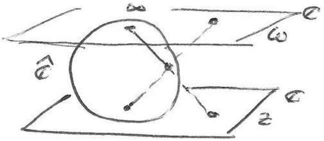
Complex Dynamics [Carleson - Gamelin]

A Normal family and Montel's theorem

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ ~~compact~~ $\simeq S^2$
compact manifold equipped with charts

$\hat{\mathbb{C}} \setminus \{\infty\} \longrightarrow \mathbb{C} \ni z$ transition map
$\hat{\mathbb{C}}^* = \hat{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{C} \ni w = 1/z$



$\hat{\mathbb{C}}$: Riemann sphere

Spherical metric (inv. under $w = 1/z$)

$$ds = \frac{|dz|}{1+|z|^2} = \frac{|dw|}{1+|w|^2} \quad (*)$$

$D \subset \hat{\mathbb{C}}$ an open subset.
$f : D \to \hat{\mathbb{C}}$ is meromorphic if it is holomorphic in a nghd of every point in some charts

ex: $f(z) = \frac{z^3}{1-z^2}$ is merom: $\hat{\mathbb{C}}$ ↺

$$z \in \mathcal{O}(1) \xmapsto{C^\omega} \frac{1}{f(z)} \in \mathcal{O}(0)$$

$$z \in \mathcal{O}(\infty) \longmapsto f(z) \in \mathcal{O}(\infty)$$

$$w = 1/z \in \mathcal{O}(0) \longmapsto \frac{1}{f(1/w)} = w(w^2-1) \in \mathcal{O}(0)$$

Def// $\mathcal{F} \subset \mathrm{Merom}(D)$ is said to form a normal family if every seq $(f_n)_{n\geq 1} \in \mathcal{F}$ admits a subseq $(f_{n_k})_{k\geq 1}$ which converges uniformly on compact subsets in D (w.r.t the spherical metric on $\hat{\mathbb{C}}$)

By Arzela-Ascoli being normal is equiv to the family being equi-cont (on compact subsets of D).

Given $f \in \mathrm{Merom}(D)$ we call:

$$D_{\hat{\mathbb{C}}} f(z) = |f'(z)| \frac{1+|z|^2}{1+|f(z)|^2}$$

the spherical derivative of f.

ex when $\tilde{f}(z) = f(1/z)$ or $\frac{1}{f(z)}$ then at the same pts: $D_{\hat{\mathbb{C}}} \tilde{f} = D_{\hat{\mathbb{C}}} f$ (follows from (*))

Thm 12 $\mathcal{F} \subset \mathrm{Merom}(D)$ is a normal family iff for every compact set $K \subset D$ the spherical deriv $D_{\hat{\mathbb{C}}} f(z)$ is uniformly bdd for $z \in K$, $f \in \mathcal{F}$.

---

proof: "⇒" if not $\exists K$ cmpt, $z_n \in K$ and $f_n \in \mathcal{F}$ s.t. $Df_n(z_n) \to \infty$. Taking subseq^s we may assume $z_n \to z^* \in K$ and $f_{n_k} \to f^*$ unif on a nghd of z^*.

We may assume $z^* \in \mathbb{C}$, $f^*(z^*) \in \mathbb{C}$ (or else compose with $z \mapsto 1/w$). There is $\delta > 0$ $\varepsilon > 0$ s.t. $f^* \overline{B(z, \delta)} \subset B(w, \varepsilon)$ so for n large $f_n \overline{B(z^*, \delta)} \subset B(w, 2\varepsilon)$.
Then $\forall\, |z - z^*| \leq \delta/2$:
$|f_n'(z)| \leq \frac{2\varepsilon}{\delta/2}$ uniformly contradicting $Df_n(z_n) \to \infty$, $z_n \to z^*$

$\oint_{\partial B} \frac{f(u)}{(u-z)^2} \frac{du}{2\pi i}$

"⇐" Arzela-Ascoli

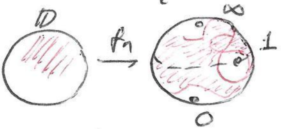
13

Thm (3.2 Carleson-Gamelin)
[Montel's theorem]

