Let ν_g be a Gibbs-like measure and define the inner prod

$$\langle A, B\rangle_\circ = \nu_g(A\cdot B) = \mu_g(A\cdot B\cdot h_g)$$

Remark allows to define L^2-prod completion but we will stay with Lip fcts.

We define $TA = A\circ T$ (called Koopman operator) and the dual operator.

$$\langle TA, B\rangle_\circ = \langle A, T^* B\rangle$$

Lemma We have
$$T^* B = \frac{1}{h_g} e^{-P_g} \mathcal{L}_g (B\cdot h_g)$$

(which is a conjugated or normalized version of $\mathcal{L}_g$ the RPF op).

proof:

$$\langle TA, B\rangle = \mu_g(A\circ T\cdot B h_g)$$
$$= e^{-P_g}\mu_g(\cancel{A}\mathcal{L}_g\, A\circ T\, B h_g)$$
$$= e^{-P_g}\mu_g(A\cdot \mathcal{L}_g(Bh_g))$$
$$= \cancel{e}\,\mu_g(A\cdot (\tfrac{1}{h_g}e^{-P_g}\mathcal{L}_g(Bh_g))\, h_g)$$
$$= \mu_g(A\cdot T^* B\, h_g)$$
$$= \langle A, T^* B\rangle$$

Lemma
We have $\underline{T^* T = \mathrm{id}}$

$\underline{T\mathbb{1} = \mathbb{1}}$, $\underline{T^*\mathbb{1} = \mathbb{1}}$

$\phi\in X$: $\cancel{K_g}$ $T^{*n}\phi = \mathbb{1}\langle \mathbb{1},\phi\rangle + R^{*n}\phi$
where $\|R^{*n}\| \le \mathcal{O}(q^n)$

proof:

$$\underset{(\text{exc})}{T^{*n}B} = \frac{1}{h_g} e^{-nP}\mathcal{L}_g^n(\cancel{B}B h_g)$$
$$= \frac{1}{h_g}\cdot h_g\mu_g(Bh_g) + \mathcal{O}(q^n)$$
$$= \mathbb{1}\cdot\nu_g(B) + \mathcal{O}(q^n)$$
$$= \mathbb{1}\cdot\langle\mathbb{1}, B\rangle + \mathcal{O}(q^n)$$

exchange

By T-invariance of ν_g
$$\nu_g(A\circ T\cdot B\circ T) = \nu_g(A,B)$$
or
$$\langle A,B\rangle = \langle TA, TB\rangle = \langle A, T^*TB\rangle$$
valid for all A, B
so $T^*T = 1$

$\mathbb{1}\circ T = \mathbb{1} = T\mathbb{1}$.
~~Then again~~
Then
$$\mathbb{1} = T^*T\mathbb{1} = T^*\mathbb{1}$$

Prop Let $\nu_g(A)=0$. Then we may write $A\in X$ in a unique way as follows:

$$A = Z + \underbrace{(T\lambda - \lambda)}_{\text{coboundary}}$$

with $Z, \lambda \in X$ and

$$T^*Z = 0$$

[If we require $\nu_g(\lambda)=0$ then also λ is unique].

proof: Suppose A has the above form. Then

$$T^*A = T^*Z + T^*(T\lambda - \lambda)$$
$$= 0 + \lambda - T^*\lambda$$

or

$$\lambda = T^*A + T^*\lambda$$
$$= R^*A + T^*\lambda$$

since $\langle 1, A\rangle = 0$. Then

$$\lambda = R^*A + R^{*2}A + \dots$$
$$= R^*(1-R^*)^{-1}A \in X.$$

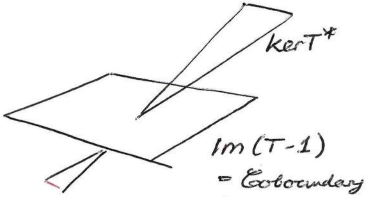
is well defined and determines λ uniquely if $\langle 1,\lambda\rangle = 0$ so $T^*\lambda = R^*\lambda$.

Then

$$Z = A + (\lambda - T\lambda)$$

verifies that

$$T^*Z = T^*A + T^*(\lambda - T\lambda)$$
$$= T^*A + T^*\lambda - \lambda$$
$$= R^*A + (R^* - 1)\lambda$$
$$= (R^* + (R^*-1)R^*(1-R^*)^{-1})A$$
$$= 0$$



not an orthogonal decomposition

Remark If $\|R^{*n}\| \le C q^{n-1}$, $n \ge 1$ then we have in X:

$$\|\lambda\|_X \le \sum_{n\ge 1} C q^{n-1} \|A\| = \frac{C\|A\|}{1-q}$$

$$\|Z\|_X \le \left[1 + (1+\|T\|)\frac{C}{1-q}\right]\|A\|$$

Remark: More generally a unique decomp:

$$A = \nu(A)\,1 + Z + (T-1)\lambda$$

with $T^*Z = 0$, $\langle 1, \lambda\rangle = 0$

Remark: Note that $T^*Z = 0$ means that Z is orthogonal to Im T

Lemma We have

$$S_n A = S_n Z + T^n \lambda - \lambda$$

$$\| S_n A \|^2 = n \| Z \|^2 + O(1)$$

proof: The 1^{st} is clear. Then

$$\langle S_n A, S_n A \rangle = \langle S_n Z, S_n Z \rangle + 2 \langle S_n Z, T^n \lambda - \lambda \rangle + \langle T^n \lambda - \lambda, T^n \lambda - \lambda \rangle$$

Now since $T^* Z = 0$ we have for $l > k$:

$$\langle T^l Z, T^k Z \rangle = \langle Z, T^{*(l-k)} Z \rangle = 0$$

so

$$\langle S_n Z, S_n Z \rangle = n \cdot \langle Z, Z \rangle .$$

Also $\langle S_n Z, T^n \lambda \rangle \cancel{=0}$

$$= \langle T^{*n} S_n Z, \lambda \rangle = 0 .$$

$$\langle T^n \lambda - \lambda, T^n \lambda - \lambda \rangle = 2 \langle \lambda, \lambda \rangle - 2 \langle \lambda, T^{*n} \lambda \rangle$$

stays unif bounded (converges to $2 \langle \lambda, \lambda \rangle$).

Finally:

$$-2 \langle S_n Z, \lambda \rangle = -2 \sum_{k=0}^{n-1} \langle T^k Z, \lambda \rangle = -2 \sum_{0}^{n-1} \langle Z, T^{*k} \lambda \rangle \xrightarrow{n} -2 \langle Z, (1 - R^*)^{-1} \lambda \rangle$$

so we have shown

$$| S_n A |^2 - n | Z |^2 \xrightarrow[n \to \infty]{} 2 \langle \lambda, \lambda \rangle - 2 \langle Z, (1 - R^*)^{-1} \lambda \rangle$$

Corollary

$$\sigma_A^2 = \lim_n \frac{1}{n} \| S_n A \|^2 = \| Z \|^2$$

We have $\sigma_A^2 = 0$ iff $\| Z \|^2 = 0$ iff

$$A = T\lambda - \lambda$$

for some $\lambda \in X$, i.e. A is a coboundary.