~~The~~ Ergodic Theory

Def 1.1 $(X,\mathcal{B})$ a measure space and μ a positive σ-finite measure on $(X,\mathcal{B})$.

A measurable map $T:X\to X$ is said to be a measure preserving transformation ~~provided~~ (MPT) provided

$$\forall A\in\mathcal{B}: \mu(T^{-1}A)=\mu(A) \Leftrightarrow \int \mathbb{1}_A\circ T\,d\mu = \int \mathbb{1}_A\,d\mu$$

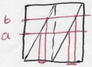
<u>Remark</u>: When $\mu(X)<+\infty$ we may normalise the measure so that $\mu(X)=1$. We shall mostly work with proba measures.

<u>Examples</u>:

1. $X=\mathbb{R}, \mathcal{B}_{\mathbb{R}}, T(x)=x+\alpha$, $\alpha\in\mathbb{R}$ preserves Lebesgue measure: $A\subset\mathcal{B}_{\mathbb{R}}$:
   $\lambda(A-\alpha)=\lambda(A)$.
   but $\lambda(\mathbb{R})=+\infty$

2. $X=\mathbb{R}/\mathbb{Z}$, λ = Lebesgue, $\alpha\in\mathbb{R}$.
   $T(x)=x+\alpha \pmod{\mathbb{Z}}$
   preserves Lebesgue and $\lambda(X)=1$

3. $X=\mathbb{R}/\mathbb{Z}$, λ = Lebesgue.
   $T(x)=\cancel{x}\,2x \pmod{\mathbb{Z}}$
   preserves Leb and $\lambda(X)=1$.



$\lambda(T^{-1}[a,b)) =$
$\lambda([\frac{a}{2},\frac{b}{2}))+\lambda([\frac{a}{2}+\frac{1}{2},\frac{b}{2}+\frac{1}{2}))$
$= 2\times(\frac{b}{2}-\frac{a}{2}) = b-a.$
$= \lambda([a,b))$

4. $(\mathcal{X}_i,\mathcal{B}_i,\mu_i)$ identical proba spaces $i\in\mathbb{N}_0$

   $\tilde{X}=\prod_i \mathcal{X}_i$, $\mathcal{B}$ generated by finite prods of meas sets (the rest being the whole space)

   $A=\prod_{i\le N} A_i \times \prod_{j>N} \mathcal{X}_j$

   $\exists$ unique proba on $\tilde{X},\mathcal{B}$ so that $\mu(A)=\prod_{i\le N}\mu_i(A_i)$

   $T:\tilde{X}\to\tilde{X}$ given by

   $T(\omega_0,\omega_1,\ldots)=(\omega_1,\omega_2,\ldots)$

   is then a MPT of μ.

   (since $\mu(T^{-1}A)=\mu(X\times A_1\times\cdots\times A_N\times X\ldots)$
   $=\mu(A_1\times\cdots\times A_N\times\ldots)$)

5. invertible version when $\mathbb{N}\to\mathbb{Z}$
   $\tilde{X}=\prod_{i\in\mathbb{Z}}\mathcal{X}$

   $T((\omega_i)_{i\in\mathbb{Z}})=(\omega_{i+1})_{i\in\mathbb{Z}}$

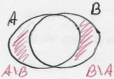
6. Chaînes de Markov avec mesure stationnaire

Null sets etc...

Def 1.2 For measurable sets A, B we define their symmetric set-difference:

$$A \triangle B = (A \cup B) \setminus (A \cap B)$$
$$= (A \setminus B) \sqcup (B \setminus A)$$
$$= (A \cap B^c) \sqcup (B \cap A^c)$$

When $\mu(A \triangle B) = 0$ we write
$A \sim B \pmod 0$ or
$A \equiv B \pmod 0$



Remark: In terms of char fct:

$$\mathbb{1}_{A \triangle B} = |\mathbb{1}_A - \mathbb{1}_B|$$

Def 1.3
Given $(X, \mathcal{B}, \mu)$ a σ-finite measure space we define a semi-metric

$$d : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_+ \cup \{+\infty\}$$

$$d(A,B) = \mu(A \triangle B) = \int_X |\mathbb{1}_A - \mathbb{1}_B| \, d\mu$$

properties:

