TD 11 : Hyperbolic dynamics, the Poincare disk and geodesic flow.

1. Define the rectangles: $R_A = [0, 1/3] \times [0, 1]$ and $R_B = [2/3, 1] \times [0, 1]$. Define the map from $R_A \cup R_B$ to the unit square:

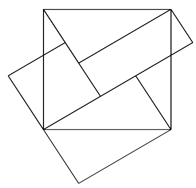
$$f(x_1, x_2) = \begin{cases} (3x_1, \frac{x_2}{3}) & , \quad x \in R_A \\ (3x_1 - 2, \frac{2 + x_2}{3}) & , \quad x \in R_B \end{cases} \tag{0.1}$$

We let $\Omega \in [0, 1] \times [0, 1]$ denote the maximal invariant set of f. One has $\Omega = f\Omega = f^{-1}\Omega$.

   (a) Show that R_A and R_B may be described as products of pieces of unstable and stable manifolds (or rather subsets of such). $\{R_A, R_B\}$ is called a Markov partition for Ω.

   (b) Show that there is a bijective map $j : \Omega \to \Sigma_2$ which conjugates (Ω, f) to the full shift (Σ_2, σ).

2. Let $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a map of the 2-torus $\mathbb{T}^2$ and $\gamma = \frac{1+\sqrt{5}}{2}$. Setting $e^u = \begin{pmatrix} 1 \\ 1/\gamma \end{pmatrix}$, $e^s = \begin{pmatrix} -1/\gamma \\ 1 \end{pmatrix}$ one has: $f(e^u) = \gamma e^u$, $f(e^s) = -1/\gamma e^s$.



   (a) Describe 3 closed rectangles $R_1, R_2, R_3 \subset \mathbb{T}^2$ (see figure) bounded by pieces of u/s-manifolds of the origin such that they have disjoint interior and their union covers $\mathbb{T}^2$.

   (b) Let t be the 3 by 3 transition matrix defined by $t_{ij} = 1$ if $f(R_i^o) \cap R_j^o \neq \emptyset$, and zero otherwise. Calculate it. Let $\Sigma_t = \{\xi \in \Sigma_3 : t_{\xi_k, \xi_{k+1}} = 1, \text{ for all } k \in \mathbb{Z}\}$. Show that (Σ_t, σ) is semi-conjugated to $(\mathbb{T}^2, f)$ through a surjective map $j : \Sigma_t \to \mathbb{T}^2$.

   (a) Show that the periodic points in the two spaces are in 1-1 correspondence except for one fixed point (which one?). Calculate $\lim_{n\to\infty} \frac{1}{n} \#\text{Fix}(f^n)$.

3. Consider the (Euclidean) d-torus and its set of closed geodesics. For each $\alpha \in \pi^1(\mathbb{T}^d)$ pick a closed geodesic γ_α. Its length depends only upon α. Let $N(r)$ be the number of γ_α of length not exceeding r. Show that $N(r) \sim r^d$.

4. Consider the closed geodesics in the double doughnut $S = \Gamma \setminus \mathbb{D}$. For each $\alpha \in \pi^1(S)$ let γ_α be the unique closed geodesic homotopic to α. Define $N(r)$ as above. Show that

$$\lim_{r\to+\infty} \frac{1}{r} \log N(r) = 1. \tag{0.2}$$