2. Birkhoff etc...

Theorem 2.1 [Birkhoff, 1931]

$(X, \mathcal{B}, \mu, \tau)$ meas pres. transf. with μ σ-finite. Let $f \in L^1(X,\mu)$. We then have

1. For μ a.e. $x \in X$ the following limit exists:
$$f^*(x) := \lim_n \frac{1}{n} \sum_{k=0}^{n-1} f \circ \tau^k(x)$$

2. $f^*(x) = f^* \circ \tau(x)$ (μ-a.e. x) and f^* is in $L^1(X,\mu)$.

3. Let $\Omega \equiv \tau^{-1}\Omega$ (mod 0). Then

a) $\int_\Omega |f^*| \leq \int_\Omega |f|$

and if in addition $\mu(\Omega) < \infty$:

b) $\int_\Omega f^* = \int_\Omega f$

c) $\frac{1}{n} \sum_0^{n-1} f \circ \tau^k \longrightarrow f^*$ in $L^1(X,\mu)$.

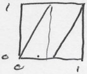
Remarks:
Suffices to show: for
$\bar{f}(x) = \overline{\lim}_n \frac{1}{n} S_n f(x)$
$\underline{f}(x) = \underline{\lim}_n \frac{1}{n} S_n f(x)$
that a.e. $\bar{f}(x) = \underline{f}(x)$.

Note that
$\frac{1}{n} S_n f(x) = \frac{1}{n} f(x) + \frac{1}{n} S_{n-1} f(\tau x)$
As $\frac{1}{n} f(x) \xrightarrow[n\to\infty]{} 0$ we get
$\bar{f}(x) = \bar{f}(\tau x)$ and
$\underline{f}(x) = \underline{f}(\tau x)$

Clearly $\underline{f}(x) \leq \bar{f}(x)$.

Corollary 2.2
If τ is μ-ergodic then $f^* = \text{const}$ (a.e.)
So for μ-a.e. $x \in X$:
$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f \circ \tau^k(x) \longrightarrow f^*$$
If $\mu(X) = 1$ ~~(prob.)~~ then
$$\lim_n \frac{1}{n} \sum f \circ \tau^k(x) \longrightarrow \int_X f(x)\,d\mu$$
for μ-a.e. $x \in X$.

example: { later...
$\tau(x) = 2x \bmod 1$, $x \in S^1$
$$\frac{1}{n}\sum_0^{n-1} f \circ \tau^k(x) \xrightarrow[n]{} \int_0^1 f(x)\,dx$$
for Lebesgue a.e. $x \in S^1$



$\tau(x) = 2x \bmod 1$

Lemma 2.3 Let $\Omega \equiv \tau^{-1}\Omega$ mod 0 (measurable σ-finite meas.) Let $f : \Omega \to [0, +\infty)$ be meas. Then

$$\int_\Omega f \geq \int_\Omega \bar{f}$$

---

proof: Let $R < \infty$, $\epsilon > 0$ and define:

$$g(x) = g_{R,\epsilon}(x) = \min\{R, (1-\epsilon)\bar{f}(x)\}$$

Then $g(x) = g \circ \tau(x)$ and when $g(x) > 0$ we have $\bar{f}(x) > g(x)$

We set (stopping time)

$$\bar{n}(x) = \bar{n}_{R,\epsilon}(x) = \inf\{k \geq 1 : S_k f(x) \geq k g(x)\}$$

When $g(x) = 0$ we have $\bar{n}(x) = 1$ and if $g(x) > 0$ then by definition of $\bar{f}$ there is k finite verifying the condition.

Thus $\bar{n}(x) < \infty$ for every $x \in \Omega$.

Let $1 \leq L < +\infty$ and denote

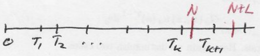
$$E = E_{L,R,\epsilon} = \{x \in \Omega : \bar{n}(x) \leq L\}$$

We set

$$T(x) = \begin{cases} \bar{n}(x) & \text{if } x \in E \\ 1 & \text{if } x \notin E \end{cases}$$

and define a (need not measurable) sequence of stopping times

$$T_0(x) = 0, \quad T_{k+1}(x) = T_k(x) + T \circ \tau^{T_k(x)}(x)$$



Let $N < \infty$. For every $x \in \Omega$ there is a maximal k s.t. $T_k(x) < N$. And then

$$N \leq T_{k+1}(x) < N + L.$$

~~For~~ We have

$$S_{N+L} f \geq S_{T_{k+1}} f = S_T f + \ldots + S_T f \circ \tau^{T_k}$$

We thus get the bound

$$S_{N+L} f \geq \sum_{T_k < N} S_T f \circ \tau^{T_k} \geq$$

$$\sum S_T(g \cdot \mathbb{1}_E) \circ \tau^{T_k} \geq S_N(g \cdot \mathbb{1}_E)$$

valid for every $x \in \Omega$.

