Sinai billiard maps with Ruelle resonances

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Abstract

We construct families of two-dimensional Sinai billiards whose transfer operators have Ruelle resonances arbitrarily close to 1. Our method involves taking a large enough cover of an initial billiard table, and relating the transfer operator of the covering table to twisted transfer operators of the initial table. We also study the distribution of these resonances which are close to 1.

Convex billiards tables are one of the classical models of chaotic dynamics, dating back to Sinai [15]. Over the years, many of their statistical properties have been proved, starting with their ergodicity [15], and up to the Central Limit Theorem, the exponential decay of correlations [18, 5] and large deviations for the collision map [14, 11].

In the last few years, the approach *via* the study of the spectral properties of the transfer operator bore fruits, with M. Demers and H.-K. Zhang constructing Banach spaces $\mathcal{B}$ on which the transfer operator acts quasi-compactly [7, 8]. This implies the previous results, and led to a finer understanding of the statistical properties of the billiard flow using Dolgopyat-type arguments [2].

Since the transfer operator acting on a suitable Banach space is quasi-compact, one can define Ruelle resonances, that is, eigenvalues of the transfer operator. There is at least one such eigenvalue (1, corresponding to constant functions). A question, asked by V. Baladi, was whether one could find billiard tables with non-trivial Ruelle resonances.

There are relatively few examples for which we are able to describe explicitly the spectrum of the transfer operator ; one such instance is given by the work of O. Bandtlow, W. Just and J. Slipantschuk on Blaschke products [4, 16]. A generic Anosov diffeomorphism of the 2-torus also admits non-trivial Ruelle resonances [1].

We prove that, for a suitable choice of billiard tables, the transfer operator admits non-trivial Ruelle resonances. In a nutshell, we fix an initial billiard table, and relate the transfer operator on Abelian covers of the billiard table to twisted transfer operators for this initial table. Then the eigenvalues of a twisted transfer operator appear as Ruelle resonances of the transfer operator on a corresponding cover. A perturbative argument finally shows that, if the cover is large enough, there must exist such resonances close to 1 (Theorem 1.1).

While we found it independently, our method is very close to the one used by D. Jakobson, F. Naud and L. Soares [10] to prove the existence of Ruelle resonances for geodesic flows on convex-cocompact surfaces of constant negative curvature. The main difference is that, for the geodesic flow in constant curvature, an approach *via* dynamical zeta functions is available, which simplifies many arguments (in particular those relying on the time-reversal symmetry). This approach is not available in the context of billiards, so we provide more elementary, and more robust, arguments.

As in [10], we also show that those resonances which are close to 1 are real, and study their distribution for families of large covers (Propositions 1.3 and 1.4).

While we expect this method to work as well with the billiard flow, some groundwork is necessary to be able to deal with the Banach spaces constructed in [2]. As a consequence, we only discuss the collision map.

The necessary background and our results are exposed in Section 1. We prove the existence of resonances in Section 2, and study their distribution in Section 3.

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1 Context and results

1.1 Sinai billiards and Ruelle spectrum

A planar Sinai billiard with finite horizon is given by a finite number of non-overlapping closed convex regions $(\Gamma_k)_{1\leq k\leq d}$ of the torus $\mathbb{T}^2$, whose boundaries $(\gamma_k)_{1\leq k\leq d}$ are $\mathcal{C}^3$ with non-vanishing curvature, and such that any line in the torus meets the interior of one of the Γ_k's (the so-called finite horizon condition).

The billiard table is $Q = \mathbb{T}^2 \backslash \bigcup_{k=1}^d \mathring{\Gamma}_k$. We consider the dynamics of a point particle moving at unit speed in Q, with specular reflection at the obstacles. The state space for this flow is three-dimensional, and there is a natural Poincaré section: the set $M := \bigcup_{k=1}^d \gamma_k \times [-\pi/2, \pi/2]$ of outward-facing unit vectors at the boundaries of the obstacles. The finite horizon condition implies that the return time to this Poincaré section is bounded. Let $T : M \to M$ be the first return map to M, which is also called the collision map.

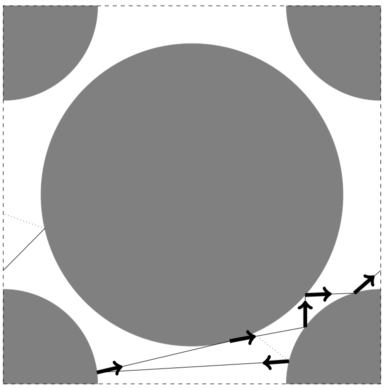

Figure 1: A trajectory in a finite horizon Sinai billiard table, with a marked outward-facing unit vector at each collision.

The map T is hyperbolic with codimension 1 singularities (which correspond to grazing trajectories on the billiard table), and a finite number of domains of continuity. It preserves a symplectic form on M, and thus the associated Liouville measure $\mu = \frac{\cos(\theta)}{2\sum_{k=1}^d |\gamma_k|} \mathrm{d}\ell \mathrm{d}\theta$ on M.

The collision map T is one of the most prominent examples of chaotic maps ; for instance, it exhibits exponential decay of correlations against Hölder observables. More precisely, for any $\eta > 0$, there exist constants $C > 0$ and $\lambda > 1$ such that

$$\left| \int_M \varphi \cdot \psi \circ T^n \, \mathrm{d}\mu - \int_M \varphi \, \mathrm{d}\mu \int_M \psi \, \mathrm{d}\mu \right| \leq C\lambda^{-n} \left\|\varphi\right\|_\eta \left\|\psi\right\|_\eta, \tag{1.1}$$

where $\|\cdot\|_\eta$ is the η-Hölder norm. Equation (1.1) can be seen as a consequence of spectral properties of the composition operator $h \mapsto h \circ T$, or its dual, the transfer operator P. The transfer operator P acts on $\mathbb{L}^1(M, \mu)$ by:

$$P(h) = h \circ T^{-1} \quad \forall h \in \mathbb{L}^1(M, \mu).$$

The operator P also acts on various function or distribution spaces; of interest to us will be the Banach spaces of anisotropic distributions $\mathcal{B}$ constructed by M. Demers and H.-K. Zhang [7, 8].

