Example: A standard family of log-Lipschitz cones.

7.8

(Ω,d) metric space of bounded diameter $D=\operatorname{diam}\Omega<+\infty$.

$X = \mathrm{Lip}(\Omega,d)$ with the norm

$$|\varphi|_{Lip} = \sup_\Omega|\phi| + \mathrm{Lip}_\Omega(\phi)$$

$$\mathrm{Lip}(\varphi) = \sup_{\substack{x\neq y\\ x,y\in\Omega}} \frac{|\phi(x)-\phi(y)|}{d(x,y)}$$

Def 9.1 We define for $a>0$ the cone

$$\mathcal{C}_a = \{\varphi : X\to\mathbb{R}_+ : \varphi_x \le e^{a d(x,y)}\varphi_y \quad \forall x,y\in\Omega\}.$$

Rem: when $\varphi>0$: $\varphi\in\mathcal{C}_a \Leftrightarrow \mathrm{Lip}(\log\varphi)\le a$

Prop 9.2 For any $a>0$, $\mathcal{C}_a$ is an $\mathbb{R}$-cone in X which is both inner and outer regular.

Furthermore, when $a>b>0$:

$$\Delta = \operatorname{diam}_{\mathcal{C}_a}\mathcal{C}_b^* \le 2\log\frac{a+b}{a-b} + 2bD < \infty$$

proof: Inner regular:

The constant fct $\mathbb{1}\in\mathcal{C}_a$.

Let $h\in X$, $|h|_\infty<\delta$, $\mathrm{Lip}\,h<\delta$:

Then
$$\begin{aligned} 1+h_x &\le 1+h_y+\delta d(x,y)\\ &\le (1+h_y)\left(1+\frac{\delta d}{1-\delta}\right)\\ &\le (1+h_y)\,e^{a d(x,y)}\end{aligned}$$

provided $\delta/(1-\delta)\le a$ or $\delta\le\frac{a}{1+a}$ so $B_X(\mathbb{1},\frac{a}{1+a})\subset\mathcal{C}_a$

Outer regular: Pick $x_0\in\Omega$ and set $\langle \ell,\phi\rangle=\phi(x_0)$. $\|\ell\|_{X'}=1$

Let $\phi\in\mathcal{C}_a$ ~~and suppose~~ ~~that sup φ ≤ φ~~

~~Suppose~~ For $x,y\in\Omega$, $d=d(x,y)$:

suppose $\varphi_x\ge\varphi_y>0$ then:

$$\frac{\varphi_x-\varphi_y}{d} \le \frac{\varphi_x-e^{-ad}\varphi_x}{d} \le \varphi_x\frac{1-e^{-ad}}{d}$$

so
$$\mathrm{Lip}\,\varphi \le \sup\phi\cdot\lim_{d\to 0}\frac{1-e^{-ad}}{d} = a\cdot\sup\phi$$

Thus
$$\begin{aligned}|\varphi|_{Lip} &\le \sup\varphi + \mathrm{Lip}\,\varphi\\ &\le (1+a)\sup\varphi\\ &\le (1+a)\,e^{+aD}\varphi(x_0)\\ &=: \frac{1}{K}\langle\ell,\phi\rangle\end{aligned}$$

with $1/K=(1+a)e^{aD}>0$.

Strict cone inclusion:

Let $\varphi^1,\varphi^2\in\mathcal{C}_b$, $b<a$.

$t\varphi^1-\varphi^2\in\mathcal{C}_a$ iff $\forall x,y$:

$t\varphi^1_x-\varphi^2_x \le e^{ad}(t\varphi^1_y-\varphi^2_y)$ or

$$t \ge \frac{\varphi^2_y e^{ad}-\varphi^2_x}{\varphi^1_y e^{ad}-\varphi^1_x} = \frac{\varphi^2_y}{\varphi^1_y}\,\frac{1-\frac{\varphi^2_x}{\varphi^2_y}e^{-ad}}{1-\frac{\varphi^1_x}{\varphi^1_y}e^{-ad}}$$

so
$$\begin{aligned} t_1=\beta(\varphi^1,\varphi^2) &\le \sup(\mathrm{RHS})\\ &\le \sup\frac{\varphi^2_y}{\varphi^1_y}\cdot\frac{1-e^{-(a+b)d}}{1-e^{-(a-b)d}}\\ &\le \sup\frac{\varphi^2_y}{\varphi^1_y}\cdot\frac{a+b}{a-b}\quad(\text{exo})\end{aligned}$$

$$\begin{aligned} t_1t_2 = \beta(\varphi^1,\varphi^2)\,\beta(\varphi^2,\varphi^1) &\le \sup\frac{\varphi^2_y}{\varphi^1_y}\frac{\varphi^1_x}{\varphi^2_x}\left(\frac{a+b}{a-b}\right)^2\\ &\le e^{2bD}\left(\frac{a+b}{a-b}\right)^2 \text{ and}\end{aligned}$$