Integrating over Ω using τ-invariance of μ:

$$(N+L)\int_\Omega f \geq \int_\Omega S_{N+L} f \geq$$

$$\int S_N(g \cdot \mathbb{1}_E) = N \int g \cdot \mathbb{1}_E$$

so

$$\int_\Omega f \geq \frac{N}{N+L} \int_\Omega g_{R,\epsilon} \mathbb{1}_{E_{L,R,\epsilon}}$$

$$\xrightarrow[N \to \infty]{} \int_\Omega g_{R,\epsilon} \mathbb{1}_{E_{L,R,\epsilon}}$$

$$\xrightarrow[\substack{L \to \infty \\ \text{monot. conv}}]{} \int_\Omega g_{R,\epsilon} \quad \text{(since stopping time finite)}$$

$$\xrightarrow[\substack{R \to \infty \\ \text{monot conv}}]{} \int_\Omega (1-\epsilon)\bar{f}$$

and $\epsilon > 0$ was arbitrary

We have $S_T f \geq S_T(g \cdot \mathbb{1}_E)$

proof:

$x \in E$: $S_T f \geq T(x) g(x) \geq S_T(g \cdot \mathbb{1}_E)$

$x \notin E$: $S_T f = S_1 f \geq 0 = S_T(g \cdot \mathbb{1}_E)$

$T(x) = 1$ $\qquad = S_T(g \cdot \mathbb{1}_E)$

Lemma 2.4 $\Omega = \tau^{-1}\Omega$ (mod 0) and $\mu(\Omega) < \infty$.
Let $f : \Omega \to [0, +\infty]$ be measurable. Then here the value ∞ is not allowed

$$\int_\Omega f \le \int_\Omega \underline{f}$$

Let $\varepsilon > 0$ and define $g(x) = g_\varepsilon(x) = \underline{f}(x) + \varepsilon$.
We have $g(x) = g \circ \tau(x)$ and $\underline{f}(x) < g(x)$ provided $\underline{f}(x) < +\infty$.

The function
$$\underline{n}(x) = n_\varepsilon(x) = \inf\{k \ge 1 : S_k f(x) \le k g(x)\}$$
is finite for every $x \in \Omega$.
(when $\underline{f}(x) = g(x) = +\infty$ then $\underline{n}(x) = 1$).

We write $E = E_{L,\varepsilon} = \{\underline{n} \le L\}$ for $L \ge 1$. and we set

$$T(x) = \begin{cases} \underline{n}(x) & \text{if } \underline{n}(x) \le L \\ 1 & \text{if } \underline{n}(x) > L. \end{cases}$$

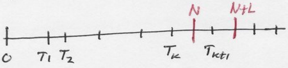
We then have $\forall x \in \Omega$:

$$\boxed{S_{T(x)}(f \cdot 1_E)(x) \le S_{T(x)}\, g(x)}$$

proof:
For $x \in E$: $S_T(f \cdot 1_E) \le S_T f \le S_T g$
For $x \notin E$: $S_T(f \cdot 1_E) = S_1(f \cdot 1_E)(x) = 0 \le S_T g$.

