TD 08 : Dimension of repellers and Gibbs measures.

Consider a collection of $k \geq 2$ disjoint closed subintervals $I_1, \ldots, I_k$ of $I = [0,1]$ and strictly positive weights $p = (p_1, \ldots, p_k)$ for which $\sum_i p_i = 1$. We write $g : \bigcup I_i \to \mathbb{R}$ for the function $g(x) = \log(p_i)$ for $x \in I_i$.

Let $\psi_i : I \to I_i$ be an affine bijective map, $i = 1, \ldots, k$ and let $T : \bigcup I_i \to I$ be the inverse (expanding) map. Let $J = \bigcap_{n \geq 0} T^{-n} I$ denote the maximal T-invariant subset of I. We consider the transfer operator acting upon $\phi \in C^0, \mathrm{Lip}(I)$ or $L^1(I)$:

$$L_g\phi(y) = \sum_{x:Tx=y} e^{g(x)}\phi(x) = \sum_{i=1}^{k} p_i \phi(\psi_i(y)), \quad y \in I.$$

1. Let ν_g be the Gibbs measure associated with g and T. Show that $\nu_g(I_i) = p_i$.
2. Show that the support of ν_g is J.
3. Calculate $P(g)$, the Lyapunov exponent Λ_{ν_g} and the dimension of the measure, $\underline{d}^*_{\nu_g} = \dim_H \nu_g$.
4. Using Jensen's inequality, show that if $\sum_i q_i = 1$, then $\sum_i p_i \log(q_i/p_i) \leq 0$.
5. Show that $\underline{d}^*_{\nu_g}$ attains its maximum value when $p_i = |I_i|^s$ where s is the unique value for which $\sum_i |I_i|^s = 1$.
6. Show that this maximum value is the Hausdorff dimension of J.