^+ = 0\}$. *Then* $T = c dd^c G_f^+$.

In other words, the Julia set carries only one closed form of bidegree $(1,1)$.

We can now prove Dujardin and Cantat's theorem.

*Proof.* Write $T^\pm := dd^c G_f^\pm$. If $\varphi \circ f = g \circ \varphi$, then

$$\begin{aligned} dd^c(G_f^+ \circ \varphi) &= \varphi^* T_g^+ \\ &= \lim_{n\to+\infty} \frac{1}{d^n} \ln_+ \|g^n \circ \varphi\| \\ &= \lim_{n\to+\infty} \frac{1}{d^n} \ln_+ \|\varphi \circ f^n\| \\ &= c T_f^+. \end{aligned}$$

Hence, $\varphi^*(T_g^+ \wedge T_g^-) = T_f^+ \wedge T_f^-$. From there, $(dd^c \max(G_f^+, G_f^-))^2 = T_g^+ \wedge T_g^-$, so that $\max(G_f^+, G_f^-) \circ \varphi \leq \max(c, c^{-1}) \max(G_f^+, G_f^-)$.

Finally, $\max(G_f^+, G_f^-) = \ln \|\varphi\| + O(1)$ is bounded by $O(\|\cdot\|) + O(1)$, so $\|\varphi\|$ has polynomial growth. So does its inverse, so $\varphi$ is a polynomial automorphism. $\square$

# References


[1] M. Baker and L. DeMarco, *Preperiodic points and unlikely intersections*, Duke Mathematical Journal **159** (2011), no. 1, pp. 1–29.

[2] F. Berteloot and V. Mayer, *Rudiments de dynamique holomorphe*, Cours Spécialisés **7** (2001), Société Mathématique de France, Paris; EDP Sciences, Les Ulis.

[3] H. Brolin, *Invariant sets under iteration of rational functions*, Arkiv för Matematik **6** (1965), pp. 103–144.

[4] X. Buff, *La mesure d'équilibre d'un endomorphisme de* $\mathbb{P}^k(\mathbb{C})$ *(d'après Briend et Duval)*, Séminaire Bourbaki. Vol. 2004/2005, Astérisque **307** (2006), Exposé no. 939, pp. 33–69.

[5] M. Klimek, *Potential theory in the complex plane*, London Mathematical Society Monographs, New Series **6**. Oxford Science Publications, 1992.

[6] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society, Student Texts, Book **28**. Cambridge University Press, 1995.