We define a ~~(measurable)~~ sequence of stopping times

$$T_0(x) = 0, \quad T_{k+1}(x) = T_k(x) + T \circ \tau^{T_k(x)}(x)$$



Let $1 \le N < +\infty$.
Then for every $x \in \Omega$ there is $k = k(x)$ maximal for which $T_k(x) < N$. And then
$$N \le T_{k+1}(x) < N + L$$
Then also for $h \ge 0$:
$$S_N h \le S_{T_{k+1}} h = S_T h + \dots + S_T h \circ \tau^{T_k}$$
We obtain
$$S_N(f \cdot 1_E) \le \sum_{T_k < N} S_T(f \cdot 1_E) \circ \tau^{T_k}$$
$$\le \sum_{T_k < N} S_T(g) \circ \tau^{T_k}$$
$$\le S_{N+L}\, g = (N+L)\, g$$
valid for every $x \in \Omega$.
Now integrate over Ω and use τ-invariance of μ to get
$$N \int f \cdot 1_E = \int S_N(f \cdot 1_E) \le (N+L) \int g \quad \text{or}$$
$$\int_\Omega f \cdot 1_{E_{L,\varepsilon}} \le \left(1 + \frac{L}{N}\right) \int_\Omega g_\varepsilon \xrightarrow{N \to \infty} \int_\Omega g_\varepsilon$$
By monotone conv, $L \to \infty$:
$$\int_\Omega f \le \int_\Omega g_\varepsilon = \int_\Omega \underline{f} + \varepsilon \mu(\Omega)$$
As $\varepsilon > 0$ was arbitrary
$$\int_\Omega f \le \int_\Omega \underline{f}$$

NB: $\mu(\Omega) < \infty$ is necessary
$\tau(x) = x + 1$ $(\mathbb{R}, \text{Leb})$
$f \ge 0$ $f \in L^1$ compact supp
$\Rightarrow \underline{f}(x) \equiv 0$ $\forall x$

Prop: 2.5 ~~Theorem~~ (Birkhoff L^1_+)

$(X, \mathcal{B}, \mu, \tau)$ meas pres transf.
μ σ-finite, $f \in \mathcal{L}^1_+(X,\mu)$
Then conclusions of Thm 2.1

1. For μ ae $x \in X$: $f^*(x) := \lim \frac{1}{n} \sum S_n f(x)$ exists.
2. $f^* = f^* \circ \tau$ (μ-a.e.) with f^* in L^1_+
3. For $\Omega \in \mathcal{B}$, $\Omega \equiv \tau^{-1}\Omega$ (mod 1)
   a) $\int_\Omega f^* \leq \int_\Omega f$

   If in addition $\mu(\Omega) < \infty$

   b) $\int_\Omega f^* = \int_\Omega f$ and

   c) $\frac{1}{n} S_n f \longrightarrow f^*$ in $L^1(X,\mu)$

---

proof: For $n \geq 1$ consider
$\Omega_n = \{\bar{f} \geq 1/n\}$. $(= \tau^{-1}\Omega_n)$
By Lemma 2.3 we have
$$\|f\|_1 \geq \int_{\Omega_n} f \geq \int_{\Omega_n} \bar{f} \geq \frac{1}{n}\mu(\Omega_n)$$
so $\mu(\Omega_n) \leq n\|f\|_1 < +\infty$.
We may then apply Lemma 2.4
$$\int_{\Omega_n} f \leq \int_{\Omega_n} \underline{f}$$
so combining:
$$\int_{\Omega_n} \bar{f} \leq \int_{\Omega_n} f \leq \int_{\Omega_n} \underline{f} \leq \int_{\Omega_n} \bar{f}$$
implies equality everywhere
As $\bar{f} - \underline{f} \geq 0$ we must have
$\bar{f} - \underline{f} = 0$ for μ ae $x \in \Omega_n$.
Whence also for μ ae x in
$$\Omega = \{\bar{f} > 0\} = \bigcup_{n \geq 1} \Omega_n$$
finally on $\Omega^c = \{\bar{f} = 0\}$
we have trivially
$0 \leq \underline{f} \leq \bar{f} = 0$ so $\bar{f} = \underline{f}$ on Ω^c

Thus for a set of full measure ~~...~~ we have
$$f^*(x) := \bar{f}(x) = \underline{f}(x)$$
As $\bar{f}$ and $\underline{f}$ are τ-invar so is f^*.
By Lemma 2.3 we have for $\Omega \equiv \tau^{-1}\Omega$ (mod 0):
$$\int_\Omega f^* = \int_\Omega \bar{f} \leq \int_\Omega f.$$

b) Suppose $\mu(\Omega) < \infty$.
Then also (Lemma 2.4)
$$\int_\Omega f \leq \int_\Omega \underline{f} \leq \int_\Omega \bar{f} = \int_\Omega f$$
so ~~again~~ equality ~~and~~ everywhere. so
$$\int_\Omega f^* = \int_\Omega f.$$