Let $\mathcal{F} \subset$ Merom(D). Suppose there are 3 points in $\hat{\mathbb{C}}$ that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

Suppose $\forall n \geq 1$:

ex: $f_n : \mathbb{D} = B_{\mathbb{C}}(0,1) \xrightarrow{\omega} \mathbb{C} \setminus \{0,1\}$

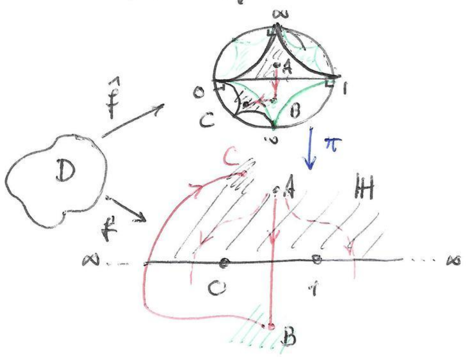


then $\exists$ a subseq: (f_{n_k}) that converges uniformly on compact sets $K \subset \mathbb{D}$ to an analytic fct $f^* : \mathbb{D} \longrightarrow \hat{\mathbb{C}}$

(the limit could take values also in $0, 1, \infty$)

(sketch of proof of Montel:)

$\exists$ universal covering map
$\pi : \Delta = B_{\mathbb{C}}(0,1) \longrightarrow \mathbb{C} \setminus \{0,1\}$



$$\begin{tikzcd} & \Delta \arrow[d, "\pi"] \\ D \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0,1\} \end{tikzcd}$$

Any $f : D \xrightarrow{C^0} \mathbb{C} \setminus \{0,1\}$ lifts to a holom map $\hat{f} : \cancel{\Delta} D \to \Delta$ which is thus unif bd.

$\mathcal{F}$ a family of merom(D) omitting $\{0,1,\infty\}$ lifts to a family $\hat{\mathcal{F}} \subset$ merom$(D;\Delta)$ which is normal.

One shows that ~~if~~ then ~~$\hat{f}_n \in \hat{\mathcal{F}}$ converges unif on compact subsets, then so does $f_n \in \mathcal{F}$~~ also $\mathcal{F}$ is normal.

Def 14 $f:\mathbb{C}\to\mathbb{C}$ polyn of degree ≥ 2.

$K(f) = \{z\in\mathbb{C} : f^n(z) \not\to \infty\}$
is called the filled Julia set of f. (compact)

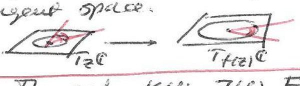
$J(f) = \partial K(f)$ is called the Julia set of f. (compact)

$F(f)=\mathbb{C}\setminus J(f)$ is called the Fatou set of f. (open)

ex: $f(z) = z^d$ $\quad d\geq 2$.

$K(f) = \{z\in\mathbb{C} : |z|\leq 1\} = \overline{\mathbb{D}}$
$J(f) = \partial K(f) = S^1$
$F(f) = \mathbb{C}\setminus S^1$

When f holomorphic $f'(z)\neq 0$:
'f'': $T_z\mathbb{C} \to T_{f(z)}\mathbb{C}$
is a conformal map, i.e. it preserves angles, of tangent vectors at z. It also maps "round" balls to "round" balls (essential for dimension theory). in the tangent space.



~~The sets~~

Prop 15 The sets $K(f)$, $J(f)$, $F(f)$ (and $U_\infty = \mathbb{C}\setminus K(f)$) are completely invariant; e.g.:

$$f K(f) = K(f) = f^{-1}K(f)$$

(clear)

Def 16 $J(f)$ is said to be uniformly hyperbolic if $\exists\, \beta>1, 0<C<\infty$ st.

$\forall z\in J(f),\ n\geq 1:\ |(f^n)'(z)|\geq C\beta^n$

ex: $J(z\mapsto z^2) = S^1$ is unif hyp
$|f'(z)| = |2z| = 2 \quad \forall z\in S^1$.

