Remarks:

More general Ruelle operator

$$\mathcal{L}_g \phi(y) = \sum_{x:\, fx=y} e^{g(x)} \phi(x)$$

e.g. $f: S^1 \to S^1$ unif expanding C^2

Any $g \in \mathrm{Lip}(X;\mathbb{R})$ gives rise to an ergodic, mixing measure μ_g.

(in general not abs cont / Leb.)

Ex (part II : Dyn Sys)

- ~~Dyn Sys~~: Hausdorff dimension of Julia sets

$z \mapsto z^2+c$
(c small)

Ex: Central Lim Thm.
and
Large deviation theorem.

Large deviations in Ergodic theory.

$(\Omega, \mathcal{B}, \mu, T)$ ergodic transf. of a proba. space.

$A : \Omega \to \mathbb{R}$ an observable. assumed $L^2(\mu)$

$S_n A = A + A\circ T + \dots + A \circ T^{n-1}$

Birkhoff sums.

$\frac{1}{n} S_n A \xrightarrow[n]{} \mathbb{E}(A) = \mu(A)$ a.e.

(Central Limit theorem studies $\frac{1}{\sqrt{n}} S_n(A - \mathbb{E}A) \xrightarrow[n]{\text{law}} N(0, \sigma^2)$ valid under some circumst.)

Rare event (large deviation) probabilities studies the behavior of

$$P\left(\tfrac{1}{n} S_n A \geq q\right) \quad \text{for } p > \mathbb{E}(A).$$

Often one has for some exponent

$$P\left(\tfrac{1}{n} S_n A \geq q\right) \sim e^{-nI(q)}$$

Cramer '38 (worked in an insurance company)

Suppose existence of

$$\phi(t) = \log \mathbb{E}(e^{tA}), \quad \forall t \in \mathbb{R}$$

and i.i.d. rand. var.

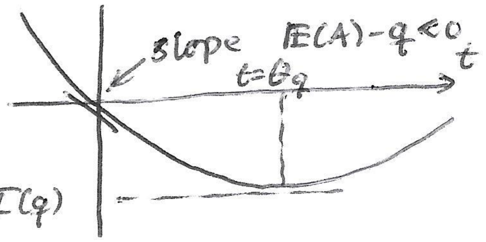
Set

$$I(q) = \inf_{t>0} (\phi(t) - tq)$$

Then

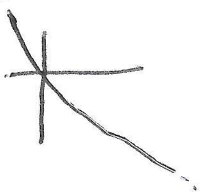
$$\lim_n \frac{1}{n} \log P\left(\tfrac{1}{n} S_n A \geq q\right) = I(q)$$

Graph of $\phi(t) - tq$ (always a convex fct)



May happen that $q >$ esssup A

Then $I(q) = -\infty$



$X = X_{\mathbb{C}} = Lip(\Omega; \mathbb{C})$

equipped with the Lipschitz norm is a Banach algebra $\forall A, B \in X$:

$$\|A \cdot B\|_{Lip} \le \|A\|_{Lip} \cdot \|B\|_{Lip}$$

$\|\mathbb{1}\|_{Lip} = 1$

$\mathbb{1}$ unit element.

$f: S^1 \to S^1$ unif exp C^2-map.

$A \in X$, $t \in \mathbb{C}$:

$$\mathcal{L}_t \phi(y) = \sum_{x: fx=y} e^{tA(x)} \cdot \frac{1}{|f'(x)|} \phi(x)$$

unif conv in $Lip(\Omega, \mathbb{C})$ $\to \sum \frac{t^n}{n!} (A(x))^n$

$t \in \mathbb{C} \mapsto \mathcal{L}_t \in L(X)$ defines an analytic family of bd lin operators.

Spectral ~~result~~ Thm

[Kato: Perturbation theory for linear operators chap VII §1]

Suppose that λ_0 is a simple eval for $\mathcal{L}_0$ with right/left evec's $h_0 \in X$, $\ell_0 \in X'$.