The anisotropic distributions on M constructed by M. Demers and H.-K. Zhang are regular in the unstable direction, and dual of regular in the stable direction. By [7, Lemma 2.1],

$$\mathcal{C}^{1/3}(M) \to \mathcal{B} \to \mathcal{C}^{1/3}(T^{-n}\mathcal{W}^s)^*,$$

where $\mathcal{C}^\gamma$ is the space of γ-Hölder functions, $\mathcal{W}^s$ is a space of stable curves, and the inclusions are continuous and injective. In addition, the injection $\mathcal{C}^1(M) \subset \mathcal{B}$ has a dense image.

The operator P acts continuously on $\mathcal{B}$. By [7, Proposition 2.3], its action is quasi-compact: 1 belongs to the spectrum of P (since $\mathbf{1} \in \mathcal{B}$ and $P(\mathbf{1}) = \mathbf{1}$), and the essential spectral radius $\rho_{\text{ess}}(P \curvearrowright \mathcal{B})$ of P acting on $\mathcal{B}$ is strictly smaller than 1. More precisely, there exists a constant $\rho_0 > 0$, depending only on the minimal travel time between obstacles and the minimal curvature of the obstacles, such that $\rho_{\text{ess}}(P \curvearrowright \mathcal{B}) \leq \rho_0 < 1$. The spectrum of P in $\overline{B}(0, \rho_0)^c$ is discrete, contained in $\overline{B}(0, 1)$, and consists in (at most) countably many eigenvalues of finite multiplicity. Such eigenvalues are *Ruelle resonances* of the transfer operator.

In addition, when acting on $\mathcal{B}$, the operator P has a spectral gap: its resonance 1 is simple and is the only resonance of modulus 1. Hence, P is the sum of the rank 1 projection $h \mapsto \int_M h \, \mathrm{d}\mu \cdot \mathbf{1}$ and of an operator of spectral radius strictly smaller than 1. The exponential decay of correlations (1.1) follows.

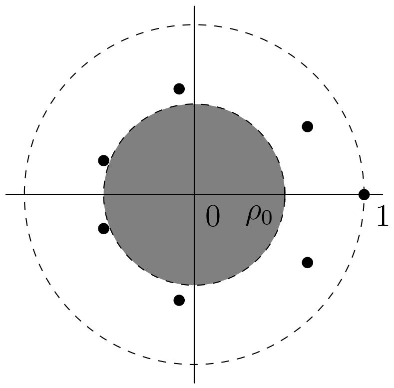

Figure 2: Spectrum of the operator P acting on $\mathcal{B}$. The set of Ruelle resonances is represented by the black dots, and may be as small as $\{1\}$. The essential spectrum is inside the gray disk. The spectrum is symmetric with respect to the real line because P is real.

If the obstacles are close or the minimal curvature of the obstacles is small, then the estimate on the essential spectral radius becomes worse, which makes it harder to find resonances. The strategy we adopt let us work with those constants being fixed, avoiding this difficulty.

1.2 Coverings of billiard tables

A Sinai billiard table admits a $\mathbb{Z}^2$ covering, that is, a $\mathbb{Z}^2$-periodic billiard table $\widetilde{Q} \subset \mathbb{R}^2$ such that the natural projection of $\widetilde{Q}$ on $\mathbb{T}^2$ is Q. In what follows, we denote with a tilde all objects related to this $\mathbb{Z}^2$-cover.

By choosing an origin and making $\mathbb{Z}^2$ act on $\widetilde{Q}$, the obstacles of $\widetilde{Q}$ can be indexed by $\mathbb{Z}^2$: we get $\mathbb{R}^2 \setminus \overset{\circ}{\widetilde{Q}} = \bigcup_{k=1}^d \bigcup_{p\in\mathbb{Z}^2} \Gamma_{k,p}$, where $\Gamma_{k,p} = \Gamma_{k,0} + p$, whence $\widetilde{M} = \bigcup_{p\in\mathbb{Z}^2} M \times \mathbb{Z}^2$.

The Liouville measure μ also lifts to a $\widetilde{T}$-invariant measure $\widetilde{\mu}$, such that any restriction of $\widetilde{\mu}$ to a fundamental domain equals μ.

The collision map $\widetilde{T}$ for $\widetilde{Q}$ is a $\mathbb{Z}^2$ extension of the collision map for Q. Given $(x,p) \in \widetilde{M} = M \times \mathbb{Z}^2$, we have $\widetilde{T}(x,p) =: (T(x), q)$, and $q-p$ depend only on x. Writing $q - p =: F(x)$, we get a function $F : M \to \mathbb{Z}^2$ such that:

$$\widetilde{T}(x,p) = (T(x), p + F(x))$$

The value $F(x)$ stays the same as long as $(x,0)$ and $\widetilde{T}(x,0)$ belong to the same two obstacles. As a consequence, F is constant on the domains of continuity of T. By the finite horizon condition, T admits finitely many domains of continuity, so that F is bounded.

The billiard map is time-reversible. The involution $\iota(\ell,\theta) := (\ell,-\theta)$ on M has the following properties:

$$\begin{aligned} \iota \circ T &= T^{-1} \circ \iota, \\ F \circ \iota &= -F \circ T^{-1}, \end{aligned}$$

and ι preserves μ. It follows that $\int_M F \, \mathrm{d}\mu = 0$.

The same construction can be used on any covering of Q. In particular, given any rank 2 lattice $\Lambda \subset \mathbb{Z}^2$, we get a billiard table Q_Λ, to which we associate a probability-preserving dynamical system $(M_\Lambda, \mu_\Lambda, T_\Lambda)$. Writing $G := \mathbb{Z}^2/\Lambda$, we have:

$$\begin{aligned} M_\Lambda &= M \times G, \\ \mu_\Lambda &= \frac{1}{|G|} \sum_{g\in G} \mu \times \delta_g, \\ T_\Lambda(x,g) &= (T(x), g + F(x)[\Lambda]). \end{aligned}$$