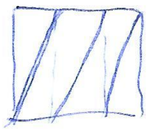
$$\Delta \le 2\log\frac{a+b}{a-b}+2bD$$

Remark: Birkhoff constant:

$$\begin{aligned}\eta = \tanh\frac{\Delta}{4} &= \frac{e^{\Delta/2}-1}{e^{\Delta/2}+1}\\ &\le \frac{\frac{a+b}{a-b}e^{bD}-1}{\frac{a+b}{a-b}e^{bD}+1}\\ &= \frac{(a+b)e^{bD}-(a-b)}{(a+b)e^{bD}+(a-b)}\end{aligned}$$

7.9

Theorem ~~7.8~~ Let $f: S^1 \circlearrowleft$ be a uniformly expanding C^2-map, i.e $\forall x \in S^1: |f'(x)| \geq \beta > 1$. Then



a) ~~Then~~ There is a unique f-invariant probability measure μ abs cont w.r.t Lebesgue. $d\mu = h(x)\,dx$

b) μ is mixing, whence ergodic

c) When ~~...~~ $\phi \in L^1(S^1)$ then for Lebesgue a.e $x \in S^1$

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x) \xrightarrow[n\to\infty]{} \int_{S^1} \phi\, d\mu$$

Proof: a) We look for $d\mu = h\,dx$ such that $\forall a \in L^\infty(S^1)$:

$$\int a\circ f\, d\mu = \int a\, d\mu \quad \Leftrightarrow$$

(*) $$\int a\circ f \cdot h\, dx = \int a \cdot h\, dx$$

Define the Ruelle transfer ~~op~~ operator: For ~~...~~ $\phi \in L^1$

$$\mathcal{L}\phi(y) = \sum_{x \in f^{-1}(y)} \frac{1}{|f'(x)|}\phi(x)$$

Then for $a \in L^\infty$, $\phi \in L^1$:

(**) $$\boxed{\int_{S^1} a\circ f(x)\,\phi(x)\,dx = \int_{S^1} a(y)\,\mathcal{L}\phi(y)\,dy}$$

(change of variables)

We want to find h s.t. $\mathcal{L}h = h$.

Lemma: (transfer property)

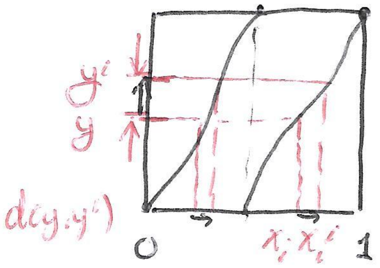
$$\mathcal{L}(a\circ f \cdot \phi) = a \cdot \mathcal{L}\phi$$

proof:

$$\mathcal{L}(a\circ f\cdot\phi)(y) = \sum \frac{1}{|f'(x)|}\, a\circ \underbrace{f(x)}_{=y}\,\phi(x) = a(y)\cdot \mathcal{L}\phi(y)$$

We may suppose that 0 is a fixed point of f

We suppose $f' > 0$



The degree of f:

$$d = \int_{S^1} f'\,dx \in \{2,3,\dots\}$$

Given y, y' at distance $d(y,y') \le 1/2$ we may pair preimages x_i, x_i' $i=1,\dots,d$ s.t.

$$d(x_i,x_i') \le \frac{1}{\lambda} d(y,y')$$

$$\left|\log \frac{|Df^k(x)|}{|Df^k(x')|}\right| = \left|\int_{x'}^{x} \frac{f''}{f'}\,d\tilde{x}\right| \le M\, d(x,x')$$

where $M = \sup_{S^1} \left|\frac{f''}{f'}\right| < +\infty$

Computing the Ruelle op:

$$\mathcal{L}\phi(y') = \sum_{\text{preim.}} \frac{1}{|Df(x_i')|}\phi(x_i')$$

$$\le \sum_{\text{preim}} \frac{e^{M d(x,x')}}{|Df(x_i)|}\phi(x_i')$$

$$\le \sum_{\text{preim}} \frac{\phi(x_i)}{|Df(x_i)|} e^{M d(x,x') + \alpha d(x,x')}$$