Then $\exists \delta > 0$: $\forall |t| < \delta$:

$\mathcal{L}_t$ has a simple eval λ_t with $t \in \mathbb{D}_\delta \to \lambda_t \in \mathbb{C}$ analytic

There are right/left evecs h_t, ℓ_t $(\ell_t(h_t) \equiv 1$ s.t

$t \in \mathbb{D}_\delta \mapsto h_t \in X$, $\ell_t \in X'$ are analytic fcts.

$\forall t \in \mathbb{R}$, $\mathcal{L}_t$ is a uniform contraction of some log-Lipschitz cone $\mathcal{C}_{a(t)}$

but cone const depends upon t.

so the sp. gap thm holds

~~PXXX~~:

$\exists \lambda(t) \in \mathbb{R}_+$ largest eigenvalue analytic in $t \in \mathbb{R}$, evecs

$h_t \in \mathcal{C}_{a(t)}$ $\ell_t \in X'$

ℓ_t strictly pos on $\mathcal{C}^*_{a(t)}$

all depending analytically upon t.

$$\lambda_t^{-n} \mathcal{L}_t^n \phi = h_t \langle \ell_t, \phi \rangle + R_t^n \phi$$

$$\|R_t^n\| \le C_t \eta_t^{n-1}, \quad n \ge 1$$

$\eta_t < 1$ all $t \in \mathbb{R}$

$$\Phi_n(t) := \frac{1}{n}\log \mathbb{E}_0(e^{tS_nA})$$

(i.i.d case: $= \log \mathbb{E}_0(e^{tA})$
but no longer the case)

$$e^{n\Phi_n(t)} = \mathbb{E}_0(e^{tS_nA})$$
$$= \ell_0(e^{tS_nA}h_0)$$
$$= \ell_0(\mathcal{L}_0^n e^{tS_nA}h_0)$$
$$= \ell_0(\mathcal{L}_t^n h_0)$$
$$= \ell_0((h_t\ell_t + R_t^n)h_0)\lambda_t^n$$
$$= \lambda_t^n(\underbrace{\ell_0(h_t)}_{>0}\underbrace{\ell_t(h_0)}_{>0} + (\ell_0 R_t^n h_0))$$
$$= \text{const}\ \lambda_t^n + O(\lambda_t^n \eta_t^n)$$

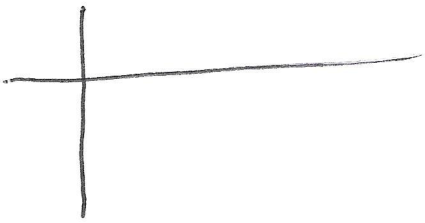
$$\phi(t) = \lim_n \Phi_n(t) = \lim_n \frac{1}{n}\log(\text{const}\ \lambda_t^n)$$
$$= \log \lambda_t$$

Thm (easy part)
$f: S^1 \to S^1$ unif exp C^2 map
$A: S^1 \to \mathbb{R}$ Lipschitz fct.
μ_0 the unique ergodic f-invar meas. abs cont/Leb.

Then $\forall q > \mathbb{E}_0(A)$:
$$\lim_n \frac{1}{n}\log \mu_0(S_nA \geq nq) \leq \inf_{t\in\mathbb{R}_+}(\log\lambda_t - t\cdot q)$$

(difficult part: we have equality)

Chernov inequality:
$$P(S_nA \geq nq) =$$
$$\mathbb{E}(\mathbb{1}_{S_nA-nq}) \leq$$
$$\mathbb{E}(e^{t(S_nA-nq)}) \quad \binom{\text{any}}{t\geq 0}$$
$$= \mathbb{E}(e^{tS_nA})e^{-tnq}$$
$$\vdots$$
$$= (\underbrace{\text{const}}_{>0}\ \lambda_t^n + O(\lambda_t^n\eta_t^n))e^{-tnq}$$
$$\simeq \text{const}\cdot e^{n(\log\lambda_t - tq)}$$



$$\frac{1}{n}\log P(S_nA \geq nq)$$
$$\leq \underbrace{\frac{1}{n}\log \text{cst}}_{\downarrow n\ 0} + \log\lambda_t - tq$$