---

c) Let $\epsilon > 0$ and let $g \in L^1_+(X,\mu)$ with g bounded and $\|f - g\|_1 \leq \varepsilon/3$.
By dominated + pointwise convergence
$$\int_\Omega \frac{1}{n} S_n g \longrightarrow \int_\Omega g^*.$$
For $n \geq N_\epsilon$
$\|\frac{1}{n} S_n g - g^*\|_1 \leq \varepsilon/3$
~~and since~~
and as $\|\frac{1}{n} S_n (f-g)\|_1 \leq \|f-g\|_1 \leq \epsilon/3$
~~and for $n \geq N_\epsilon$~~
~~$\|\frac{1}{n} S_n f - f^*\|_1 \leq 3 \cdot \epsilon/3$~~ $= \varepsilon$

$\exists N_\epsilon$: $n, m \geq N_\epsilon$ $\|\frac{1}{n} S_n g - \frac{1}{m} S_m g\| \leq \epsilon/3$
so $\|\frac{1}{n} S_n f - \frac{1}{m} S_m f\| \leq \varepsilon$ Cauchy

Proof of Thm 2.1.

For $f \in \mathcal{L}^1(X,\mu)$ write

$f = f_+ - f_-$ $\quad f_+ = f \vee 0, \; -f_- = f \wedge 0$

Then $f_+, f_- \in \mathcal{L}^1_+$ and verifies the condition for Prop 2.5.

$$\frac{1}{n} S_n f_+ \longrightarrow f_+^* \text{ (} \cancel{\;} \text{)}$$

$$\frac{1}{n} S_n f_- \longrightarrow f_-^*$$

with convergence pointwise a.e in X and in L^1 on invariant sets of bd measure.

So there is a set of full measure Λ so that

$$\begin{aligned} f^*(x) &:= f_+^*(x) - f_-^*(x) \\ &= \lim \tfrac{1}{n} S_n f_+ - \lim \tfrac{1}{n} S_n f_- \\ &= \lim \tfrac{1}{n} S_n (f_+ - f_-) \\ &= \lim \tfrac{1}{n} S_n f(x) \end{aligned}$$

$$\begin{aligned} f^*(\tau x) &= f_+^*(\tau x) - f_-^*(\tau x) \\ &= f_+^*(x) - f_-^*(x) = f^*(x) \end{aligned}$$

Let $\Omega = \tau^{-1}\Omega$ then

$$\begin{aligned} \int_\Omega |f^*| &= \int_\Omega |f_+^* - f_-^*| \le \int_\Omega f_+^* + f_-^* \\ &\le \int_\Omega f_+ + f_- = \int_\Omega |f| \end{aligned}$$

If $\mu(\Omega) < \infty$:

$$\int_\Omega f^* = \int_\Omega f_+^* - f_-^* = \int_\Omega f_+ - f_- = \int_\Omega f.$$

$$\begin{aligned} \tfrac{1}{n} S_n f &= \tfrac{1}{n} S_n f_+ - \tfrac{1}{n} S_n f_- \\ &\longrightarrow f_+^* - f_-^* \text{ (in } L^1) \\ &= f^* \end{aligned}$$

Theorem 2.6 L^p Ergodic Thm (v. Neumann $p=2$)

Let $1 \le p < \infty$ and let $(X, \mathcal{B}, \mu, \tau)$ be a meas pres. transf.

We assume $\mu(X) = 1$ here.

Given $f \in L^p(X,\mu)$ there is $f^* \in L^p(X,\mu)$ such that

1) $f^* = f^* \circ \tau$

2) $\| \frac{1}{n} S_n f - f^* \|_p \xrightarrow[n]{} 0$

<u>proof</u>: For g bd measurable we have by Birkhoff

$$\frac{1}{n} S_n g \xrightarrow[n]{} g^* \text{ a.e. pointwise}$$

and by dominated conv.

$$\frac{1}{n} S_n g \xrightarrow[n]{} g^* \text{ in } L^p$$

For $f \in L^p(X,\mu)$, $\epsilon > 0$ there is g bd meas with $\|f - g\|_p \le \epsilon/3$

Then $\| \frac{1}{n} S_n (f-g) \|_p \le \epsilon/3$, $\forall n$ as well. Since

$$\frac{1}{n} S_n g \longrightarrow g^* \text{ in } L^p$$

$(\frac{1}{n} S_n g)$ is Cauchy in L^p.