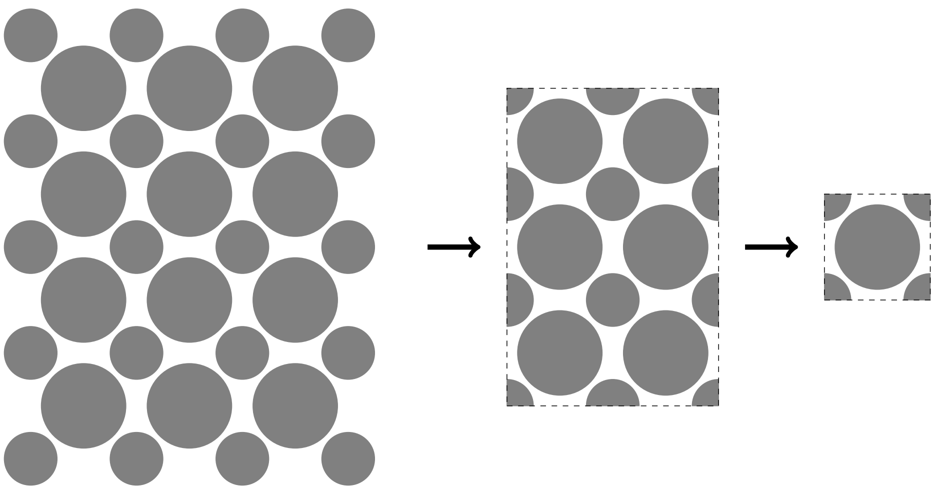

Figure 3: The $\mathbb{Z}^2$-covering $\widetilde{Q}$ of Q, and an intermediate covering Q_Λ with $\Lambda = 2\mathbb{Z} \oplus 3\mathbb{Z}$.

We denote by P_Λ the transfer operator associated with $(M_\Lambda, \mu_\Lambda, T_\Lambda)$, and $\mathcal{B}_\Lambda$ the Banach space as constructed in [7] for the system $(M_\Lambda, \mu_\Lambda, T_\Lambda)$. By construction, for any $h \in \mathcal{B}_\Lambda$,

$$\|h\|_{\mathcal{B}_\Lambda} = \max_{g\in G} \left\| h \mathbf{1}_{M\times\{g\}} \right\|_{\mathcal{B}}.$$

1.3 Results

We shall prove that, when Λ is large enough, the Sinai billiard on the table Q_Λ has non-trivial Ruelle resonances. More precisely,

Theorem 1.1. *There exists* $\delta > 0$ *such that* $\mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda) \subset \overline{B}(0, 1-\delta) \cup [1-\delta, 1]$ *for all lattices* Λ.

In addition, there exist positive constants $c < C$ *such that:*

$$c|G| < |\{\textit{Ruelle resonances in } [1-\delta, 1], \ \textit{with multiplicities}\}| < C|G|$$

In particular, whenever Λ *is sparse enough,* P_Λ *admits non-trivial Ruelle resonances.*

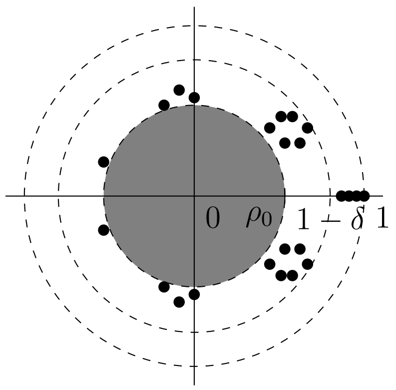

Figure 4: Spectrum of the operator P_Λ acting on $\mathcal{B}_\Lambda$ for a sparse enough lattice Λ. The spectrum is still symmetric with respect to the real line. The resonances on the segment $[1-\delta, 1]$ are guaranteed to exist; the others may or may not exist.

Now, let us focus on the distribution of these resonances. Let ρ_0 be the upper bound on $\rho_{\mathrm{ess}}(P \curvearrowright \mathcal{B})$ given by [7, Proposition 2.3]. For $\Lambda < \mathbb{Z}^2$ a rank 2 lattice, define the *spectral measure* of P_Λ as:

$$\nu_\Lambda := \frac{1}{|G|} \sum_{\substack{\lambda \text{ resonance of } P_\Lambda \\ |\lambda| > 1-\rho_0}} \delta_\lambda,$$

where the sum is taken with multiplicity. Then $\rho_{\mathrm{ess}}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda) \leq \rho_0$, so that ν_Λ is a Radon measure on $\overline{B}(0, \rho_0)^c$. Our next proposition, which is a variant of [10, Theorem 1.3], states that, for any sequence (Λ_N) of lattices, the sequence (ν_{Λ_N}) of spectral measures admits a converging subsequence.

Proposition 1.2. *For any sequence of rank* 2 *lattices* $\Lambda_N < \mathbb{Z}^2$, *there exists a subsequence* $(\Lambda_{N_k})_{k \geq 0}$ *and a Radon measure* ν *such that* $\nu_{\Lambda_{N_k}} \to \nu$ *for the vague topology, i.e.*

$$\lim_{k \to +\infty} \int_{\overline{B}(0,\rho_0)^c} f \, \mathrm{d}\nu_{\Lambda_{N_k}} = \int_{\overline{B}(0,\rho_0)^c} f \, \mathrm{d}\nu$$

for all $f \in \mathcal{C}_c(\overline{B}(0, \rho_0)^c, \mathbb{C})$.

For specific choices of a sequence of lattices (Λ_N), we can express explicitly the limit of (ν_{Λ_N}) near 1, similarly to what was done in [10, Section 3.2]. Let $\Lambda_N^{(1)} := N\mathbb{Z} \times \mathbb{Z}$ and $\Lambda_N^{(2)} := (N\mathbb{Z})^2$. We write $\nu_N^{(1)} := \nu_{\Lambda_N^{(1)}}$ and likewise $\nu_N^{(2)} := \nu_{\Lambda_N^{(2)}}$. For the sequence of lattices $(\Lambda_N^{(1)})$, we get the following statement:

Proposition 1.3. *Let δ be as in Theorem 1.1. There exists $\delta_0 \in (0, \delta]$ and a finite measure $\nu^{(1)}_{|[1-\delta_0,1]}$ on $[1-\delta_0, 1]$ such that:*

$$\lim_{N \to +\infty} \nu^{(1)}_{N|[1-\delta_0,1]} = \nu^{(1)}_{|[1-\delta_0,1]},$$

where the convergence is in $\mathcal{C}([1-\delta_0, 1], \mathbb{C})^*$. Moreover,

$$\frac{\mathrm{d}\nu^{(1)}_{|[1-\delta_0,1]}}{\mathrm{d}x} \sim_{x \to 1^-} \frac{1}{\pi\sqrt{2\Sigma_{11}}} \cdot \frac{1}{\sqrt{1-x}}. \tag{1.2}$$

In Equation (1.2), the constant Σ_{11} is the upper-left coefficient of the covariance matrix Σ associated with the diffusion of the billiard on the table $\widetilde{Q}$ (see Equation (2.2)). Replacing $\Lambda^{(1)}_N$ by $\mathbb{Z} \times N\mathbb{Z}$ only changes the constant in Equation (1.2), where Σ_{11} becomes Σ_{22}.

