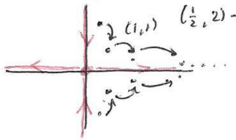
5 Hyperbolic Dynamics

M a d-dim Riem. mfld.
$f \in \mathrm{Diff}^1(M)$.

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$ Eucl. metric
$f(x,y) = (2x, y/2)$

$(\frac{1}{2}, 2) \to (1,1) \to (2, \frac{1}{2}) \to (4, \frac{1}{4})$



Only $(0,0)$ has a bounded forward and backward orbit.

stable mfld of $(0,0)$

$\forall\, \xi \in \{0\} \times \mathbb{R}: f^n(\xi) \xrightarrow[n \to +\infty]{} (0,0)$ $W^s(0,0)$

$\forall\, \xi \in \mathbb{R} \times \{0\}: f^n(\xi) \xrightarrow[n \to -\infty]{} (0,0)$ $W^u(0,0)$

unstable mfld of $(0,0)$

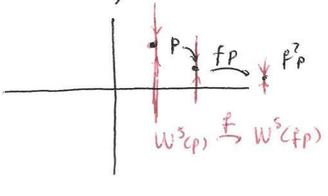
More generally:
Let $p \in M$ be any point.

$W^s(p) = \{ \xi \in M : d(f^n(\xi), \cancel{f^n(p)}) \xrightarrow[n \to +\infty]{} 0 \}$
is the stable "manifold" of p.

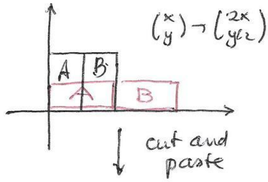
$W^u(p) = \{ \xi \in M : d(f^n \xi, f^n p) \xrightarrow[n \to -\infty]{} 0 \}$
the unstable "manifold" of p.

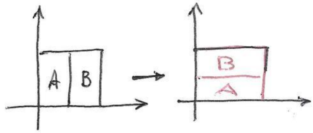
$W^s_{\epsilon}(p) = \{ \xi \in M : d(f^n \xi, f^n p) \le \epsilon \ \forall n \ge 0$
$d(f^n \xi, f^n p) \xrightarrow[n \to +\infty]{} 0 \}$

ϵ-local stable "manifold" of p.
similarly for $W^u_\epsilon(p)$



Ex: More interesting dynamics

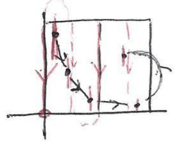




$f: [0,1) \times [0,1) \circlearrowleft$

$$f(x,y) = \begin{cases} (2x, y/2) & 0 \le x < \frac{1}{2} \\ (2x-1, \frac{y}{2} + \frac{1}{2}) & \frac{1}{2} \le x < 1 \end{cases}$$

but discontinuities



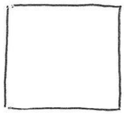
$\exists\ (\mathbb{T}^2, f) \longrightarrow (\Sigma_2, \sigma)$

not good: but not surjective: no orbits in $\Sigma_2(\{A,B\})$ that ends or begin with $\overline{B} = \ldots BBBB\ldots$

$\exists\ (\Sigma_2, \sigma) \longrightarrow (\mathbb{T}^2, f)$

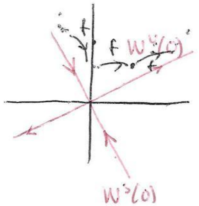
ok: ~~surjective but not continuous~~ ~~ate~~ not injective but cont.

Ex preserving continuity $\mathrm{Diff}^1(\mathbb{T}^2)$



$$f\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

mod $\underline{1}$



$$1+\gamma=\gamma^2$$

$$\gamma=\frac{1+\sqrt5}{2}=1,68\ldots>1$$

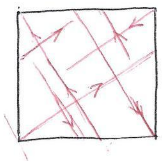
eigenvalues of $M=\begin{pmatrix}1&1\\1&0\end{pmatrix}$

$\lambda_1=\gamma$ and $-1/\gamma=\lambda_2$

$v_1=\begin{pmatrix}\gamma\\1\end{pmatrix}$ $\quad v_2=\begin{pmatrix}-1\\\gamma\end{pmatrix}$

$f:\mathbb{Z}^2\longrightarrow\mathbb{Z}^2$ so acts on the quotient as a cont map (Diff^ω)