There is N_ϵ s.t. $\forall n, m \ge N_\epsilon$:

$$\| \frac{1}{n} S_n g - \frac{1}{m} S_m g \| \le \epsilon/3$$

So for such n, m:

$$\| \frac{1}{n} S_n f - \frac{1}{m} S_m f \| \le \epsilon/3 \times 3 = \epsilon$$

So $(\frac{1}{n} S_n f)$ is Cauchy in L^p and by completeness:

$$f^* = \lim \frac{1}{n} S_n f \in L^p(X,\mu).$$

Since

$$\frac{1}{n} S_n f - \frac{1}{n} S_{n-1} f \circ \tau = \frac{1}{n} f$$

and $1/n f \to 0$ in L^p we get

$$f^* - f^* \circ \tau = 0 \text{ in } L^p.$$

Unique ergodicity of an irrational rotation

$T_\alpha(x) = x + \alpha \mod \mathbb{Z}$

$T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

$\alpha \in \mathbb{R}\setminus\mathbb{Q}$

Thm

Let $(S^1, \mathcal{B}_{S^1})$ be the unit circle $(\mathbb{R}/\mathbb{Z})$ with its Borel σ-algebra. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then $\forall f \in C(S^1)$, $x \in S^1$:

$$\lim \frac{1}{n}\sum_{k=0}^{n-1} f\circ T_\alpha^k(x) = \int_{S^1} f(x)\,dx$$

In particular, μ = Lebesgue is the unique invariant Borel measure under T_α.

Proof: Let

$e_k(x) = \exp(2\pi i x k)$.

Then for $k \neq 0$

$$\frac{1}{N} S_n e_k(x) = \frac{1}{N}\sum_{n=0}^{N-1} e_k(x)\cdot e_k(n\alpha)$$

$$= \frac{1}{N} e_k(x) \sum_{n=0}^{N-1} e_k(\alpha)^n$$

$$= \frac{1}{N} e_k(x) \frac{e_k(N\alpha)-1}{e_k(\alpha)-1}$$

$$\xrightarrow[N\to\infty]{} 0$$

~~So~~ For $k=0$ $\quad \frac{1}{N} S_n e_0(x) = \cancel{1_0} = 1$

So for any trigonometric polynomial $P \in \{trig\}$

$\cancel{\frac{1}{N}S_N} P = \sum_{\text{fini}} c_k e_k$

$$\frac{1}{N} S_N P \longrightarrow c_0 = \int_{S^1} P$$

Now for $f \in C(S^1)$, $\epsilon > 0$

$\left|\frac{1}{N} S_N(P-f)\right| \le |P-f|_\infty$

so if we approximate f by $P \in$ trigo then $|f-P| < \varepsilon$ and

$\left|\frac{1}{N} S_N f - \int f\right| \le$

$\left|\frac{1}{N} S_N P - \frac{1}{N} S_N f\right| +$

$\left|\frac{1}{N} S_N P - \int P\right| +$

$\left|\int P - f\right|$ and

$\limsup_N \left|\frac{1}{N} S_N f - \int f\right| \le 2\epsilon$ with $\epsilon > 0$ arbitrary

~~Since~~

Now let μ be any T_α invariant Borel measure. Then for μ ae $x \in S^1$; $f \in C(S^1)$

$\frac{1}{N} S_N f(x) \longrightarrow \int f d\mu$

but we have seen that

$\frac{1}{N} S_N f(x) \longrightarrow \int f dx$

So $\int f d\mu = \int f dx$ $\forall f \in C(S^1) \Rightarrow d\mu = \text{Leb.}$

Irrational rotation
unique ergodicity

Lemma
$R_\alpha(x) = x+\alpha \mod \mathbb{Z}$ with $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is ergodic for Lebesgue measure. λ

proof Let $A = R_\alpha^{-1}A$ and decompose $\mathbb{1}_A$ in $L^2(S^1,\lambda)$ as
$$\mathbb{1}_A = \sum_{\mathbb{Z}} c_k e_k$$
with $e_k(x) = \exp(2\pi i k x)$
Then
$$\chi_A \circ R_A = \chi_A \Leftrightarrow$$
$$\sum_{\mathbb{Z}} c_k e_k \cdot e^{2\pi i k\alpha} \underset{L^2}{=} \sum_{\mathbb{Z}} c_k e_k \Leftrightarrow$$
$$c_k(e^{2\pi i k\alpha} - 1) = 0 \quad \forall k \Leftrightarrow$$
$$c_k = 0 \quad \forall k \in \mathbb{Z}\setminus\{0\} \text{ so}$$
$\chi_A \underset{L^2}{=} c_0 \cdot \mathbb{1}_A$ and either
$A \equiv \emptyset \pmod 0$ and $c_0 = 0$ or
$A \equiv S^1 \pmod 0$ and $c_0 = 1$.