We get an analogous statement for the sequence $(\Lambda^{(2)}_N)$:

Proposition 1.4. *Let δ be as in Theorem 1.1. There exists $\delta_0 \in (0, \delta]$ and a finite measure $\nu^{(2)}_{|[1-\delta_0,1]}$ on $[1-\delta_0, 1]$ such that:*

$$\lim_{N \to +\infty} \nu^{(2)}_{N|[1-\delta_0,1]} = \nu^{(2)}_{|[1-\delta_0,1]},$$

where the convergence is in $\mathcal{C}([1-\delta_0, 1], \mathbb{C})^*$. Moreover,

$$\lim_{x \to 1^-} \frac{\mathrm{d}\nu^{(2)}_{|[1-\delta_0,1]}}{\mathrm{d}x} = \frac{1}{2\pi\sqrt{\det(\Sigma)}}. \tag{1.3}$$

2 Existence of resonances

In this section, we prove a weaker version of Theorem 1.1:

Proposition 2.1. *Let (M, μ, T) be a finite horizon Sinai billiard. Let U be a neighborhood of 1 in $\mathbb{C}$. There exists a constant $c(U) > 0$ such that, for any rank 2 lattice $\Lambda < \mathbb{Z}^2$, the spectrum of P_Λ acting on $\mathcal{B}_\Lambda$ admits at least $c(U)|G|$ Ruelle resonances (with multiplicities) in U.*

In particular, if Λ is sparse enough, then P_Λ admits non-trivial Ruelle resonances in U.

Theorem 1.1 shall follow from Proposition 2.1 and some additional results on the localization of Ruelle resonances proved in Section 3.

2.1 Spectral decomposition of $\mathcal{B}_\Lambda$

The map $M_\Lambda \to M$ is a Galois covering, the deck transformations being translations $\tau_g : M_\Lambda \to M_\Lambda$ given by $\tau_g(x, g') = (x, g' + g)$, for all $g \in G$. All these translations are measure-preserving and commute with T_Λ:

$$\tau_g \circ T_\Lambda(x, g') = (T(x), g' + F(x) + g[\Lambda]) = T_\Lambda \circ \tau_g(x, g').$$

As a consequence, G acts on $\mathcal{B}_\Lambda$ by pre-composition:

$$(\tau_g h_\Lambda)(\varphi) := h_\Lambda(\varphi \circ \tau_{-g})$$

for all $h_\Lambda \in \mathcal{B}_\Lambda$ and $\varphi \in \mathcal{C}^1(M_\Lambda, \mathbb{C})$, and this action commutes with P_Λ.

Let $g \in G$. Since τ_g and P_Λ commute, the eigenspaces of τ_g are P_Λ-invariant. As this holds for all $g \in \tau_g$, the intersections of the eigenspaces for $(\tau_g)_{g \in G}$ are P_Λ-invariant. But these eigenspaces are given by the characters of G:

$$\mathcal{B}_{\Lambda,\chi} := \{h \otimes \chi : \ h \in \mathcal{B}_\Lambda, \ \chi \in \widehat{G}\},$$

where $(h \otimes \chi)(\varphi_\Lambda) = \sum_{g \in G} \chi(g) h(\varphi_\Lambda(\cdot, g))$ for all $\varphi_\Lambda \in \mathcal{C}^1(M_\Lambda, \mathbb{C})$. Note that we can define maps $\Pi_{\Lambda,\chi} : \mathcal{B}_\Lambda \to \mathcal{B}$ by:

$$\Pi_{\Lambda,\chi}(h_\Lambda)(\varphi) := |G|^{-1} h_\Lambda(\varphi \otimes \overline{\chi}) \quad \forall \varphi \in \mathcal{C}^1(M, \mathbb{C}),$$

and the space $\mathcal{B}_{\Lambda,\chi}$ can be written as:

$$\mathcal{B}_{\Lambda,\chi} = \bigcap_{\substack{\chi \in \widehat{G} \\ \chi' \neq \chi}} \mathrm{Ker}(\Pi_{\Lambda,\chi'}).$$

In particular, the spaces $\mathcal{B}_{\Lambda,\chi}$ are closed, and there is a P_Λ-invariant splitting:

$$\mathcal{B}_\Lambda = \bigoplus_{\chi \in \widehat{G}} \mathcal{B}_{\Lambda,\chi}.$$

2.2 Spectrum of P_Λ

Each subspace $\mathcal{B}_{\Lambda,\chi}$ is isomorphic to $\mathcal{B}$, for instance *via* $\Pi_{\Lambda,\chi}$ and its right inverse $h \mapsto h \otimes \chi$. The action of P_Λ on $\mathcal{B}_{\Lambda,\chi}$ is thus conjugated with the action of some operator $P_{\Lambda,\chi}$ on $\mathcal{B}$, which can be made explicit. For all $h \in \mathcal{B}$ and $\varphi \in \mathcal{C}^1(M, \mathbb{C})$,

$$\begin{aligned}
P_{\Lambda,\chi}(h)(\varphi) &= [\Pi_{\lambda,\chi} P_\Lambda(h \otimes \chi)](\varphi) \\
&= \frac{1}{|G|}[P_\Lambda(h \otimes \chi)](\varphi \otimes \overline{\chi}) \\
&= \frac{1}{|G|}(h \otimes \chi)(\varphi \otimes \overline{\chi} \circ T_\Lambda) \\
&= \frac{1}{|G|}(h \otimes \chi)[(\overline{\chi}(F) \cdot \varphi \circ T) \otimes \overline{\chi}] \\
&= h(\chi(-F) \cdot \varphi \circ T) \\
&= [P(\chi(-F)h)](\varphi).
\end{aligned}$$

