Thm 4 Let $f_j \in \mathcal{F}^k(\xi_j, \xi_{j+1})$, $j \in \mathbb{Z}$ be a sequence of hyperbolic maps between rectangles. Γ_j the associated graph transfm

Then $\exists !\ \phi_j^* \in K^u_{L=1}(\xi_j)$, $j \in \mathbb{Z}$ such that

$$\phi^*_{j+1} = \Gamma_j(\phi_j^*).$$

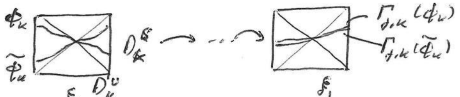
In addition each

$$\phi_j^* : D_j^u \longrightarrow D_j^s$$

is of class C^k. $\mathcal{G}(\phi_j^*)$ is called the local unstable manifold at ξ_j

Proof: For fixed $j \in \mathbb{Z}$ consider $k < j$ and pick

$$\phi_k \in K^u_{L=1}(\xi_k)$$



The map $\Gamma_{j,k} = \Gamma_{k-1} \circ \dots \circ \Gamma_j : K^u_{L=1}(\xi_k) \to K^u_{L=1}(\xi_j)$ is a uniform Lipschitz contraction so picking any initial sequence (ϕ_k), $k \in \mathbb{Z}$ $\phi_k \in K^u_{L=1}(\xi_k)$

$$\phi_j^* = \lim_{k \to -\infty} \Gamma_{j,k}(\phi_k) \in C^0(D^u_j; D^s_j)$$

converges uniformly (seq is Cauchy) (indep of choice of (ϕ_k)) and $\Gamma_j(\phi_j^*) = \phi^*_{j+1}$

When each $\Gamma_{j,k}(\phi_k)$ has Lipschitz constant $\leq L = 1$ then so does the limit. Thus $\phi_j^* \in K^u_{L=1}(\xi_j)$

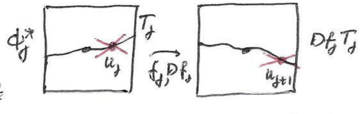
Remark: $Df_j : T\mathcal{G}(\phi_j) \to T\mathcal{G}(\phi_{j+1})$ is expanding. The inverse is contracting: For $k<j$:

$$\| Df_k^{-1} \circ \dots \circ Df_{j-1}^{-1} |_{T\mathcal{G}(\phi_j)} \| \leq C\theta^{j-k}$$

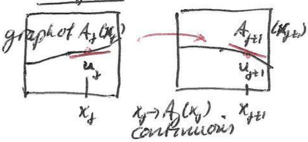
Sketch of C^k-proof:

C^1:

Consider a collection of tangent planes $T_j(u_j)$ at each $\mathcal{G}(\phi_j^*)$



1. Show that Df_j is a contraction on tangent planes.
2. The fixed pts yield a collection of invariant tangent planes (morally tangent to $\mathcal{G}(\phi_j^*)$) 1-jet $\simeq$ Taylor order 1



3. Write

$$\phi_j(x_j + z_j) = \phi_j(x_j) + A(x_j) \cdot z \quad (+\ \underline{\epsilon_j(z)})\ \text{rest}$$

a priori only Lipschitz

4. Iterate by graph transf inverse

$$(\Gamma\phi_j)_{j-1}(x_j + z) = \phi_j(x_j) + A(x_j) \cdot z \quad (+\ O_j(z) + \theta\epsilon_{j-1}(\theta z_{j-1}))$$

near rest

$$(\Gamma^{j-k}\phi_k)(x_j + z) = \phi_j(x_j) + A(x_j) z + \underset{\text{unif in } k}{O_j(z)} + \underset{\overrightarrow{\text{unif}}\ 0}{\theta^{j-k}\epsilon_k(\theta^{j-k} z_{j-k})}$$

$\Rightarrow$ Fixed point is C^1.

5. Repeat with higher order Taylor (k-jets) induction in k. //

Anosov maps.