Triangular ineq:

$$d(A,C) = \int |\mathbb{1}_A - \mathbb{1}_B + \mathbb{1}_B - \mathbb{1}_C|$$
$$\leq \int |\mathbb{1}_A - \mathbb{1}_B| + |\mathbb{1}_B - \mathbb{1}_C|$$
$$= d(A,B) + d(B,C)$$

Complement invariance

$$d(A^c, B^c) = \int |\mathbb{1}_{A^c} - \mathbb{1}_{B^c}|$$
$$= \int |(\mathbb{1} - \mathbb{1}_A) - (\mathbb{1} - \mathbb{1}_B)|$$
$$= \int |\mathbb{1}_A - \mathbb{1}_B| = d(A,B)$$

Intersections ineq

$$d\Big(\bigcap_{\mathbb{N}} A_i, \bigcap_{\mathbb{N}} B_i\Big) =$$
$$\int \Big|\prod_{\mathbb{N}} \mathbb{1}_{A_i} - \prod_{\mathbb{N}} \mathbb{1}_{B_i}\Big| \leq$$
$$\int \sum_{\mathbb{N}} |\mathbb{1}_{A_i} - \mathbb{1}_{B_i}| =$$
$$\sum_{\mathbb{N}} d(A_i, B_i)$$

union ineq

$$d\Big(\bigcup A_i, \bigcup B_i\Big) =$$
$$d\Big(\bigcap A_i^c, \bigcap B_i^c\Big) \leq$$
$$\sum d(A_i^c, B_i^c) =$$
$$\sum d(A_i, B_i)$$

When $T : X \to X$ is measure preserving

$$d(T^{-1}A, T^{-1}B) = \int |\mathbb{1}_A \circ T - \mathbb{1}_B \circ T|$$
$$= \int |\mathbb{1}_A - \mathbb{1}_B| \, d\mu$$
$$= d(A,B)$$

Prop 1.4 If $A \sim B_i \ \forall i \in \mathbb{N} \Rightarrow$

$$A \sim \bigcap_{\mathbb{N}} B_i \sim \bigcup_{\mathbb{N}} B_i$$

Proof eg $d\Big(A, \bigcap_{\mathbb{N}} B_i\Big) \leq$

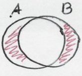
$$\sum_{\mathbb{N}} d(A, B_i) = 0$$

same for union.

$(X, \mathcal{B}, \mu)$ σ-finite measure space.

$d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}_+$ semi-metric

$$d(A,B) = \mu(A \triangle B) = \mu(A \setminus B) + \mu(B \setminus A)$$
$$= \int_X |\mathbb{1}_A - \mathbb{1}_B| \, d\mu$$



also equals $\mu(A \cap B^c) + \mu(B \cap A^c)$

The metric is T-invariant

$$d(T^{-1}A, T^{-1}B) = \int |\mathbb{1}_A \circ T - \mathbb{1}_B \circ T|$$
$$= \int |\mathbb{1}_A - \mathbb{1}_B| = d(A,B)$$

Triangular ineq:

$$d(A,C) = \int |\mathbb{1}_A - \mathbb{1}_B + \mathbb{1}_B - \mathbb{1}_C|$$
$$\leq d(A,B) + d(B,C)$$

Complement

$$d(A^c, B^c) = \int |\mathbb{1}_{A^c} - \mathbb{1}_{B^c}|$$
$$= \int |(\mathbb{1} - \mathbb{1}_A) - (\mathbb{1} - \mathbb{1}_B)|$$
$$= \int |\mathbb{1}_A - \mathbb{1}_B| = d(A,B)$$

Intersections $(A_i)_{i \in \mathbb{N}}$, $(B_i)_{i \in \mathbb{N}}$

$$d(\cap A_i, \cap B_i) = \int |\prod \mathbb{1}_{A_i} - \prod \mathbb{1}_{B_i}|$$
$$\leq \int \sum_j |\mathbb{1}_{A_j} - \mathbb{1}_{B_j}|$$
$$= \sum_j d(A_j, B_j)$$

Note that if e.g. $\prod \mathbb{1}_{A_i}(x) = 1$, $\prod \mathbb{1}_{B_i}(x) = 0$ then there is j with $\mathbb{1}_{A_j}(x) = 1$ but $\mathbb{1}_{B_j}(x) = 0$

Unions

$$d(\cup A_i, \cup B_i) = d(\cap A_i^c, \cap B_i^c)$$
$$\leq \sum d(A_j^c, B_j^c)$$
$$= \sum d(A_j, B_j)$$

Prop 1.5 $(X,\mathcal{B},\mu)$ σ-finite
$T: X \to X$ meas pres transf.