Hence, $P_{\Lambda,\chi}(h) = P(\chi(-F)h)$. More intuitively, for $h \in \mathcal{C}^0(M, \mathbb{C})$, we have $P_{\Lambda,\chi}(h \otimes \chi) = (h \otimes \chi) \circ T_\Lambda^{-1}$, with:

$$T_\Lambda^{-1}(x, p) = (T^{-1}(x), p - F \circ T^{-1}(x)),$$

from which the same result follows. As a consequence,

$$\mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda) = \bigcup_{\chi \in \widehat{G}} \mathrm{Sp}(P_{\Lambda,\chi} \curvearrowright \mathcal{B}), \tag{2.1}$$

where the union is taken with multiplicities.

Note that the estimate given in [7, Proposition 2.3] on the essential spectral radius of P also holds for P_Λ, for all Λ. There are two ways to prove this theorem:

- That bound only depends on the minimal curvature of the obstacles, the minimal free path length of the bouncing particle, and on the choice of ε_0. These quantities are the same for the billiard table M and for all its covers M_Λ.
- The estimates on the essential spectral radius of [7, Proposition 2.3] generalize to the weighted operators $P_{\Lambda,\chi}$, with no dependence on the character χ. Equation (2.1) yields the claim.

2.3 Perturbations of transfer operators

The function $F : M \to \mathbb{Z}^2$ satisfies the assumptions of [8, Lemma 5.4]. Hence, the family of twisted transfer operators:

$$P_w(h) := P(e^{i\langle w,F\rangle}h),$$

acting on $\mathcal{B}$, depends analytically on $w \in 2\pi\mathbb{T}^2$.

As 1 is an isolated eigenvalue of P with eigenfunction $\mathbf{1}$, there exists a neighbourhood U of 0 in $2\pi\mathbb{T}^2$ and analytic functions $w \mapsto \lambda_w \in \mathbb{C}$, and $w \mapsto g_w \in \mathcal{B}$ defined on U such that $g_w(\mathbf{1}) = 1$ and $P_w(h) = \lambda_w g_w$.

By a classical computation, appearing for instance in the Nagaev-Guivarc'h proof of the Central Limit Theorem for Markov chains [12, 13, 9],

$$\lambda_w = 1 - \frac{\Sigma(w,w)}{2} + O(|w|^3) \tag{2.2}$$

near 0, where Σ is a bilinear form. As Σ is the matrix of covariance for the central limit theorem for $(\sum_{k=0}^{n-1} F \circ T^k)$ and such a central limit theorem for a finite horizon Sinai billiard has a non-degenerate limit law, F is not a coboundary and Σ is positive definite. In particular, up to taking a smaller neighborhood U, we shall assume that $|\lambda_w| < 1$ for $w \neq 0$.

2.4 Ruelle resonances for Sinai billiards

We now prove Proposition 2.1.

Proposition 2.1. Let U be a neighborhood of 1 in $\mathbb{C}$. Let V be a neighborhood of 0 in $2\pi\mathbb{T}^2$ on which the main eigenvalue λ_w of P_w is well-defined and belongs to U. Let W be a neighborhood of 0 such that $W - W \subset V$. Let $\Lambda < \mathbb{Z}^2$. Note that $\widehat{G} = \widehat{(\mathbb{Z}^2/\Lambda)} < \widehat{\mathbb{Z}^2} = 2\pi\mathbb{T}^2$. Then:

$$|G| \operatorname{Vol}(W) = \int_{2\pi\mathbb{T}^2} \sum_{\chi\in\widehat{G}} \mathbf{1}_{\chi+W} \,\mathrm{d}\operatorname{Vol} \leq 4\pi^2 \max_{2\pi\mathbb{T}^2} \sum_{\chi\in\widehat{G}} \mathbf{1}_{\chi+W}. \tag{2.3}$$

In addition, for all $\chi' \in 2\pi\mathbb{T}^2$,

$$\sum_{\chi\in\widehat{G}} \mathbf{1}_{\chi+W}(\chi') = |\widehat{G} \cap (\chi' - W)|.$$

Assume that $\widehat{G} \cap (\chi' - W)$ is non-empty, and let χ_0 be one of its elements. The function $\chi \mapsto |\widehat{G} \cap (\chi - W)|$ is $\widehat{G}$-invariant, so $|\widehat{G} \cap (\chi' - W)| = |\widehat{G} \cap (\chi' - \chi_0 - W)|$. But $\chi_0 \in \chi' - W$, so $\chi' - \chi_0 \in W$ and $\chi' - \chi_0 - W \subset W - W \subset V$. Hence, $|\widehat{G} \cap (\chi' - W)| \leq |\widehat{G} \cap V|$. This inequality is also true if $\widehat{G} \cap (\chi' - W)$ is empty, so Equation (2.3) implies:

$$|G| \operatorname{Vol}(W) \leq 4\pi^2 |\widehat{G} \cap V|. \tag{2.4}$$

Finally, given any $\chi = e^{i\langle w,\cdot\rangle} \in \widehat{G} \cap V$, the operator $P_w = P_{\Lambda,\overline{\chi}}$ admits λ_w as a Ruelle resonance. By construction, $\lambda_w \in U$, and λ_w is also a resonance of P_Λ by Equation (2.1). Hence, P_Λ admits at least $|\widehat{G} \cap V| \geq \frac{\operatorname{Vol}(W)}{4\pi^2} |G|$ Ruelle resonances in U (with multiplicities). □

3 Distribution of resonances

We now focus on the properties of the non-trivial resonances constructed in Section 2. We shall finish the proof of Theorem 1.1 in Subsections 3.1 and 3.2, where we describe more precisely the main eigenvalue λ_w of the twisted transfer operator P_w. We shall prove Propositions 1.2, 1.3 and 1.4 in Subsection 3.3.