$$= \mathcal{L}\phi(y)\, e^{(M+\alpha) d(x,x')}$$

$$\le \mathcal{L}\phi(y) \cdot e^{\frac{1}{\lambda}(M+\alpha) d(y,y')}$$

$$(*) \Rightarrow \lambda^{-n}\int\phi - \int h\,\langle m,\phi\rangle = O(q^{n-1})$$

$$\lambda^{-n}\int\phi \xrightarrow[n]{} \underbrace{\int h}_{>0}\cdot\langle m,\phi\rangle$$

$$\Rightarrow \left\{ \lambda = 1,\ \int h\cdot 1 = \int 1 = 1,\ \langle m,\phi\rangle = \int\phi \right.$$

insert $\int\mathcal{L}(a\circ f\cdot h)$

Choose $\alpha > \frac{M}{\lambda-1}$ and set

$$\sigma = \frac{1}{\lambda}\left(1+\frac{M}{\alpha}\right) < 1$$
$$= \frac{1}{\alpha}\cdot\frac{\alpha+M}{\lambda} < 1$$

Then we have

$$\mathcal{L}\phi(y') \le \mathcal{L}\phi(y)\, e^{\sigma\alpha\, d(y,y')}$$

so

$$\mathcal{L} : C_\alpha^* \longrightarrow C_{\sigma\alpha}^*$$

We may apply the spectral gap thm $\Rightarrow$

$\exists!\ \lambda > 0$, $\boxed{h \in C^*_{\alpha\sigma}}$, normalize so $\int h\,dx = 1$, $m \in \mathrm{Lip}(S^1)'$ such that $\mathcal{L}h = \lambda h$

better: $m(1)=1$

$$(*) \quad |\lambda^{-n}\mathcal{L}^n\phi - h\langle m,\phi\rangle|_{\mathrm{Lip}} \le C q^{n-1}|\phi|_{\mathrm{Lip}}$$

On the other hand from (**) setting $a \equiv 1$ we get

$$\int\phi = \int\mathcal{L}\phi = \int\mathcal{L}^2\phi = \dots$$

with $\phi = h$: $\lambda\int h = \int h_{>0} \Rightarrow \boxed{\lambda = 1}$

replace by

Then for any ϕ

$$\int\phi = \int\mathcal{L}^n\phi \longrightarrow \int h\cdot\langle m,\phi\rangle = \langle m,\phi\rangle$$

so we may identify $m \in \mathrm{Lip}(S^1)'$ as ~~an~~ the Lebesgue measure

$$\boxed{dm = dx \quad \text{on } S^1}$$

we have already $\int h\,dx = \langle m,h\rangle = 1$

We claim that

$$d\mu = h(x)\,dx$$

is an f-invariant measure

Use (**) for $\phi \in \mathrm{Lip}\, S^1$; $\mathcal{L}h = \lambda h$.

$$\int a\circ f\cdot h\,dx = \int a\cdot\mathcal{L}h\cdot dx = \int a\cdot h\,dx$$

or

$$\boxed{\int_{S^1} a\circ f\,d\mu = \int_{S^1} a\,d\mu}$$

μ is mixing!

For $a \in L^\infty(S^1)$, $\phi \in \mathrm{Lip}\, S^1$

$\int a\circ f^n \cdot \phi \, d\mu = \int a\circ f^n \, \phi h \, dx = \int \mathcal{L}^n(a\circ f^n \cdot \phi h) dx$

$= \int a \cdot \mathcal{L}^n(\phi h) \, dx.$

So

$\left| \int a\circ f^n \phi \, d\mu - \int a \, d\mu \int \phi \, d\mu \right| =$

$\left| \int a \cdot [\mathcal{L}^n(\phi h) - h \cdot \int \phi h \, dx] \, dx \right| \le$

$\int |a| \cdot C\eta^{n-1} \cdot |\phi h|_{Lip} \, dx \xrightarrow[n\to\infty]{} 0$ mixing expon. fast on Lip (S^1)

For $a \in L^\infty(S^1)$, $b \in L^1(S^1)$, $\epsilon > 0$ find $\phi \in \mathrm{Lip}(S^1)$ with

$|b - \phi|_{L^1(d\mu)} \le \epsilon$

Then

$\overline{\lim_{n\to\infty}} \left| \int a\circ f^n \, b \, d\mu - \int a \, d\mu \int b \, d\mu \right|$

$\le \epsilon \cdot |a|_\infty$

and $\epsilon > 0$ was arbitrary.

