5 F.C. KINGMAN 1968 KO *

Theorem 5.1 (Subadditive Erg Thm)
Let $(X, \mathcal{B}, \mu)$ be a proba space and $T: X \to X$ an MPT. Let $(f_n)_1^\infty$ be a sequence of meas fcts $f_n: X \to \mathbb{R} \cup \{-\infty\}$ satisfying the conditions

(a) $f_1^+ \in L^1(\mu)$

(b) $\forall k, n \geq 1$: $f_{n+k} \leq f_n + f_k \circ T^n$ a.e.

Then there exist a meas fct $\underline{f}: X \to \overline{\mathbb{R}}$ such that

1. $\underline{f}^+ \in L^1_\mu$, $\underline{f} = \underline{f} \circ T$ a.e.
2. $\lim \frac{1}{n} f_n = \underline{f}$ a.e and for any T-invar $A = T^{-1}A \pmod 0$
3. $\lim_n \frac{1}{n} \int_A f_n = \inf_n \frac{1}{n} \int_A f_n = \int_A \underline{f} \in [-\infty, +\infty)$

Remarks: (a), (b) $\Rightarrow$ $f_n^+ \in L^1_\mu$ $\forall n \geq 1$
Thus either $f_n \in L^1_\mu$ or $\int f_n = -\infty$ and the same for $\underline{f}$.

Proof: ...

R1*

Lemma

$\underline{f} \leq \underline{f} \circ T \Rightarrow \underline{f} = \underline{f} \circ T$ a.e.

proof for $r < s$ let

$E_{r,s} = \{\underline{f} < r < s < \underline{f} \circ T\}$

$= \{\underline{f} < r\} \cap \{s < \underline{f} \circ T\}$

Now $\{s < \underline{f} \circ T\} \subset \{s < \underline{f}\}$

$\mu(s < \underline{f} \circ T) = \mu(s < \underline{f})$

so $\{s < \underline{f} \circ T\} = \{s < \underline{f}\} \pmod 0_\mu$

and

$E_{r,s} \equiv \{\underline{f} < r\} \cap \{s < \underline{f}\} \pmod 0_\mu$

$= \emptyset.$

$\{\underline{f} < \underline{f} \circ T\} = \bigcup_{\substack{r<s \\ r,s \in \mathbb{Q}}} E_{r,s}$ null set

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$\underline{f}(x) = \liminf_{n \to \infty} \frac{1}{n} f_n(x) \in [-\infty, +\infty]$

$f_{n+1}(x) \leq f_1(x) + f_n \circ T(x)$

$\frac{1}{n+1} f_{n+1}(x) \leq \frac{1}{n+1} f_1(x) + \frac{n}{n+1} \frac{1}{n} f_n \circ T(x)$

Case: $f_1(x) = -\infty$. Then $f_{n+1}(x) = -\infty \ \forall n \geq 1$

and $\underline{f}(x) \lim_n \frac{1}{n} f_n(x) = -\infty \leq \underline{f} \circ T(x)$

Case: $f_1(x) > -\infty$. Let (n_k) be a subseq. s.t. $\frac{1}{n_k} f_{n_k}(T(x)) \to \underline{f}(Tx)$

Then

$\frac{1}{n_k+1} f_{n_k+1} \leq \frac{1}{n_k+1} f_1(x) + \frac{n_k}{n_k+1} \cdot \frac{1}{n_k} f(Tx)$

$\downarrow k \qquad \downarrow k$

$\underline{f}(x) \leq \liminf_k \frac{1}{n_k+1} f_{n_k+1} \leq 0 + 1 \cdot \underline{f}(Tx)$

Conclusion: $\exists X' \subset X$ of full measure

$\underline{f}(x) = \underline{f}(Tx) \quad \forall x \in X'$

~~We assume here $f>0$~~ so $\underline{f} = \underline{f} \circ T$ K2 *

Lemma: Let g be a T-invariant function which is in L^1_μ. Suppose that $\forall x \in X$ either

$\underline{f}(x) < g(x) < +\infty$ or

$\underline{f}(x) = g(x) = +\infty$.

Then for a.e. $x \in X$:

$\bar{f}(x) \leq g(x)$ and if $A = T^{-1}A$:

$\limsup_n \frac{1}{n} \int_A f_n \leq \int_A g$

Proof: For $x \in X$ s.t. $\underline{f}(x) < g(x) < \infty$ we write (always finite)

$n(x) = \inf\{k \geq 1 : f_k(x) \leq k \cdot g(x)\} < +\infty$.