Suppose $B \in \mathcal{B}$ with

$$B \equiv T^{-1}B \pmod 0$$

Then there is $A \in \mathcal{B}$ with

$A \equiv B \pmod 0$ and $A = T^{-1}A$.

For A one may take:

$$A = \bigcap_{N \geq 0} \bigcup_{k \geq N} T^{-k}B$$

proof: We denote $A_N = \bigcup_{k \geq N} T^{-k}B$

and then $A = \bigcap_{N \geq 0} A_N$.

Since $T^{-1}A_N = A_{N+1} \subset A_N$
we have

$$T^{-1}A = \bigcap_{N \geq 0} T^{-1}A_N = \bigcap_{N \geq 0} A_{N+1} = \bigcap_{N \geq 0} A_N = A$$

so $A = T^{-1}A$ is genuinely invariant (not just mod 0).

Concerning measure

$B \sim T^{-1}B \Rightarrow T^{-1}B \sim T^{-2}B \Rightarrow \cdots$

so

$B \sim T^{-1}B \sim T^{-2}B \sim T^{-3}B \sim \cdots$

and for all N:

$B \sim A_N = \bigcup_{k \geq N} T^{-k}B$

and then

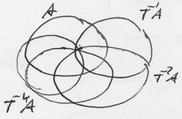
$B \sim A = \bigcap_{N \geq 0} A_N$

(by prop 1.4)

Poincaré Recurrence.

Thm 1.6
Let $T: X \to X$ be a meas pres. transf of a proba space $(X, \mathcal{B}, \mu)$.
Let $A \in \mathcal{B}$ with $\mu(A) > 0$.
Then ~~for~~ μ-a.e. $x \in A$ returns ∞ often to A, i.e. there is a sequence $n_1 < n_2 < n_3 < \dots$ such that $T^{n_i}(x) \in A$.

$\mathrm{Card}\{k \in \mathbb{N} : T^k x \in A\} = +\infty.$



proof

$\mu(\bigcup_{k \geq 0} T^{-k}A) \leq 1$

For $N \geq 0$ define the ~~dist~~ decr. sequence
$$A_N = \bigcup_{k \geq N} T^{-k}A$$
$$A_0 \supset A_1 \supset A_2 \supset \dots$$
and set
$$\Omega_\infty = \bigcap_{N \geq 0} A_N = \bigcap_{N \geq 0} A_{N+1}$$
$$= \bigcap_{N \geq 0} T^{-1}A_N = T^{-1}\Omega_\infty$$

Given $x \in \Omega_\infty$: Let $n_0 = 0$ and define recursively given n_k:

~~$n_{k+1} = \inf\{k \geq n_k : T^k x \in A\}$~~

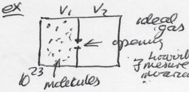
As $x \in A_{n_k+1}$ ~~then~~ the set $\{k > n_k : T^k x \in A\}$ is non-empty. and we set
$$n_{k+1} = \inf\{k > n_k : T^k x \in A\}$$
So every $x \in \Omega_\infty$ ~~not~~ enters A infinitely often.

Concerning measure
$$\infty > \mu(A_0) = \mu(T^{-1}A_0) = \mu(A_1)$$
$$= \mu(T^{-1}A_1) = \mu(A_2)$$
$$= \dots$$
so $\mu(A_k \setminus A_{k+1}) = 0 \ \forall k$.
and
$$\infty > \mu(A_0) = \mu(\bigcap_N A_N)$$
$$= \mu(\Omega_\infty)$$
Thus $\mu(A_0 \setminus \Omega_\infty) = 0$
and $= \sum_{k \geq 0} \mu(A_k \setminus A_{k+1})$

$\mu(A \cap \Omega_\infty) =$
$\mu(A \cap A_0) = \mu(A)$
since $A \subset A_0$ //

so μ-a.e $x \in A$ returns to A ∞ often

ex



all molecules in V_1 at time 0
will diffuse between V_1, V_2
but return to at some time t a the state where all mol are again in V_1

but $t \simeq$?