Thm 17 Let $J(f)$ be unif hyp. Then for every $z\in J(f), r>0$:
$\mathcal{U} = \bigcup_{n\geq 0} f^n B(z,r)$
omits at most one pt in $\mathbb{C}$
~~In particular, $f: J\to J$ is topol. mixing~~
If $a\in\mathbb{C}$ is not in $\mathcal{U}$ then $f^{-1}(a) = \{a\}$ so $f(z) = a+(z-a)^d$ and a is a stable fixed pt. (so $\notin J$).

proof: Since $f^n(z)$ stays unif bd the conformal derivative

$$D_{\hat{\mathbb{C}}} f^n(z) = |f^{n\prime}(z)| \frac{1+|z|^2}{1+|f^n(z)|^2}$$

diverges as $n\to\infty$.
By thm 12 f^n cannot be equicontinuous on $B(z,r)$
$(f^n)|_{B(z,r)}$ not equi-cont
$\Rightarrow$ Montel: at most two points in $\hat{\mathbb{C}}$ can be omitted and one is ∞

If a is omitted then so are $f^{-1}(a)$ which must equal $\{a\}$. This is possible only if
$f(z) = a+(z-a)^d$.

ex: $f(z)=z^2$ $\quad \bigcup_{n\geq 0} f^n B(z,r) = \mathbb{C}^*$

Thm 18 $J(f)$ unif hyp.
Then $\forall r>0\ \exists N=N(r)$ st:

$\forall z\in J(f)$: $f^N B(z,r) \supset J(f)$. $(*)$

proof: f is an open map.
(maps open sets to open sets).
$J(f)$ compact $\subset \bigcup_{n\geq 0} f^n B(z,r)$

$\Rightarrow$ $J(f) \subset \bigcup_{n\geq 0}^{N} f^n B(z,r)$

for some $N = N(z,r)$

Suppose $(*)$ is false. Then $\forall k\geq 1$:
$\exists z_k\in J(f)$, $w_k\in J(f)$, $n_k\to\infty$:

$$w_k \notin f^{n_k} B(z_k, r)$$

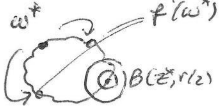
extracting subsequences
$w_{k_l}\to w^*$, $z_{k_l}\to z^*$ we
may assume: $\forall l\geq l_0$:

$$w^* \notin f^{n_{k_l}} B(z^*, r/2)$$

But then $f^{n_{k_l}-1} B(z^*, r/2) \not\supset \underbrace{f^{-1}(w^*)}$

so $\left(f^{n_{k_l}-1}|_{B(z^*,r/2)}\right)_{l\geq l_0}$ Card ≥ 2

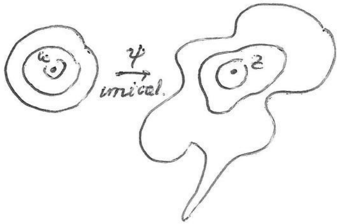
must be a normal family
contrary to thm 17.



Lemma 19 (Distortion Lemma)
Let $\psi: B(w,\delta) \to \mathbb{C}$ be
holomorphic and
univalent (1-1).
Write $z=\psi(w)$, $\lambda=\psi'(w)\neq 0$
Then

$$B\left(z, \lambda\tfrac{2\delta}{9}\right) \subset \psi B\left(w, \tfrac{\delta}{2}\right) \subset B(z, \lambda\cdot 2\delta)$$

proof (C-G, Thm 1.6)



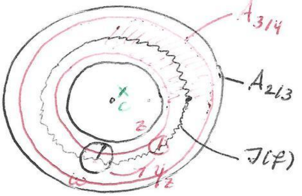
Case study:

$f_c(z) = z^2 + c, \quad |c| < \frac{1}{10}$

One has:

$|z| \leq 3/4 \Rightarrow |f_c(z)| < \frac{2}{3}$

$|z| \geq 4/3 \Rightarrow |f_c(z)| > 3/2$



Annuli, $0 < \rho < 1$:

$A_\rho = \{z : \rho < |z| < \frac{1}{\rho}\}$

$f^{-1} \bar{A}_{2/3} \subset A_{3/4}$

$J(f) = \bigcap_{n \geq 1} f^{-n} \bar{A}_{2/3}$

For $z \in \bar{A}_{3/4}$ we have

$|f'(z)| = 2|z| \geq 2 \cdot 3/4 = 3/2 =: \beta$

Any local inverse (there are two for each ω)