Let $L \geq 1$ and define a stopping time:

$\tau_L(x) = \begin{cases} 1 & \text{if } n(x) > L \\ n(x) & \text{if } n(x) \leq L \end{cases}$

$B_L = \{x : n(x) > L\}$

stopping time sequence: elaborate

$n_0 = n \quad n_{k+1} = n_k + \tau_L \circ T^{n_k}$

we have $1 \leq n_{k+1} - n_k \leq L$ so $n_k \to \infty$ as $k \to \infty$.

Given $N > 0$ we let ~~K_N~~

$k_N = \inf\{k \geq 1 : n_{k+1} \geq N\}$

then for $k = k_N$: $N - L \leq n_k < N$

Define the auxilary fct:

$F_L(x) = \mathbb{1}_{B_L}(x) \cdot (f_1(x) - g(x))^+ \geq 0$.

We claim that

$\forall x \in X$ with $\underline{f}(x) < +\infty$: $n = \tau_L(x)$:

~~$f_{\tau_L(x)}(x) \leq S_{\tau_L(x)} g(x) + S_{\tau_L(x)} F_L(x)$~~

~~$n = \tau_L(x)$:~~ $f_n(x) \leq S_n(g + F_L)(x)$

proof $x \in B_L$

$f_1(x) = g(x) + (f - g)$

$\leq g + (f-g)^+$

$x \notin B_L$

$f_n \leq S_n g \leq S_n(g + F_L^+)$

Then

$f_N(x) \leq f_{n_1} + f_{n_2 - n_1} \circ T^{n_1} + \dots + f_{n_k - n_{k-1}} \circ T^{n_{k-1}} + f_{N - n_k} \circ T^{n_k}$

$\leq S_{n_k}(g + F_L) + S_{N - n_k} f_1^+ \circ T^{n_k}$

elaborate $S_{n_1} + S_{n_2 - n_1} + \dots + S_{N - n_k} f_1^+ \circ T^{n_k}$

so that

$\frac{1}{N} f_N \leq \frac{n_k}{N} \frac{1}{n_k} S_{n_k}(g + F_L) + \frac{1}{N} S_L f_1^+ \circ T^{N-L}$ (*)

Since g and F_L are in L^1_μ, $\frac{1}{n_k} S_{n_k}(g + F_L) \to g^* + F_L^*$ where g^* and F_L^* are the Birkhoff limits a.e.

Also $\frac{1}{N} f_1^+ \circ T^L \to 0$ a.e so the last term tends to zero as $N \to \infty$. Therefore

$\bar{f} = \limsup \frac{1}{N} f_N \leq g^* + F_L^*$

Now $F_L \xrightarrow[L \to \infty]{} 0$ pointwise everywhere so also $F_L^* \to 0$ a.e. and therefore

$\bar{f}(x) \leq g^*(x)$

By (*) we also see that

$\int_A \frac{1}{N} f_N \leq \int_A \frac{1}{n_k} S_{n_k}(g + F_L) + \int_A \frac{1}{N} S_L f_1^+ \circ T^{N-L}$

$\leq \int g + F_L + \frac{L}{N} f_1^+$

so

$\limsup \int \frac{1}{n} f_n \leq \int g + F_L$

and again taking $L \to \infty$ we get the result.

K3 *

Let $g = g_{\epsilon,M} = \max\{\underline{f}+\epsilon, -M\}$

Then $\underline{f} < g$ whenever $\underline{f} < +\infty$.

and since

$$f_n(x) \le f_1^+ + f_1^+\circ T + \dots + f_1^+\circ T^{n-1}$$

$$\frac{1}{n} f_n(x) \le \frac{1}{n} S_n f_1^+ \xrightarrow[n]{\text{Birkh.}} (f_1^+)^* \text{ a.e.}$$

we have

Point 1. $\underline{f}(x) \le \overline{f}(x) \le (f_1^+)^*(x)$ a.e. which is in L^1_μ

There is a set of full measure $\Omega_{\epsilon,M}$ for which $\overline{f}(x) \le g_{\epsilon,M}(x)$

Then $\Omega = \bigcap_{M=1} \Omega_{1/M,M}$ is of full measure and $\forall x \in \Omega$, $M \ge 1$

$$\overline{f}(x) \le \max\{-M, \underline{f}(x) + \tfrac{1}{M}\}$$

which implies

$$\overline{f}(x) = \underline{f}(x) \in [-\infty, +\infty)$$

Also for $A = T^{-1}A$ ($\le \int g_{\epsilon,M}$ $\forall \epsilon, M$ so taking ...)

$$\limsup_N \int_A \frac{1}{N} f_N \le \int_A \underline{f} \quad (**)$$

← insert here

~~Since ...~~

~~...~~

Clearly

$$\int_A \inf_{k>n} \frac{1}{k} f_k \le \inf_{k>n} \int_A \frac{1}{k} f_k \le I_A^*$$