Return time

$(X, \mathcal{B}, \mu, T)$ measures transf
μ σ-finite measure.

In the following let

$\boxed{A \in \mathcal{B} \qquad 0 < \mu(A) < +\infty}$

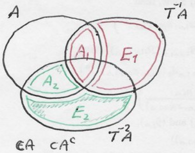
Def 1.7 For $\mu(A) \in (0, +\infty)$ we define the first entrance time to A: $\forall x \in X$

$$n_A(x) = \inf\{k \geq 1 : T^k x \in A\}$$

$$n_A : X \mapsto \mathbb{N}^* \cup \{+\infty\}$$

We have:

$$\{n_A = 1\} = T^{-1}A$$
$$\{n_A = 2\} = T^{-2}A \cap T^{-1}A^c$$
$$\{n_A = 3\} = T^{-3}A \cap T^{-2}A^c \cap T^{-1}A^c$$
$$\vdots$$



$\in A \quad \subset A^c$

$$T^{-1}A = A_1 \sqcup E_1 \quad \{n_A = 1\} \quad \mu(A) = \mu(T^{-1}A) = \mu(A_1) + \mu(E_1)$$
$$T^{-1}E_1 = A_2 \sqcup E_2 \quad \{n_A = 2\} \quad \mu(E_1) = \mu(T^{-1}E_1) = \mu(A_2) + \mu(E_2)$$
$$\vdots$$
$$T^{-1}E_{n-1} = A_n \sqcup E_n \quad \{n_A = n\} \quad \mu(E_{n-1}) = \mu(T^{-1}E_{n-1}) = \mu(A_n) + \mu(E_n)$$

$$(*) \quad \mu\Big(\bigcup_{k=0}^{n} T^{-k}A\Big) = \mu(A) + \sum_{k=1}^{n-1} \mu(E_k) = \sum_{k=1}^{n} (\mu(A_k) + \mu(E_k))$$

$$\Rightarrow \mu(A) = \Big(\sum_{k=1}^{n} \mu(A_k)\Big) + \mu(E_n)$$

$$\mu(A) \geq \mu(E_1) \geq \mu(E_2) \geq \dots$$

[Kac]

Thm 1.8 Suppose $\mu(X) = 1$, $0 < \mu(A) < 1$. Then for μ-a.e. $x \in A$: $n_A(x) < +\infty$

$$\frac{1}{\mu(A)} \int_A n_A \, d\mu = \frac{\mu(\bigcup T^{-k}A)}{\mu(A)}$$

In particular if T is μ ergodic or $\bigcup T^{-k}A \equiv X$.

not defined yet

$$\frac{1}{\mu(A)} \int n_A \, d\mu = \frac{1}{\mu(A)}$$

proof: Let $B = \bigcup_{k \geq 0} T^{-k}A$

We have
$$1 \geq \mu(B) = \mu(A) + \sum \mu(E_n)$$
showing that $\lim_n \mu(E_n) = 0$

Therefore
$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$$
implying $n_A < +\infty$ for a.e. $x \in A$

Also $\mu(E_n) = \sum_{k > n} \mu(A_k)$

$$\mu(B) = \sum_{n \geq 0} \mu(E_n)$$

$$\int_A n_A \, d\mu = \sum_{n \geq 1} n \cdot \mu(A_n) = \sum_{n \geq 1} \Big(\sum_{k \geq n} \mu(A_k)\Big) = \sum_{n \geq 1} \mu(E_{n-1}) = \mu(A) + \sum_{n \geq 1} \mu(E_n) = \mu(B).$$

T-μ ergodic $\Rightarrow \mu(B) = 1$

T-invariance.

Lemma 1.69 A measure μ on X is T-invariant iff for every $f \in \mathcal{L}^\infty(X,\mathcal{B}) \cap \mathcal{L}^1(X,\mu)$ ← not necessary if μ is proba

$$\int f\,d\mu = \int f\circ T\,d\mu. \quad (*)$$

Moreover, if μ is T-invariant then $(*)$ holds $\forall f \in \mathcal{L}^1(X,\mu)$.