$\forall \omega \in \bar{A}_{2/3},\ z \in f^{-1}(\omega)\ (\subset A_{3/4})$:

$|\psi_z'(\omega)| = \frac{1}{|f'(z)|} \leq 2/3$

$\Rightarrow \forall \omega \in \bar{A}_{2/3},\ \delta_0 = 3/4 - 2/3 = 1/12$:

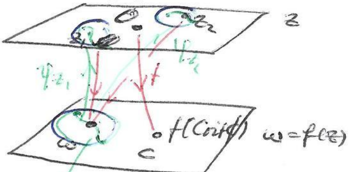
$\psi_z B(\omega, \delta_0) \subset B(z, \frac{2}{3} \delta_0)$

(contracting map)

$1/\beta = 2/3$-Lipschitz

$f(z) = z^2 + c$

$\{0\} = \mathrm{Crit} f \quad f'(0) = 0$



(inverse fct thm)

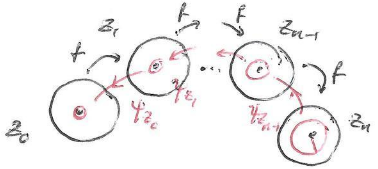
path avoiding c $\omega(t)$ $t \in [0,1]$ lifts to unique path $z_1(t)$, $z_2(t)$

simply connected domain avoiding $\{c\} = f(\mathrm{Crit} f)$ $B(\omega, \delta)$ lifts to unique s.c. preimages $\psi_{z_1} B(\omega, \delta)$, $\psi_{z_2} B(\omega, \delta)$

ψ_{z_1} and ψ_{z_2} are univalent

$f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{c\}$

is a degree 2 covering map.



$z_n = f^n(z_0) \qquad (z_0 \in J(f))$

$\psi_{z_k}: B(z_{k+1}, \delta_0) \longrightarrow B(z_k, \delta_0)$
local inverse,
holom + univalent

$\psi^{(n)}_{z_0}: B(z_n, \delta_0) \longrightarrow B(z_0, \delta_0)$
holom + univalent

$\psi^{(n)}_{z_0} = \psi_{z_0} \circ \dots \circ \psi_{z_{n-1}}$

$\lambda_n(z_0) = (f^n)'(z_0)$

$1/\lambda_n(z_0) = (\psi^{(n)}_{z_0})'(z_n)$

By the distortion lemma, we get for $0 < r < \delta_0/2$ and writing

$$B_n(z_0, r) = \psi^{(n)}_{z_0}(B(z_n, r))$$

(the n'th Bowen Ball at z_0)

(*)
$$B(z_0, \tfrac{4}{9} \cdot \tfrac{r}{\lambda_n}) \subset B_n(z_0, r) \subset B(z_0, 4 \cdot \tfrac{r}{\lambda_n(z_0)})$$

<u>Lemma 20</u> Let $\delta_1 = \frac{\delta_0}{9M}$.
where $M = \sup_{A_{3/4}} |f'|$

For $z \in J(f)$, $r > 0$ let $n = n(z, r) \geq 1$ be the maximal integer for which $B(z, r) \subset B_n(z_q, \delta_0/2)$

Then $B_n(z_q, \delta_1/2) \subset B(z, r)$
and $\delta_1 \leq \frac{1}{2} \lambda_n(z_q) \cdot r \leq \delta_0$

---

proof:

Since $B(z, r) \not\subset B_{n+1}(z, \delta_0/2)$ we must have (by (*))

$$\cancel{B(z}\ r > \frac{4}{9} \frac{\delta_0/2}{\lambda_{n+1}} \geq \frac{\cancel{4}\, 2\delta_0}{9M\lambda_n} \text{ or}$$

$$\frac{1}{2}\lambda_n \cdot r \geq \frac{\delta_0}{9M} = \delta_1$$

Also
$B(z, r) \subset B_n(z, \delta_0/2) \subset B(z, \frac{2\delta_0}{\lambda_n})$
implies $\frac{1}{2}\lambda_n \cdot r \leq \delta_0$ //