Thm $R_\alpha(x) = x+\alpha \mod \mathbb{Z}$ with $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ is uniquely ergodic: ~~We have $\forall x \in S^1$~~ on $(S^1, \mathcal{B}_{S^1})$

Let $f \in C(S^1)$. Then we have $\forall x \in S^1$: (not just a.e)
$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ R_\alpha^k \longrightarrow \int_0^1 f(t)\,dt$$
(so Leb. is the unique erg meas)

proof By Birkhoff for Leb a.e $x \in S^1$
$$\frac{1}{n}\sum f \circ R_\alpha^k(x) \longrightarrow \int_0^1 f(t)\,dt$$
Let $\epsilon > 0$ and let $\delta > 0$ be st
$d(x,y) < \delta \Rightarrow |fx - fy| < \epsilon$
Then since $d(R_\alpha^k x, R_\alpha^k y) < \delta$:
$$\left|\frac{1}{n}\sum f \circ R_\alpha^k(x) - \frac{1}{n}\sum f \circ R_\alpha^k(y)\right| < \epsilon$$

If $y \in S^1$ find $x_0 \in S^1$ with $d(y,x_0) < \delta$. Then
$$\limsup_n \left|\frac{1}{n}\sum_0^{n-1} f \circ R_\alpha^k - \int_0^1 f\,d\lambda\right| \le \epsilon$$
and $\epsilon > 0$ was arbitrary.

Now let μ be any R_α invariant Borel proba meas. Then for $f \in C(S^1)$.
Birkhoff for μ implies
$\forall \mu$ ae $x \in S^1$: $\frac{1}{n}S_n f(x) \to \int f d\mu$
but $\frac{1}{n}S_n f(x) \to \int f\,d\lambda$
so $\int f\,d\mu = \int f\,d\lambda \quad \forall f \in S^1$
and $\mu = \lambda$.

Gelfand's problem

(Needs unique ergodicity or ~~at least~~ uniform orbital density)

Sequence of powers of 2: $(2^k)_{k\geq 0}$

1 2 4 8 16 32 64 128 256 ...

First digit $(D_k)_{k\geq 0}$

1 2 4 8 1 3 6 1 2 ....

Gelfand's question: What is the asymptotic density of 7 in the sequence $(D_k)_{k\geq 0}$? $D_{46} = 7$ first occurrence

$$P_7(N) = \#\{0 \leq k < N : D_k = 7\}$$

$$\lim_{N\to\infty} \frac{P_7(N)}{N} = \underbrace{?}_{\text{answer}} = \log_{10} 8/7 = 0{,}0579919...$$

$$\log_{10} 2^k = \text{integer} + \underset{x\in[0,1)}{\text{fractional part}}$$

$$D_k = 7 \iff x \in [\log 7, \log 8)$$

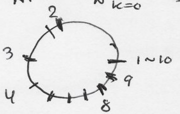
$$\log_{10} 2^{k+1} = \log_{10} 2^k + \log_{10} 2 \quad / \bmod \mathbb{Z}$$

$$(k+1)\log_{10} 2 = k \log_{10} 2 + \log_{10} 2 \quad / \bmod \mathbb{Z}$$

$$T_\alpha(x) = x + \underbrace{\log_{10} 2}_{=\alpha} \mod \mathbb{Z}$$

$$f(x) = 1\!\!1_{[\log 7, \log 8)} = 1\!\!1_{I_7}$$

$$\frac{P_7(N)}{N} = \frac{1}{N} S_N f(x) = \frac{1}{N}\sum_{k=0}^{N-1} 1\!\!1_{I_7}(T_\alpha^k x) \longrightarrow \int_{S^1} 1\!\!1_{I_7}\, dx = \log_{10} 8 - \log_{10} 7$$



Notions of mixing

Theorem $(X,\mathcal{B},\mu,T)$ a meas preserving transformation. μ probability meas.