"$\Leftarrow$" Let $B \in \mathcal{B}$ with $\mu(B) < \infty$. Then $\mu(B) = \int 1_B\,d\mu = \int 1_B\circ T\,d\mu = \mu(T^{-1}B)$ so μ is invariant.

"$\Rightarrow$" Suppose $\mu(B) = \mu(T^{-1}B)$ $\forall B \in \mathcal{B}$. Then $(*)$ holds for finite linear comb. of integrable indicator fcts.

When $f \in \mathcal{L}^1_+(X,\mu)$ we may find a sequence of simple fcts (f_n) increasing to f, $f_n \nearrow f$. Then also $f_n\circ T \nearrow f\circ T$ and

$\int f\,d\mu = \lim \int f_n = \lim \int f_n\circ T = \int f\circ T.$

by monotone convergence.

Remark It suffices to look at invariance for a semi-algebra $\mathcal{S} \subset \mathcal{B}$ generating $\mathcal{B}$.

For ex: $\{[a,b) : -\infty < a < b < +\infty\} = \mathcal{S}$ generates $\mathcal{B}_{\mathbb{R}}$.

Ergodicity

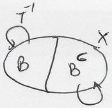
1.10

Def: $(X,\mathcal{B},\mu,T)$ a measure preserving transformation of a σ-finite measure space.

T is said to be ergodic w.r.t μ iff

$\forall B\in\mathcal{B}: \ T^{-1}B = B \Rightarrow \mu(B)=0$ or $\mu(B^c)=0$.

We say also that μ is an ergodic measure for T.



If $B = T^{-1}B$ then also $B^c = T^{-1}B^c$ so X decomposes or partitions into invariant subsystems

Thm 1.11 For a meas. pres. transf. $(X,\mathcal{B},\mu,T)$ the following is equivalent

1) T is ergodic
2) $\forall A\in\mathcal{B}: A\equiv T^{-1}A$ mod $0 \Rightarrow \mu(A)=0$ or $\mu(A^c)=0$.
3) For $f: X\to \mathbb{R}$ (or $\mathbb{C}$) measurable $f\circ T = f$ a.e $\Rightarrow f = \text{const}$ a.e.

When μ is finite (proba) the above is also equiv to

4) $\forall A\in\mathcal{B}$, $\mu(A)>0$: $\mu(\bigcup_{k\geq 0} T^{-k}A)=1$.
5) $\forall A,B\in\mathcal{B}$, $\mu(A)>0, \mu(B)>0 \Rightarrow \exists n\geq 1: \mu(T^{-n}A\cap B)>0$.

Proof: $1\Rightarrow 2$: If $A\equiv T^{-1}A$ mod 0 there exist $B\equiv A$ mod 0, $B=T^{-1}B$
$\Rightarrow \mu(B)=0$ or $\mu(B^c)=0$
$\Rightarrow \mu(A)=0$ or $\mu(A^c)=0$

$2\Rightarrow 1$: automatic

$2\Rightarrow 3$: $f = f\circ T$ a.e. (values in $\mathbb{R}$)
For every $u\in\mathbb{R}$: $A_u = \{f>u\}$ is T-invar (mod 0) so $\mu(A_u)=0$ or $\mu(A_u^c)=0$
Let $s = \inf\{u: \mu(A_u)=0\}$. Then s must be finite and
$\mu(f\neq s) \leq \sum_{n\geq 1} \mu(|f-s| \geq \frac{1}{n}) = 0$ so $f = s$ a.e.

$3\Rightarrow 2$: $A\equiv T^{-1}A$ mod 0 $\Rightarrow$
$1\!\!1_A\circ T = 1\!\!1_A$ a.e $\Rightarrow$
$1\!\!1_A = \text{const} = 1$ ou 0 a.e.
so $\mu(A)=0$ or $\mu(A^c)=0$.