M compact Riem mfld dim $d \geq 2$.
charts in $\mathbb{R}^d = \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$ (splitting)
with product norm
: $|(x,y)| = \max\{|x|_{\mathbb{R}^{d_u}}, |y|_{\mathbb{R}^{d_s}}\}$.
Let $R>0$
We consider an atlas of M consisting of C^k charts $\{(U_p, \pi_p)\}_{p\in M}$ where for each $p \in M$:

$$\pi_p : U_p \to B(0,R) = D^u_p \times D^s_p$$

and when $\pi_p U_p \cap \pi_q U_q \neq \emptyset$

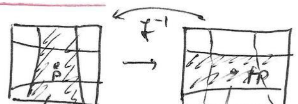
$$|\pi_p \circ \pi_q^{-1}(z) - (\pi_p(q) \ldots)| \leq C\,d(p,q))$$

converges unif to the identity map as $p \to q$

Def 5 A map $f \in \mathrm{Diff}^k(M)$ is said to be uniformly hyperbolic with unstable dim d_u, stable dim d_s
if M admits a product atlas as above and the induced map

$$\tilde{f}_p : \pi_p(U_p) \to \pi_{fp}(U_{fp}) \subset \mathbb{R}^d$$

is hyperbolic between $D^u_p \times D^s_p$ and $D^u_{fp} \times D^s_{fp}$ | Def 1



$\pi_p(U_p \cap f^{-1}U_{fp})$ $\quad$ $\pi_{fp}(fU_p \cap U_{fp})$

and that

$$f^{-1} : \pi_{fp}(fU_p \cap U_{fp}) \to \pi_p(U_p \cap f^{-1}U_{fp})$$

is hyperbolic between

$$D^u_{fp} \times D^s_{fp} \text{ and } D^u_p \times D^s_p$$

with u,s interchanging roles (s/u)

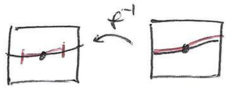
Let $p \in M$ and $\{f^j p\}_{j\in\mathbb{Z}}$ the orbit of p.
By Thm 4 there is a unique family of graphs of $\phi^u_j : D^u_j \to D^s_j$ of class C^k s.t.:

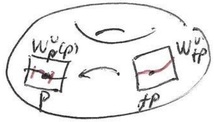
$$\tilde{f}^{-1}_{f^{j+1}p} : \mathcal{G}(\phi^u_{j+1}) \to \mathcal{G}(\phi^u_j)$$

and is a contraction

$$W^u_{f^jp} = \pi^{-1}_{f^jp}(\mathcal{G}(\phi^u_j))$$

is a local unstable mfld at f^jp





$$f^{-1} : W^u_{(f^{j+1}p)} \to W^u_{(f^jp)}$$

Similarly $\exists$ a unique family $\phi^s_j : D^s_j \to D^u_j$ class C^k

$$W^s_{(f^jp)} = \pi^{-1}_{f^jp}(\mathcal{G}(\phi^s_j))$$

s.t. $\forall j \in \mathbb{Z}$

$$f : W^s(f^jp) \to W^s(f^{j+1}p)$$

is a contraction

When f is unif. hyper there exists s/u C^k manifolds

Thm 6 The s/u-manifolds depend Hölder continuously upon $p \in M$.

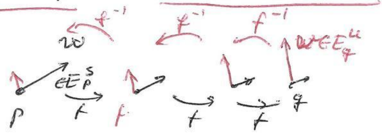
---

proof: (mostly admitted)
Given local unstable mfld $W^u(p)$, we may extend it by looking at $f^k W^u(f^{-k}p)$
similarly for s-mfld

Def 7: $f \in \mathrm{Diff}^u(M)$ is an Anosov map if there is a continuous splitting of the tangent bundle $TM = E^u \oplus E^s$ s.t. $\exists$ consts $C < \infty$, $\theta < 1$:

$\forall v \in E_p^u$: $\|Df_p^{-n} v\| \leq C\theta^n |v|$, $n \geq 0$

$\forall w \in E_p^s$: $\|Df^n w\| \leq C\theta^n |w|$, $n \geq 0$



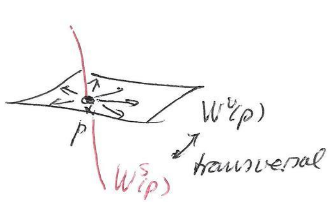
Thm 8: $f \in \mathrm{Diff}^u(M)$ is Anosov iff it ~~attracts~~ is ~~a~~ uniformly hyperbolic.