Then μ is ergodic iff

$\forall A,B\in\mathcal{B}$: $\lim \frac{1}{n}\sum_{k=0}^{n-1}\mu(T^{-k}A\cap B) = \mu(A)\,\mu(B)$

proof: Suppose T ergodic.
Let $f = 1\!\!1_A$. By Birkhoff.

$$\frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_A\circ T^k \xrightarrow[n\to\infty]{} c = \int f = \mu(A)$$

μ-a.e. So μ-a.e:

$$\frac{1}{n}\sum 1\!\!1_A\circ T^k\cdot 1\!\!1_B \xrightarrow[n]{} \mu(A)\cdot 1\!\!1_B$$

By dominated convergence:

$$\frac{1}{n}\sum \mu(T^{-k}A\cap B) = \int \frac{1}{n}\sum 1\!\!1_A\circ T^k 1\!\!1_B\,d\mu \xrightarrow[n]{} \mu(A)\mu(B)$$

Def We say that T is

* weak-mixing iff $\forall A,B\in\mathcal{B}$
$$\frac{1}{n}\sum_{0}^{n-1}|\mu(T^{-k}A\cap B)-\mu(A)\mu(B)| \xrightarrow[n]{} 0$$
* strong-mixing iff $\forall A,B\in\mathcal{B}$
$$\mu(T^{-n}A\cap B)-\mu(A)\mu(B)\xrightarrow[n]{} 0$$

Prop Obviously

Strong-mix $\Rightarrow$ weak-mix $\Rightarrow$ ergodic

Theorem $(X,\mathcal{B},\mu,T)$ μ proba is strongly mixing iff

$\forall a,b\in L^2(X,\mu)$

$$\int_X a\circ T^n\cdot b\,d\mu \to \int_X a\,d\mu\cdot\int_X b\,d\mu$$

(asymptotic independence)

"$\Rightarrow$"

Proof: True for $a = 1\!\!1_A$ and $b = 1\!\!1_B$ whence also for finite linear combinations.

$a = \sum c_k 1\!\!1_{A_k}$ $\quad b = \sum d_k 1\!\!1_{B_k}$

Now by T invariance of μ we have

$$\left|\int (a\circ T^k\cdot b - \tilde a\circ T^k\cdot \tilde b)\,d\mu\right|$$

$$\le \int |a\circ T^k - \tilde a\circ T^k|\,|b| + |\tilde a\circ T^k|\cdot|b-\tilde b|\,d\mu$$

$$\le \|a-\tilde a\|_2\|b\|_2 + \|\tilde a\|_2\|b-\tilde b\|_2$$

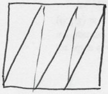
and the result follows by approx.

Ergodicity and mixing for the doubling map

More generally: $p \in \mathbb{Z}$, $|p| \geq 2$.

$T(x) = T_p(x) = p \cdot x \mod \mathbb{Z}$

T_3



$T^n(x) = p^n \cdot x \mod \mathbb{Z}$.

λ = Lebesgue measure

Suppose A, B measurable.

<u>exo</u> show that if $A = T^{-1}A$
then $\lambda(A) = 0$ or 1
using L^2-Fourier expansion

We show here:

| T is strongly mixing
| w.r.t Lebesgue measure

Let $a = \sum a_k e_k$, $b = \sum b_n e_n$
L^2-fcts.

$e_n = \exp(2\pi i k x)$

$e_k \circ T^n = e_{kp^n}$

$$\int_0^1 a \circ T^n \cdot b \, d\lambda =$$

$$\int \sum a_k e_{kp^n} \, b_\ell e_\ell \, d\lambda =$$

$$\underbrace{a_0 \cdot b_0}_{\int a \int b} + \sum_{k \neq 0}^{\infty} a_k \cdot b_{-kp^n}$$

So

$$\left| \int a \circ T^n b \, d\lambda - \int a \int b \right| \leq$$

$$\left| \sum_{k \neq 1} a_n b_{-kp^n} \right| \leq \left(\sum_{k \neq 1} |a_k|^2 \right)^{1/2} \times \underbrace{\left(\sum_{k \neq 1} |b_{-kp^n}|^2 \right)^{1/2}}_{\xrightarrow[n \to \infty]{} 0}$$

Cauchy-Schwartz