3.1 Aperiodicity of the Sinai billiard

As a step-stone to Theorem 1.1, we shall prove that Sinai billiards are aperiodic. While this is already known [17], the following argument is reasonably short. We write characters in exponential form: $\chi = e^{i\langle w, \cdot \rangle}$. Let $H := \{(\rho, w) \in \mathbb{S}_1 \times 2\pi\mathbb{T}^2 |\ \rho \in \mathrm{Sp}(P_w \curvearrowright \mathcal{B})\}$. We claim:

Lemma 3.1. *The set H is a subgroup of $\mathbb{S}_1 \times 2\pi\mathbb{T}^2$.*

Proof. Note that, by considering $H \cap (\mathbb{S}_1 \times \{0\})$, this lemma implies that the peripheral spectrum of P is a subgroup of $\mathbb{S}_1$, which is well-known [7, Lemma 5.2]. The proof of Lemma 3.1 mimics the proof of the later fact.

Up to straightforward modifications, the proof of [7, Lemma 5.1] can be generalized to prove that, for any $(\rho, w) \in H$, the corresponding Jordan block is trivial and any associated eigendistribution belongs to $\mathbb{L}^\infty(M, \mu) \cdot \mathrm{d}\mu$. In addition, whenever k is an eigendistribution associated with $(\rho, w) \in H$,

$$P_w(k \ \mathrm{d}\mu) = e^{i\langle w, F\rangle \circ T^{-1}} k \circ T^{-1} \ \mathrm{d}\mu = \rho k \ \mathrm{d}\mu,$$

and thus

$$\rho k = e^{i\langle w, F\rangle \circ T^{-1}} k \circ T^{-1}. \tag{3.1}$$

Taking absolute values in Equation (3.1), we get $|k| \circ T^{-1} = |k|$; as the Sinai billiard is ergodic, $|k|$ is constant, and k does not vanish.

Taking the complex conjugate in Equation (3.1), we get $\overline{\rho}\overline{k} = e^{i\langle -w, F\rangle \circ T^{-1}} \overline{k} \circ T^{-1}$, so $\overline{k} \ \mathrm{d}\mu$ is an eigendistribution for P_{-w} for the eigenvalue $\overline{\rho}$. In addition, $\overline{k} \ \mathrm{d}\mu \in \mathcal{B}$. Hence, $(\overline{\rho}, -w) \in H$.

Let $k_1 \ \mathrm{d}\mu$, $k_2 \ \mathrm{d}\mu$ be two eigendistributions corresponding to (ρ_1, w_1), $(\rho_2, w_2) \in H$ respectively. Then, again using Equation (3.1),

$$e^{i\langle w_1 + w_2, F\rangle \circ T^{-1}} (k_1 k_2) \circ T^{-1} = e^{i\langle w_1, F\rangle \circ T^{-1}} k_1 \circ T^{-1} \cdot e^{i\langle w_2, F\rangle \circ T^{-1}} k_2 \circ T^{-1} = \rho_1 k_1 \rho_2 k_2 = (\rho_1 \rho_2) k_1 k_2.$$

As neither k_1 not k_2 vanish, their product $k_1 k_2$ does not vanish either, so $k_1 k_2$ is an eigendistribution for the eigenvalue $\rho_1 \rho_2$ of $P_{w_1 + w_2}$. In addition, by the argument of [6, Lemma 5.5], $k_1 k_2 \ \mathrm{d}\mu \in \mathcal{B}$. Hence, $(\rho_1 \rho_2, w_1 + w_2) \in H$. □

All the arguments above are standard (they only use properties of the action of P_w on $\mathcal{B}$), and apply to a much wider class of dynamical systems. For Sinai billiards, Lemma 3.1 can be strengthened:

Lemma 3.2. *For a finite horizon Sinai billiard, $H = \{(1, 0)\}$.*

Proof. By the discussion in Subsection 2.1, there exists a neighborhood V of 0 in $2\pi\mathbb{T}^2$ and a neighborhood U of 1 in $\mathbb{C}$ such that, for all $w \in V$, we have $\mathrm{Sp}(P_w \curvearrowright \mathcal{B}) \cap U = \{\lambda_w\}$. Since $\lambda_w = 1 - \frac{\Sigma(w,w)}{2} + O(|w|^3)$, if $w \in V \setminus \{0\}$, then P_w has no eigenvalue of modulus 1 in U. Hence, $(1, 0)$ is isolated in H. The group H is discrete, and thus finite.

Assume that H is not trivial, and let $(\rho, w) \in H \setminus \{(1, 0)\}$. Since H is finite, (ρ, w) has finite order. Hence, there exists a rank 2 lattice $\Lambda < \mathbb{Z}^2$ such that $w \in \widehat{G}$. Then $\rho \in \mathrm{Sp}(P_{\Lambda,\chi} \curvearrowright \mathcal{B})$ for

$\chi = e^{-i\langle w,\cdot\rangle} \in \widehat{G}$, so $\rho \in \mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda)$. Then the transfer operator P_Λ for the Sinai billiard Q_Λ has non-trivial peripheral spectrum. If $\rho = 1$, then $1 \in \mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda)$ has multiplicity at least 2, which contradicts the ergodicity of Sinai billiards. If $\rho \neq 1$, then ρ is a non-trivial root of the unit, which contradicts the fact that Sinai billiards are mixing. $\square$

The Ruelle spectrum of $w \mapsto P_w$ depends continuously on w. By Lemma 3.2, there exists $\delta > 0$ and a neighborhood V of 0 in $2\pi\mathbb{T}^2$ such that P_w has a resonance of modulus larger than $1-\delta$ if and only if $w \in V$, and under this condition the resonance is λ_w.

By Equation (2.1), the spectrum of P_Λ acting on $\mathcal{B}_\Lambda$ is the union of the spectra of the operators P_w for $w \in \widehat{G}$. Hence, for this value of $\delta > 0$ and all rank 2 lattices $\Lambda < \mathbb{Z}^2$:

$$\mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda) \subset \overline{B}(0, 1-\delta) \cup \{\lambda_w : \ w \in V\}. \tag{3.2}$$

A further consequence is that P_Λ admits at most $|G|$ resonances (with multiplicities) in $\overline{B}(0,1-\delta)^c$. With Proposition 2.1, this implies that the number of Ruelle resonances in the annulus $\{1-\delta < |z| \leq 1\}$ is a $\Theta(|G|)$.