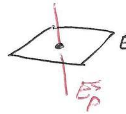
"$\Leftarrow$" $E_p^u = TW^u(p)$ tangent space of the unst. mfld.

$E_p^s = TW^s(p)$ — " — st. mfld.

contraction along W^u and W^s yields the Anosov property

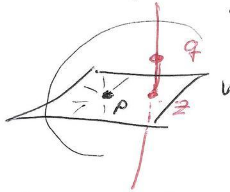
"$\Rightarrow$" Use $E_p^u \times E_p^s$ to construct suitable rectangles (complication: one needs adapting the metric or the proofs of thm 3 & 4)





$E_p^u \oplus E_p^s = \mathbb{R}^d$

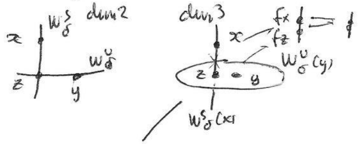
Given nearby point p and q $d(p,q) < \delta$ small enough



there exists a unique point $z = [q,p]$ s.t.

$\{z\} = W_\delta^s(q) \cap W_\delta^u(p)$

For $x, y \in M$ close enough there is a unique intersection between
$W^s_\delta(x)$ and $W^u_\delta(y)$



dynamically:

$\forall n \geq 0$: $d(f^n x, f^n z) \leq \delta$ and $\xrightarrow[n\to\infty]{} 0$

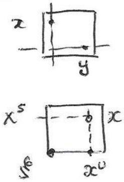
$d(f^{-n} y, f^{-n} z) \leq \delta$ and $\xrightarrow[n\to\infty]{} 0$

We write $z = [x, y]_\delta$ for the intersection point

Def 9 (Given $\delta > 0$) we call $R \subset M$ a hyperbolic rectangle (size $< \delta$) iff $R = \mathrm{Cl}\,\mathrm{Int} R$ and $\mathrm{diam} R < \delta$ ~~and~~ and

$\forall x, y \in R$: $[x, y]_\delta \in R$

We write $W^s_R(x) = W^s_\delta(x) \cap R$, $W^u_R(y) = W^u_\delta(y) \cap R$ ← closed in $W^u_\delta(y)$



Fix $\xi \in R$. Then $\forall x \in R$:

$$\left.\begin{aligned} x^u &= [x, \xi]_\delta \in W^u_R(\xi) \\ x^s &= [\xi, x]_\delta \in W^s_R(\xi) \end{aligned}\right\} (*)$$

$x \in R \longmapsto (x^u, x^s) \in W^u_R(\xi) \times W^s_R(\xi)$

Conversely with x^u, x^s as in $(*)$

$[x^u, x^s]_\delta = x \in R$

$\mathrm{Cl}\,\mathrm{Int}\, W^u_R(\xi) = W^u_R(\xi)$ (topol in $W^u_\delta(\xi)$)

$\mathrm{Cl}\,\mathrm{Int}\, W^s_R(\xi) = W^s_R(\xi)$

Product structure

Prop 10 Given $\xi \in R$: $R \simeq W^u_R(\xi) \times W^s_R(\xi)$

A collection of hyp rectangles $\{R_1, \dots, R_d\}$ is said to be Markov iff 1) $\mathring{R}_i \cap \mathring{R}_j = \emptyset$ $\forall i \neq j$, $\bigcup R_i = M$

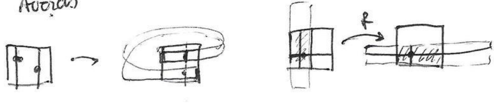
and 2) $\forall i, j \in \{1, \dots, d\}$ either $\mathring{R}_i \cap f^{-1}\mathring{R}_j = \emptyset$ ($A_{ij} = 0$)

or $\mathring{R}_i \cap f^{-1}\mathring{R}_j \neq \emptyset$ (set $A_{ij} = 1$) and in that case:

