Local dimension

For $z \in J(f)$, $r > 0$:

$$\log \nu_g(B(z,r)) = -nP(g) + S_n g(z) + O(1)$$
$$\log r = -\log \lambda_n(z) + O(1)$$

with $n = n(z,r) \longrightarrow \infty$ as $r \to 0^+$; so

$$\frac{\log \nu_g(B(z,r))}{\log r} = \frac{P(g) - \frac{1}{n} S_n g(z) + O(\frac{1}{n})}{\frac{1}{n}\log \lambda_n(z) + O(\frac{1}{n})}$$

By Birkhoff's Erg thm there is $A \subset J(f)$ of full ν_g measure s.t.

$$\forall z \in A: \quad \frac{1}{n} S_n g(z) \xrightarrow[n]{} \int g \, d\nu_g$$
$$\frac{1}{n} \cancel{\log} S_n(\log|f'|) \xrightarrow[n]{} \int \log|f'| d\nu_g$$

Since $n(z,r) \to +\infty \Leftrightarrow r \to 0^+$:

$\forall z \in A$:

$$\underline{d}_{\nu_g}(z) = \underline{\lim} \frac{\log \nu_g B(z,r)}{\log r} = \frac{P(g) - \int g \, d\nu_g}{\int \log|f'| d\nu_g}$$

Since $\nu_g(A) = 1$, and $\mathbb{C} \simeq \mathbb{R}^2$ has the Besicovitch property, Theorem 10 yields:

$$\dim_H \nu_g = \frac{P(g) - \int g \, d\nu_g}{\int \log|f'| d\nu_g}$$

By the variational principle for Gibbs measures

$$P(g) - \int g d\nu_g = h_{\nu_g}$$ metric entropy.

The quantity

$$\Lambda(\nu_g) = \int_{J(f)} \log|f'| \, d\nu_g$$

is called the Lyapunov exponent of f w.r.t. ν_g

Thm 23 For the Gibbs measure ν_g associated with $g \in Lip(J(f_c), d)$ for a uniformly ~~expanding~~ hyperbolic Julia set we have

$$\dim_H \nu_g = \frac{h_{\nu_g}}{\Lambda(\nu_g)}$$

h_{ν_g} = entropy of ν_g

$\Lambda(\nu_g)$ = Lyapunov exponent

Bowen's formula.

$J(f)$ unif hyp. Julia set.

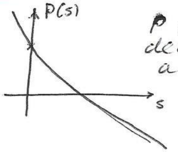
Weight: $g_s(z) = -s\log|f'|$

$$\mathcal{L}_{g_s}\phi(w) = \sum_{z: fz=w} \frac{1}{|f'(z)|^s}\phi(z)$$

$$P(s) := P(-s\log|f'|)$$

$\boxed{\text{Thm 24 } s^* = \dim_H J(f_c) \in (0,2] \text{ is the unique value for which } P(s^*)=0}$

proof:



P is strictly decreasing as a fct of $s \in \mathbb{R}$

Recall $\forall z \in J(f): |(f^n)'(z)| \geq C\beta^n$

$$s_1 > s_2: \underbrace{\mathcal{L}^n_{g_{s_1}}\mathbb{1}}_{e^{nP(s_1)+O(1)}\mathbb{1}} \leq \frac{1}{(C\beta^n)^{s_1-s_2}} \underbrace{\mathcal{L}^n_{g_{s_2}}\mathbb{1}}_{e^{nP(s_2)+O(1)}\mathbb{1}}$$

$$\Rightarrow \quad P(s_1) \leq P(s_2) - (s_1-s_2)\log\beta$$

$$P(s_1) + s_1\log\beta \leq P(s_2) + s_2\log\beta$$

$P(0) = \log 2 > 0$
P continuous (in fact analytic)
shows: $\exists!\, s^*>0: P(s^*)=0$

$$\text{Lemma 22} \Rightarrow \nu_{g_{s^*}}(B(z,r)) = e^{S_n g_{s^*}(z)+O(1)} = \frac{1}{|(f^n)'(z)|^{s_*}}\cdot e^{O(1)}$$

since also $r = \frac{1}{|(f^n)'(z)|^{s_*}} e^{O(1)}$
we get as $r\to 0, n\to\infty$

$$\nu_{g_{s^*}}(B(z,r)) = r^{s^*}\cdot e^{O(1)}$$

for every $z \in J(f)$. $\nu_{g_{s^*}}$ is s^*-Ahlfors regular so

$$s^* = \dim(\nu_{g_{s^*}}) = \dim_H \operatorname{supp}\nu_{g_{s^*}} = \dim_H J(f)$$

Remarks:
- The s^*-dim Hausdorff measure is equivalent with ν_{g^*}
- There are (not unif hyperbolic) Julia sets with $\dim_H J(f) = 2$.

For other values of s:
Analytic perturbation theory showed:

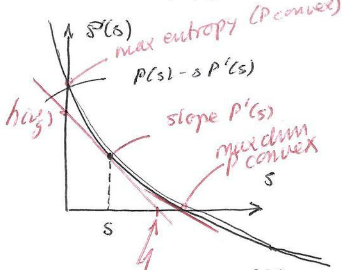
$$P'(s) = \nu_g(-\log|f'|) = -\Lambda(\nu_g)$$

Lyapunov exponent

$$P(g_s) - \int g_s\, d\nu_g = P(s) + s\int \log|f'|\, d\nu_g = P(s) - sP'(s).$$

We conclude:

$$\dim_H(\nu_{g_s}) = \frac{\overbrace{P(s)-sP'(s)}^{h_{\nu_g}}}{\underbrace{-P'(s)}_{\Lambda(\nu_g)}} = s - \frac{P(s)}{P'(s)}$$



$$\Lambda(\nu_g) = \frac{h(\nu_g)}{\dim_H \nu_g}$$