For $A, B \in \mathcal{B}(S^1)$ we have

$\mu(f^{-n}A \cap B) = \int \chi_A \circ f^n \cdot \chi_B \, d\mu$

$\xrightarrow[n]{} \int \chi_A d\mu \int \chi_B d\mu$

$= \mu(A)\mu(B)$

so f is μ-mixing whence μ ergodic

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a+c) μ is abs cont wrt Lebesgue whence the unique ~~each~~ f-ergodic measure with this property.

Since $0 < h < \infty$ unif on S^1

μ-a.e $=$ leb-a.e so with $b \in L^1(S^1, d\mu)$:

By Birkhoff for leb-ae $x \in S^1$, ~~be~~

$\frac{1}{n}\sum_{0}^{n-1} b\circ f^k(x) \xrightarrow[n\to\infty]{} \int b \, d\mu = \int_{S^1} b \, h \, dx$

General weights.

Thm 7.10 Let $f: S^1 \circlearrowleft$ be a unif expanding C^1-map ($|f'| \geq A > 1$) and $g \in X = Lip(S^1)$. Define the transfer operator $\phi \in C^0$

$$\mathcal{L}_g \phi(y) = \sum_{x: f(x)=y} e^{g(x)} \phi(x)$$

Then

1) There are $\lambda > 0$, $h \in X_+$, $dm \in M^1_+(S^1)$ such that $\int h\, dm = 1$ and
$$\mathcal{L}_g h = \lambda h, \quad m \mathcal{L}_g = \lambda m$$
~~Corresponding to~~

2) $d\nu = h\, dm$ is T-invariant ergodic, mixing.

3) For $\phi \in X$, $n \geq 1$ one has
$$\| \lambda^{-n} \mathcal{L}_g^n \phi - h \int \phi\, dm \|_X \leq C \eta^{n-1} \|\phi\|_X$$
for some $C < \infty$, $0 < \eta < 1$

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proof: C_a (log-Lipsch family) is conformly contracted for $a > 0$ large enough.
$$\mathcal{L}_g C_a^* \subset C_{\sigma a}^* \quad 0 < \sigma < 1.$$
$\Rightarrow$ spectral gap on $X = Lip$:
$$\| \lambda^{-n} \mathcal{L}_g^n \phi - h \langle m, \phi \rangle \|_X \leq C \eta^{n-1} \|\phi\|_X$$

Here $m \in X'$ is a bd linear functional on $X = Lip(S^1)$.
~~Normalize so~~
$\mathcal{L}_g$ is positive, $\mathbb{1} = \text{const fct} \in C_a$
Normalize m s.t. $\langle m, \mathbb{1} \rangle = 1$
For $\phi \in X$, $|\phi(x)| \leq 1$ we have
$$-\mathbb{1} \leq \phi \leq \mathbb{1} \Rightarrow$$
$$-\lambda_g^{-n} \mathcal{L}_g^n \mathbb{1} \leq \lambda_g^{-n} \mathcal{L}_g^n \phi \leq \lambda_g^{-n} \mathcal{L}_g^n \mathbb{1}$$
$\rightarrow$ $-h \langle m, \mathbb{1} \rangle \leq h \langle m, \phi \rangle \leq h \langle m, \mathbb{1} \rangle$

$\Rightarrow$ $|\langle m, \phi \rangle| \leq \langle m, \mathbb{1} \rangle = 1$

More generally for any $\phi \in X$:
$$|\langle m, \phi \rangle| \leq \|\phi\|_\infty$$
As X is dense in C^0, m extends as a lin funct on C^0 with the same bound. Riesz $\Rightarrow$
There is a corresponding measure $dm \in M^1_+(S^1)$.
(We normalized s.t. $\int dm = 1$)

Define $d\nu = h\, dm$ which is a proba.
For $\phi \in X$:
$$\int \phi \circ T\, d\nu = \int \phi \circ T \cdot h\, dm = \lambda^{-1} \int \mathcal{L}_g(\phi \circ T\, h)\, dm = \lambda^{-1} \int \phi \cdot \mathcal{L}_g h\, dm = \int \phi\, h\, dm = \int \phi\, d\nu$$
so T-invariant

Proof of mixing follows the same lines as before.