$f:\mathbb{R}^2/\mathbb{Z}^2=\mathbb{T}^2\longrightarrow\mathbb{T}^2$



γ irrational

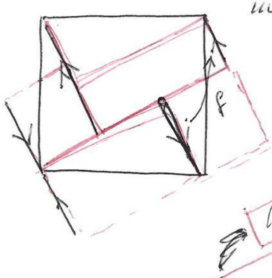
$\Rightarrow$

$W^s(0)$ and $W^u(0)$ are dense in $\mathbb{T}^2$

~~The~~ Creating a Markov partition: Use $W^{s/u}(0)$ to cut $\mathbb{T}^2$ into pieces

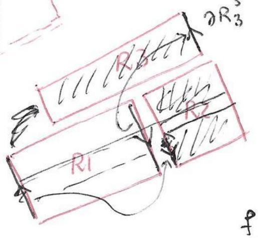
$(\Sigma_A,\sigma)\xrightarrow{\pi}(\mathbb{T}^2,f)$

injective except on $\pi^{-1}W^s(0)\cup\pi^{-1}W^u(0)$



$f\,W^s(0)=W^s(0)$

$f^{-1}W^u(0)=W^u(0)$



$R1\to R2\cup R3$

$R2\to R1$

$R3\to R1\cup R2$

$$\begin{array}{c|c} & \begin{matrix}1&2&3\end{matrix}\\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}0&1&1\\1&0&0\\1&1&0\end{pmatrix}\end{array}=:A$$

$f(\cup\partial R_i^s)\subset\cup\partial R_i^s$

$f^{-1}(\cup\partial R_i^u)\subset\cup\partial R_i^u$

§ Hyperbolic maps between rectangles

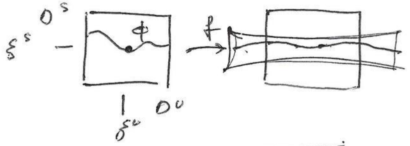
E Banach space which may be written as a direct product

$E = E^u \times E^s$, $|(x,y)| = |x|_{E^u} \vee |y|_{E^s}$ equiv to $\|\xi\|_E$

A Product-ball

$$B(\xi,R) = B_{E^u}(\xi^u,R) \times B_{E^s}(\xi^s,R) = D^u \times D^s$$

is called a rectangle centered at $\xi = (\xi^u, \xi^s)$.



Def 1 Fix $0<\theta<1$, $0<\alpha<\frac{1-\theta}{4}$, $R>0$.

$\mathcal{F}$: All C^{1} maps $f: B_R(\xi_0) \to E$ such that for $\xi = \xi_0 + (x,y)$; $\xi_1 = f(\xi_0)$

$$f(\xi) = \xi_1 + \begin{pmatrix} \Lambda_1 x \\ \Lambda_2 y \end{pmatrix} + \begin{pmatrix} \delta f_1(x,y) \\ \delta f_2(x,y) \end{pmatrix}$$

with $\Lambda_1 \in GL(E^u)$ $\|\Lambda_1^{-1}\| \le \theta$

$\Lambda_2 \in L(E^s)$ $\|\Lambda_2\| \le \theta$

$\sup_{B(\xi_0,R)} \|\delta f_i'\| \le \alpha$, $\delta f_i(0,0) = 0$

$\mathcal{F}(\xi_0, \xi_1)$ collection of such maps.

Def 2 For $L \le 1$, $\xi_0 \in E$, $B(\xi_0, R) = D^u \times D^s$ we define

$$K_L^u(\xi_0) = \{\phi : D^u \to D^s,\ Lip(\phi) \le L,\ \phi(\xi_0^u) = \xi_0^s\}$$

we denote

$$\mathcal{G}(\phi) = \{(x, \phi(x)) : x \in D^u\}$$

the graph of $\phi \in K_L^u(\xi_0)$

Prop 3: Let $f \in \mathcal{F}(\xi_0, \xi_1)$, $L \le 1$

$\forall \phi_0 \in K_L^u(\xi_0)$

$$f(\mathcal{G}(\phi_0)) \cap B(\xi_1, R)$$

is the graph of an element

$$\phi_1 = \Gamma_{\xi_0}(\phi_0) \in K_L^u(\xi_1)$$

The map

$$\phi_0 \mapsto \Gamma(\phi_0)$$

is a Lipschitz contraction