When μ is a proba. meas:

$2\Rightarrow 4$: $B = \bigcup_{k\geq 1} T^{-k}A \supseteq T^{-1}B$
As $\mu(B) = \mu(T^{-1}B) < \infty$ we must have $B\equiv T^{-1}B$ mod 0 $\Rightarrow$
$\mu(B)=0$ or $\mu(B^c)=0$.
Here $\mu(B)>0$ so $\mu(B^c)=0$

$4\Rightarrow 5$ We have $\mu(\bigcup T^{-k}A)=1$ and then
$0<\mu(B) = \mu(\bigcup_{k\geq 1} B\cap T^{-k}A)$
$\leq \sum_{k\geq 1}\mu(B\cap T^{-k}A)$ so
at least one has meas >0.

$5\Rightarrow 2$ $A\equiv T^{-1}A$ mod 0 $\Rightarrow$
$A\equiv T^{-k}A$ mod 0
Take $B = A^c \overset{(5)}{\Rightarrow} \mu(B\cap A)=0$
so $\mu(A)=0$ or $\mu(B)=0$
or else 5) would fail.

Circle rotations

$S^1 = \mathbb{R}/\mathbb{Z}$., $\alpha \in \mathbb{R}$,

$T := T_\alpha : x \mapsto x+\alpha \pmod{\mathbb{Z}}$

Thm 1.12 [Jacobi] The orbits of T are dense iff α is irrational.

proof: If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then

$T_\alpha^q(x) = x + q\alpha = x + p \equiv x \mod \mathbb{Z}$

so the orbit contains (at most) q points of any x.

Suppose $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then the orbit of any $x \in S^1$ must ~~contain~~ consist of distinct points: If not for $n \neq m$:

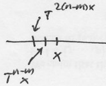
$T^n x \equiv T^m x \mod \mathbb{Z} \Rightarrow$

$n\alpha = m\alpha + p \quad p \in \mathbb{Z} \Rightarrow$

$\alpha = \frac{p}{n-m} \in \mathbb{Q}$.

Let $x \in S^1$.
Given $\varepsilon > 0$ there are n, m so that $0 < d(T^n x, T^m x) < \varepsilon$

Then also $0 < d(T^{n-m}x, x) < \varepsilon$.



And $T^{k(n-m)}x$ will be ε-dense in S^1.

Thm 1.13 Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then Lebesgue measure is ergodic for T_α.

Proof: Suppose $B \in \mathcal{B}_{S^1}$ is T_α-invariant. Then $1_B = 1_B \circ T_\alpha$ a.e. so as elements of

$H = L^2(S^1, \mathcal{B}_{S^1}, \lambda)$

the elements are identical.

Now

$\mathcal{U}f = f \circ T$

is a unitary operator on H: $\forall f \in H$:

$\int_{S^1} |f \circ T|^2 d\lambda = \int |f|^2 d\lambda$

(first show this on $C^0(S^1)$ then use approx.)

By Fourier expansion

$1_B(x) = \sum_{n \in \mathbb{Z}} c_n e_n(x)$ (in H)

with

$e_n(x) = \exp(2\pi i n x)$

and $1_B \circ T_\alpha = \sum c_n e^{2\pi i n \alpha} e_n(x)$

$c_n = \langle e_n, 1_B \rangle$

$= \langle e_n \circ T, 1_B \circ T \rangle$ unitarity of $\mathcal{U}$

$= \langle e_n \circ T, 1_B \rangle$ hypoth on B

$= e^{2\pi i n\alpha} \langle e_n, 1_B \rangle$

$= e^{2\pi i n\alpha} c_n$

$\forall n \ \ c_n(e^{2\pi i n\alpha} - 1) = 0$

$\alpha \notin \mathbb{Q} \Rightarrow c_n \equiv 0 \quad \forall n \neq 0$.

so $1_B = c_0 \cdot 1$ a.e.

so either $c_0 = 0$, $\mu(B) = 0$

or $c_0 = 1$, $\mu(B) = 1$ and $\mu(B^c) = 0$

[von Neumann] Thm 1.14 ~~Theo Lemma~~ Thm in Hilbert space

(Mean Ergodic ~~Theorem~~)

H Hilbert space, $\mathcal{U}\in L(H)$:

$\forall f\in H: \quad |\mathcal{U}f| \le |f|$.