But $\frac{1}{n} f_n \le (f_1^+)^*$

so ~~either~~ if $\inf_{k>n} \frac{1}{k} f_k$ is integrable for ~~some~~ $n \ge N_0$ then ~~and~~ by monotone convergence

$\int \lim_n \inf_{k>n} \frac{1}{k} f_k$

$$= \lim_n \int \inf_{k>n} \frac{1}{k} f_k \le \liminf_{k>n} \int \frac{1}{k} f_k$$

$$= \int \underline{f} \qquad = I_A^*$$

For $A = T^{-1}A$

$f_{n+m} \le f_n + f_m \circ T^n \Rightarrow$

$\int_A f_{n+m} \le \int_A f_n + \int_A f_m \Rightarrow$

$I_{n+m} \le I_n + I_m$ subadd

$\Rightarrow I^* = \lim \frac{1}{n} I_n = \inf_{n\ge 1} \frac{1}{n} I_n$

By (**)

$$I_A^* = \int_A \underline{f} = \int_A \overline{f}$$

$$= \inf_{n\ge 1} \frac{1}{n} \int_A f_n$$

Added later:

If we do not assume $\inf_{k>n} \frac{1}{k} f_k$ integrable then let

$f_n^{(M)} = \max\{f, -nM\}$

for $M \in \mathbb{N}$. This is sub-add and the above holds for $f_n^{(M)}$ so there is a limit a.e $\underline{f}^{(M)}$ of $\frac{1}{n} f_n^{(M)}$ and

$$\int_A \underline{f}^{(M)} = \inf_{n\ge 1} \frac{1}{n} \int_A f_n^{(M)}$$

One has $\forall M \ge 1$

$\underline{f}^{(M)} = \max\{\underline{f}(x), -M\}$

a.e. for some $f: X \to \mathbb{R} \cup \{-\infty\}$

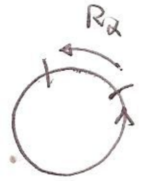
and

$$\int_A \underline{f} = \inf_M \int_A \underline{f}^{(M)} \quad \text{(monot conv)}$$

$$= \inf_{M,n} \frac{1}{n} \int f_n^{(M)}$$

$$\text{(monotone conv)} = \inf_n \frac{1}{n} \int f_n$$

ex K



$x \in S^1$ $\quad \alpha \in \mathbb{R} \setminus \mathbb{Q}$

$R_\alpha(x) = x + \alpha$ uniquely ergodic

Soit $M : S^1 \to M_{n,n}(\mathbb{R})$ measurable.

$\log^+ \|M_x\| \in L^1(S^1)$ $\qquad M_x^n = M_{R_\alpha^{n-1}(x)} \cdots M_x$ product along orbit

$f_n(x) = \log \| M_{R_\alpha^{n-1}(x)} \cdots M_x \| = \log \| M_x^n \|$

$f_{n+m}(x) \leq f_n(x) + f_m \circ R_\alpha^n(x)$

Later on: Oseledec and Lyapunov exponents

~~$\|A_{n-1} A_n\|$~~

$\|AB\| \leq \|A\| \|B\|$

$\|A_{n+m-1} \cdots A_n \cdot A_{n-1} \cdots A_1\| \leq$

$\|A_{n+m-1} \cdots A_n\| \cdot \|A_{n-1} \cdots A_1\|$

$f_1^+(x) = \log^+ \|M_x\| = \max\{0, \log\|M_x\|\}$

$\in L^1_\mu$ if eg every M is bounded

Even if $f_1 \in L^1_\mu$ we need not have $f_n \in L^1_\mu$

ex: $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\log \| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \| = -\infty$!!!

Kingman $\Rightarrow$

$\frac{1}{n} \log \|M_x^n\| \xrightarrow[n \to \infty]{a.s.}$ const. ~~$f(x) = f(R_\alpha(x))$~~

by (unique) ergod. $f(x) = $ const.

So

$\|M_x^n\|^{1/n} \xrightarrow{a.s} \exp(C)$

$\in [0, +\infty[$

$C = \inf_{n \geq 1} \frac{1}{n} \int \log \|M_x^n\| dx$

$\exp C = \inf e^{\frac{1}{n} \int \log \|M_x^n\| dx}$

$$\text{const} = \lim_{n \to +\infty} \int_0^1 \frac{1}{n} \log \|M_x^n\| dx$$

$$= \inf_{n \geq 1} \frac{1}{n} \int_0^1 \log \|M_x^n\| dx$$