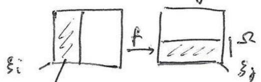
$\forall x \in R_i, y \in fx \in R_j$: $f W^u_{R_i}(x) \cap R_j = W^u_{R_j}(y)$

$f^{-1} W^s_{R_j}(y) \cap R_i = W^s_{R_i}(x)$

Avoids



R_{i_0} $\quad R_{j_1}$ $\quad A_{ij}=1$



$R_i \cap f^{-1}R_j \cong \Omega^{*}_{ij} \times W^s_{R_i}(\xi_i)$

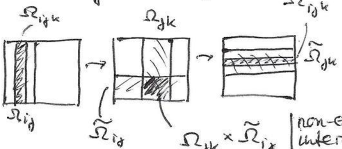
$\Omega_{ij} \subset W^u_{R_i}(\xi_i)$

$fR_i \cap R_j \simeq W^u_{R_j}(\xi_j) \times \tilde{\Omega}_{ij}$

$\tilde{\Omega}_{ij} \subset W^s_{R_j}(\xi_j)$

Composing hyperb. rect maps

$A_{ij}=1 \quad A_{jk}=1$



$\Omega_{jk} \times \tilde{\Omega}_{ij}$ | non-empty interior $=\mathrm{Cl}\,\mathrm{Int}$ |

$f^2 : R_i \cap f^{-1}R_j \cap f^{-2}R_k \longrightarrow f^2R_i \cap fR_j \cap R_k$

$\simeq \Omega_{ijk} \times W^s_{R_i}(\xi_i) \qquad \simeq W^u_{R_k}(\xi_k) \times \tilde{\Omega}_{ijk}$

$\to \Omega_{jk} \times \tilde{\Omega}_{ij} \to$

$W^u_{R_j}(\xi_k) \times \tilde{\Omega}_{ijk}$

Let $Z^{m,n}_A \subset \Sigma_A$ consisting of $[m,n]$-cylinders: $m \le n$

$[i_m, \dots i_n]_{m,n}$ (i_m: m'th coor)

subsets of M

Def 11 $\pi : Z^{m,n}_A \longrightarrow \mathcal{P}(M)$

$$\pi([i_m,\dots,i_n]_{m,n}) = \underbrace{f^m R_{i_m} \cap \dots \cap f^n R_{i_n}}_{R^{(m,n)}_{i_m \dots i_n}}$$

Prop 12 Each $R = R^{(m,n)}_{i_m \dots i_n} = \pi([i_m,\dots,i_n]_{m,n})$ is a hyperbolic rect ($R = \mathrm{Cl}\,\mathrm{Int}\,R$)

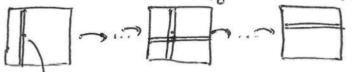
Every $\mathring{\pi}(Z) \cap \mathring{\pi}(Z') = \emptyset$ for $Z \ne Z'$

and $\bigcup_{Z \in Z_{m,n}} \pi(Z) = M$

Furthermore: $f \circ \pi(Z) = \pi \circ \underset{\text{shift}}{\sigma}(Z)$

Prop 13 $\sup_{Z \in Z_{m,n}} \operatorname{diam} \pi(Z) \longrightarrow 0$ $\begin{pmatrix} m \to -\infty \\ n \to +\infty \end{pmatrix}$

$R_{i_{m<0}} \qquad R_{i_0} \qquad R_{i_{n \ge 0}}$



$W^s_{R_{i_m}}(\xi)$ $\quad |f^m W^s_{R_{i_m}}(\xi)| \le C\theta^{|m|}$

$|f^{-n} W^u_{R_{i_n}}(\xi_n)| \le C_0 \theta^n$

so

$\operatorname{diam} \pi Z \le C \max(\theta^{|m|}, \theta^n) \to 0$

Prop 14

There is a unique continuous surjective map

$\pi : \Sigma_A \longrightarrow M$

Given by

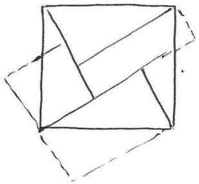
$$\pi(\xi) = \bigcap_{\substack{m \le 0 \\ n \ge 0}} \pi([\xi_m, \dots \xi_n]_{m,n})$$

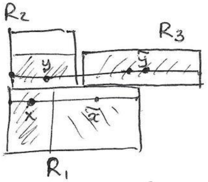
decr. intersection of non-empty compact.