3.2 Realness of the resonances

Let $\delta > 0$ be small enough that Equation (3.2) holds. Let $V := \{w \in 2\pi\mathbb{T}^2 : \ |\lambda_w| > 1-\delta\}$. Up to taking a smaller value of δ, we may assume that $\lambda_\cdot$ is continuous on V. Let $w \in V$, and $h_w \in \mathcal{B}$ an eigendistribution for P_w. Since the operator P is real,

$$P_{-w}(\overline{h}_w) = P(e^{-i\langle w,F\rangle}\overline{h}_w) = \overline{P(e^{i\langle w,F\rangle}h_w)} = \overline{\lambda}_w\overline{h}_w,$$

so $\overline{\lambda}_w \in \mathrm{Sp}(P_{-w} \curvearrowright \mathcal{B})$. But $|\overline{\lambda}_w| > 1-\delta$ and the only eigenvalue of P_{-w} of modulus larger than $1-\delta$ is λ_{-w}. Hence, $\lambda_{-w} = \overline{\lambda}_w$ for all $w \in V$.

We recall that the billiard map is time-reversible. The involution $\iota(\ell,\theta) = (\ell,-\theta)$ satifies:

$$\begin{aligned} \iota \circ T &= T^{-1} \circ \iota, \\ F \circ \iota &= -F \circ T^{-1}, \end{aligned}$$

Let $\mathcal{B}^*$ be the dual of $\mathcal{B}$. Informally, the space $\mathcal{B}^*$ contains distributions which are regular in the direction of the stable cones of T, and irregular in the unstable cones. Then $\mathrm{Sp}(P_w^* \curvearrowright \mathcal{B}^*) = \mathrm{Sp}(P_w \curvearrowright \mathcal{B})$. In addition, for all φ, $\psi \in \mathcal{C}^1(M,\mathbb{C})$:

$$\int_M P_w^*(\varphi) \cdot \psi \, \mathrm{d}\mu = \int_M \varphi \cdot P_w(\psi) \, \mathrm{d}\mu = \int_M e^{i\langle w,F\rangle} \varphi \circ T \cdot \psi \, \mathrm{d}\mu. \tag{3.3}$$

Let $\widetilde{\mathcal{B}}$ be the image of $\mathcal{B}$ under precomposition by the involution ι. Again, informally, $\widetilde{\mathcal{B}}$ contains distributions which are regular in the direction of the stable cones of T, and irregular in the unstable cones. Let us define $\widetilde{P}_w(\varphi) := (P_w(\varphi \circ \iota)) \circ \iota$, and extend this operator by continuity to $\widetilde{\mathcal{B}}$. Then

$\mathrm{Sp}(\widetilde{P}_w \curvearrowright \widetilde{\mathcal{B}}) = \mathrm{Sp}(P_w \curvearrowright \mathcal{B})$. In addition, for all φ, $\psi \in \mathcal{C}^1(M,\mathbb{C})$:

$$\begin{aligned}
\int_M \widetilde{P}_w(\varphi) \cdot \psi \, \mathrm{d}\mu &= \int_M P_w(\varphi \circ \iota) \cdot \psi \circ \iota \, \mathrm{d}\mu \\
&= \int_M e^{i\langle w, F\rangle} \varphi \circ \iota \cdot \psi \circ \iota \circ T \, \mathrm{d}\mu \\
&= \int_M e^{i\langle w, -F\circ T^{-1} \circ \iota\rangle} \varphi \circ \iota \cdot \psi \circ T^{-1} \circ \iota \, \mathrm{d}\mu \\
&= \int_M e^{-i\langle w, F\circ T^{-1}\rangle} \varphi \cdot \psi \circ T^{-1} \, \mathrm{d}\mu \\
&= \int_M e^{-i\langle w, F\rangle} \varphi \circ T \cdot \psi \, \mathrm{d}\mu. \qquad (3.4)
\end{aligned}$$

To sum up:

$$\begin{aligned}
\mathrm{Sp}(P_{-w}^* \curvearrowright \mathcal{B}^*) &= \mathrm{Sp}(P_{-w} \curvearrowright \mathcal{B}), \\
\mathrm{Sp}(\widetilde{P}_w \curvearrowright \widetilde{\mathcal{B}}) &= \mathrm{Sp}(P_w \curvearrowright \mathcal{B}),
\end{aligned}$$

and the operators P_{-w}^* and $\widetilde{P}_w$ coincide on $\mathcal{C}^1$ functions by Equations (3.3) and (3.4).

We would like to show that the spectra of P_{-w}^* and $\widetilde{P}_w$ coincide, at least outside of $\overline{B}(0,\rho_0)$. Unfortunately, a result such as [3, Lemma A.1] is not directly applicable, because we don't know whether $\mathcal{C}^1(M,\mathbb{C})$ is dense in $\mathcal{B}^*$. We use instead an ad hoc argument, and show the weaker result $\lambda_w = \lambda_{-w}$.

The space $\widetilde{\mathcal{B}}$ is defined as the completion of $\mathcal{C}^1(M,\mathbb{C})$ for the norm $\|\cdot\|_{\widetilde{\mathcal{B}}}$. Hence, $\mathcal{C}^1(M,\mathbb{C})$ is dense for the strong topology on $\widetilde{\mathcal{B}}$. In addition, by [7, Lemma 3.9], $\mathcal{C}^1(M,\mathbb{C})$ maps continuously into $\mathcal{B}^*$, and this map is injective (since there is an injective embedding $\mathcal{C}^\gamma \hookrightarrow \mathcal{B}$).

Let Π_{-w} be the (rank 1) spectral projection of P_{-w} onto the eigenspace corresponding to the eigenvalue λ_ε, and $Q_{-w} := P_{-w} - \lambda_w \Pi_{-w}$. The spectral radius of Q_{-w} is no larger than $1-\delta$, so let $\delta' < \delta$ with $|\lambda_{-w}| > 1-\delta'$. Then, for all φ, $\psi \in \mathcal{C}^1(M,\mathbb{C})$,

$$\int_M \varphi \cdot P_{-w}^n(\psi) \, \mathrm{d}\mu = \lambda_{-w}^n \Pi_{-w}(\psi)(\varphi) + O\left((1-\delta')^n \|\varphi\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1}\right).$$

By density, there exists a $\mathcal{C}^1$ function ψ such that $\Pi_{-w}(\psi) \neq 0$ in $\mathcal{B}$. Following the construction in [8, Lemma 3.8], and noticing that the test function can be chosen smooth (for instance by mollification), we get a function $\varphi \in \mathcal{C}^1$ such that $\Pi_{-w}(\psi)(\varphi) \neq 0$.