Let $Z = \ker(1-\mathcal{U})$ and let $P: H\to Z$ be the orthogonal projection. Then

$$\forall f\in H: \quad \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{U}^k f \xrightarrow[n\to\infty]{} Pf.$$

Proof: $\|\mathcal{U}^*\| = \|\mathcal{U}\| \le 1$:

$|\mathcal{U}^*f|^2 = \langle \mathcal{U}^*f, \mathcal{U}^*f\rangle = \langle \mathcal{U}\mathcal{U}^*f, f\rangle \le \|\mathcal{U}\|\,|\mathcal{U}^*f|\,|f|$ or

$|\mathcal{U}^*f| \le |f|$ so $\|\mathcal{U}^*\| \le \|\mathcal{U}\| \le 1$

(by symmetry $\|\mathcal{U}\| \le \|\mathcal{U}^*\|$)

$\ker(1-\mathcal{U}^*) = \ker(1-\mathcal{U}) = Z$:

Let $h = \mathcal{U}h$. Then

$$|\mathcal{U}^*h - h|^2 = \langle \mathcal{U}^*h, \mathcal{U}^*h\rangle + |h|^2 - 2\langle \mathcal{U}^*h, h\rangle \le 2|h|^2 - 2\langle h, \mathcal{U}h\rangle = 0$$

so $h\in\ker(1-\mathcal{U}^*)$

By symmetry we have $=$.

$Z = \ker(1-\mathcal{U}^*) = \mathrm{im}(1-\mathcal{U})^{\perp}$:

$\forall y\in H: \quad \langle (1-\mathcal{U})y, h\rangle = 0 \Leftrightarrow$

$\forall y\in H: \quad \langle y, (1-\mathcal{U}^*)h\rangle = 0 \Leftrightarrow$

$(1-\mathcal{U}^*)h = 0$

$$\boxed{\text{Conclusion} \quad H = \ker(1-\mathcal{U}) \oplus \overline{\mathrm{im}(1-\mathcal{U})}}$$

If $f = (1-\mathcal{U})y$ ~~then~~, $y\in H$ then

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{U}^k f = \frac{1}{n}(f - \mathcal{U}^n f) \xrightarrow[n]{} 0 \text{ in } H.$$

For $f\in\overline{\mathrm{im}(1-\mathcal{U})}$ let $f_\delta \in \mathrm{im}(1-\mathcal{U})$ and $f_\delta \to f$. Then if $|f_\delta - f| < \varepsilon$

$$\left|\frac{1}{n}\sum_0^{n-1}\mathcal{U}^k f\right| \le |f - f_\delta| + \underbrace{\frac{1}{n}\left|\sum \mathcal{U}^k f_\delta\right|}_{\to 0}$$

so $\limsup \left|\frac{1}{n}\sum_0^{n-1}\mathcal{U}^k f\right| \le \varepsilon$ (arbitrary)

Clearly if $\mathcal{U}h = h$ then

$\frac{1}{n}\sum_0^{n-1}\mathcal{U}^k h = h$ so.

Given any $f\in H$

$$\frac{1}{n}\sum \mathcal{U}^k f = \cancel{\tfrac{1}{n}}Pf + \frac{1}{n}\sum \mathcal{U}^k(1-P)f \xrightarrow[n\to\infty]{} Pf$$

//

Thm 1.15 Mean ergodic thm

$(X,\mathcal{B},\mu)$ σ-finite measure space, $T: X\to X$ measure preserving and $f\in L^2(\mu)$. Then $\exists!\,\bar f\in L^2(\mu)$ with $\bar f = \bar f\circ T$ a.e. and

$$\lim_n \left\|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k - \bar f\right\|_2 = 0$$

proof

$\mathcal{U}f = f\circ T$

is a unitary operator:

$\int |\mathcal{U}f|^2 d\mu = \int |f\circ T|^2 d\mu = \int |f|^2 d\mu$

by T-invariance of μ.

Then in L^2:

$$\frac{1}{n}\sum_0^{n-1} f\circ T^k \xrightarrow[n]{} Pf = \bar f$$

where $\mathcal{U}\bar f = \bar f = \bar f\circ T$ a.e.

Let $\mu(X) = 1$:

Corollary $(X,\mathcal{B},\mu,T)$ is ergodic iff 1 is a simple eigenvalue of $\mathcal{U}f = f\circ T$ on L^2. and $\bar f = 1$ a.e is