One has

$$f\pi(\xi) = \pi(\sigma(\xi))$$

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$





$\mathring{R}_1 \cap f^{-1}\mathring{R}_2 = \mathring{R}_{12} \neq \emptyset$

$\mathring{R}_1 \cap f^{-1}\mathring{R}_3$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ mod $\mathbb{Z}\times\mathbb{Z}$

Fixed pt

$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Leftrightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ [$\mathbb{Z}$]

$\Leftrightarrow x = y = 0$ [$\mathbb{Z}$]

corresponds to the
two cycle : $1 \to 2 \to 1 \to \dots$

Prop 15 : Let $X = f^p(X)$ $p \geq 2$
be a primitive p-cycle.
Then $\exists!\, \xi \in \Sigma_A$: $\pi(\xi) = X$

$x \in R_i$ say. Then if $x \in \partial R_i$
then $x \in W^u(0)$ or $x \in W^s(0)$
so $f^n x \to 0$ for $n \to +\infty$ or $-\infty$
impossible. So $x \in \mathring{R}_i$
similarly for every k: $f^k x \in \mathring{R}_{i_k}$

Gibbs measures on (Σ_A, σ)

$\theta < 1$. $d(\xi, \eta) = \sup \{ \theta^{|n|} : \xi_n \neq \eta_n \}$

$Z = [\xi_m, \dots, \xi_n] = \{ \eta \in \Sigma_A : \eta_m = \xi_m, \dots, \eta_n = \xi_n \}$

$\mathcal{Z}_{m,n}$ collection of (m,n)-cylinders

A ~~topol. mixing~~ mixing

Thm 16 Let $g \in Lip(\Sigma_A, d)$. Then $\exists ! P(g) \in \mathbb{R}$, $\mu_g \in M_+^1(\Sigma_A, \sigma)$ which is shift invariant st: $\exists C < \infty$ mixing, erg.

$\forall Z \in \mathcal{Z}_{m,n}$, $m \le 0 \le n$; $x \in Z$:

$$\mu_g(Z) \overset{\cdot}{\underset{C}{=}} e^{-nP(g)} S_m^n g(x)$$

where $S_m^n g(x) = \sum_{k=m}^{n} g \circ \sigma^k(x)$

$g \overset{Lip}{\in} \Sigma_A \rightsquigarrow g^+ \overset{Lip}{\in} \Sigma_A^+$

$\rightsquigarrow \mu_{g^+} \in M_+^1(\Sigma_A^+, \sigma^+)$ (Ruelle) mixing

$\rightsquigarrow \mu_g \in M_+^1(\Sigma_A, \sigma)$

Thm 17 [Sinai-Bowen-Ruelle] Let (M, f) be an ~~transitive~~ topol. mixing Anosov map of a compact Riem mfld M with volume element $d\lambda$. Then

$$f_*^n d\lambda \xrightarrow[n \to +\infty]{} \mu_{SRB}$$

where μ_{SRB} is a mixing ($\Rightarrow$ erg) f-invar proba measure.
One has for $\phi \in C(M)$

For λ-a.e. $x \in M$:

$$\frac{1}{n} S_0^n \phi(x) \xrightarrow[n \to +\infty]{} \int_M \phi \, d\mu_{SRB}$$

$g = -\log |\det Df_{|E^u}|$

$\hat{g} \in Lip(\Sigma_A) \rightsquigarrow$

$\mu_{\hat{g}} \in M_+^1(\Sigma_A, \sigma) \rightsquigarrow$

$\mu_g \in M_+^1(M, f)$ has the desired property.

Recall for an exp. map in dim 1:

$$\mathcal{L}\phi(y) = \sum_{x: f(x) = y} \frac{1}{|f'(x)|} \phi(x)$$

"proof": $\exists (\Sigma_A, \sigma)$ A mixing and a surjective map $\pi : \Sigma_A \to M$, $f \circ \pi = \pi \circ \sigma$