In addition, for the functions φ and ψ constructed above,

$$\begin{aligned}
\int_M \varphi \cdot P_{-w}^n(\psi) \, \mathrm{d}\mu &= \int_M (P_{-w}^*)^n(\varphi) \cdot \psi \, \mathrm{d}\mu \\
&= \int_M \widetilde{P}_w^n(\varphi) \cdot \psi \, \mathrm{d}\mu \\
&= \lambda_w^n \widetilde{\Pi}_w(\psi)(\varphi) + O\left((1-\delta')^n \|\varphi\|_{\mathcal{C}^1} \|\psi\|_{\mathcal{C}^1}\right),
\end{aligned}$$

where $\widetilde{\Pi}_w$ is the eigenprojection of $\widetilde{P}_w$ corresponding to the eigenvalue λ_w. Hence, $\Pi_{-w}(\psi)(\varphi) = \widetilde{\Pi}_w(\varphi)(\psi) \neq 0$ and $\lambda_w = \lambda_{-w}$.

The function $w \mapsto \lambda_w$ is even on a neighborhood of zero, and since $\lambda_w = \lambda_{-w} = \overline{\lambda}_w$, it is also real-valued. As a consequence, for the same value $\delta > 0$, for all Λ:

$$\mathrm{Sp}(P_\Lambda \curvearrowright \mathcal{B}_\Lambda) \subset \overline{B}(0, 1-\delta) \cup [1-\delta, 1].$$

This finishes the proof of Theorem 1.1.

Remark 3.3 (Real eigendistributions). *The eigenvalues $\lambda_w \in [1-\delta, 1]$ are real, and the corresponding eigenspaces are 2-dimensional. These eigenspaces are spanned by pairs of eigendistributions $\{h^{\Re}_{\Lambda,\chi}, h^{\Im}_{\Lambda,\chi}\}$, which are real (in that $h^{\Re}_{\Lambda,w}(\varphi)$ and $h^{\Im}_{\Lambda,w}(\varphi)$ are both real for any real test function φ) and can be chosen as:*

$$\begin{aligned} h^{\Re}_{\Lambda,\chi} &:= \Re(h_w \otimes \chi) \\ h^{\Im}_{\Lambda,\chi} &:= \Im(h_w \otimes \chi), \end{aligned}$$

where $\chi = e^{i\langle w,\cdot\rangle}$ and h_w is an eigendistribution for P_w.

3.3 Distribution of the resonances

Now, we shall discuss the convergence of the spectral densities and prove Propositions 1.2, 1.3 and 1.4.

Proof of Proposition 1.2. Let ρ_0 be the upper bound on $\rho_{\text{ess}}(P \curvearrowright \mathcal{B})$ given by [7, Proposition 2.3]. Since $w \mapsto (P_w)_{w\in 2\pi\mathbb{T}^2}$ is a continuous family of transfer operators with $\rho_{\text{ess}}(P_w \curvearrowright \mathcal{B}) \leq \rho_0$, by continuity of the spectrum, for any compact $K \subset \overline{B}(0,\rho_0)^c$, the function $w \mapsto \nu_w(K)$ is bounded, where:

$$\nu_w = \sum_{\substack{\lambda \text{ resonance of } P_w \\ |\lambda| > 1-\rho_0}} \delta_\lambda,$$

and where the resonances are counted with multiplicity.

The spectral decomposition yields, for any rank 2 lattice Λ,

$$\nu_\Lambda = \frac{1}{|G|} \sum_{w\in\widehat{G}} \nu_w,$$

whence $\Lambda \mapsto \nu_\Lambda(K)$ is also bounded uniformly in Λ. Compactness of subprobability measures on K yields the existence of limit distributions of (ν_{Λ_N}) on any compact K, and a diagonal argument yields Proposition 1.2. □

Propositions 1.3 and 1.4 follow from the discussion after [10, Theorem 3.1], with some care to make the constants explicit.

Proof of Proposition 1.3. Recall that $\Lambda_N^{(1)} = N\mathbb{Z} \otimes \mathbb{Z}$. By the aforementioned discussion, there exists $\delta_0 > 0$ such that, for all $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ supported on $[1-\delta_0, 1]$,

$$\lim_{N\to+\infty} \int_{1-\delta_0}^{1} f \, \mathrm{d}\nu_N^{(1)} = \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda_{te_1}) \, \mathrm{d}t,$$

with $e_1 = (1,0)$. The constant 2π comes from the different parametrization we use for t. By the Morse lemma, there exists a $\mathcal{C}^1$ diffeomorphism of the real line Ψ such that $\Psi'(0) = 1$ and:

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda_{te_1}) \, \mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} f\left(1 - \frac{\Sigma_{11} t^2}{2}\right) |\Psi'(t)| \, \mathrm{d}t.$$

Let $\varepsilon > 0$. If f is supported on a small enough neighborhood of 1,

$$\left| \frac{1}{2\pi} \int_{\mathbb{R}} f(\lambda_{te_1}) \, \mathrm{d}t - \frac{1}{2\pi} \int_{\mathbb{R}} f\left(1 - \frac{\Sigma_{11} t^2}{2}\right) \, \mathrm{d}t \right| \leq \varepsilon \|f\|_\infty ,$$

and:

$$\frac{1}{2\pi} \int_{\mathbb{R}} f\left(1 - \frac{\Sigma_{11} t^2}{2}\right) \, \mathrm{d}t = \frac{1}{\pi} \int_0^{+\infty} f\left(1 - \frac{\Sigma_{11} t^2}{2}\right) \, \mathrm{d}t = \int_0^{+\infty} f(1-u) \frac{1}{\pi\sqrt{2\Sigma_{11} u}} \, \mathrm{d}u,$$

finishing the proof of Proposition 1.3. □

Proposition 1.4 follows from the same kind of computations.

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