TD 06 : Pressure expansion and CLT.

The pressure function.

Let (Ω, d) be a metric space and assume in the following that T is a uniformly expanding and uniformly mixing map of Ω as defined in the lectures. We write: $X_{\mathbb{R}} = \mathrm{Lip}(\Omega, \mathbb{R})$ (real valued) and $X_{\mathbb{C}} = \mathrm{Lip}(\Omega; \mathbb{C})$ (complex valued). For $g \in X_{\mathbb{R}}$ or $g \in X_{\mathbb{C}}$ we define the transfer operator acting upon $X_{\mathbb{C}}$:

$$L_g \phi(y) = \sum_{x:Tx=y} e^{g(x)} \phi(x).$$

By $\rho(L_g)$ we denote the spectral radius of L_g when acting upon $X_{\mathbb{C}}$.

1. Let $A(x) = b(Tx) - b(x) + c, \ \forall\, x \in \Omega$ for some $b \in X_{\mathbb{R}}$ and $c \in \mathbb{R}$. One says that A is cohomologous to a constant or if $c = 0$ that A is a co-boundary.

   Calculate the spectral radius of L_{g+tA} for $t \in \mathbb{C}$.

2. Let $h \in X$, $\mu \in M^1_+(\Omega)$ be the right and left eigenvectors of L_g corresponding to the leading eigenvalue $\lambda > 0$. Let $d\nu = h\ d\mu$ be the associated Gibbs measure. Define the Hilbert space $H = L^2(\Omega, \nu)$ and let $\langle \cdot, \cdot \rangle$ denote the associated inner product. Define (slight abuse of notation) $T\phi = \phi \circ T$.

   (a) Let T^* be the dual of T in H. Derive an expression for T^* in terms of L_0, h and λ.

   (b) Show that T^*T is the identity on H.

   (c) Show that $T\mathbf{1} = \mathbf{1}$ and that $T^*\mathbf{1} = \mathbf{1}$. Show that TT^* is a projection.

   (d) Let $Z = \{\phi \in H : \langle \mathbf{1}, \phi \rangle = 0\}$. Show that $(T^*)^n \phi \to \mathbf{1}\langle \mathbf{1}, \phi \rangle$.

   (e) Given $A \in Z$ show that we may find $\psi, b \in X$ so that $A = \psi + (Tb - b)$ and $T^*\psi = 0$. (Hint: Suppose A has this form and deduce what b must look like).

   (f) Show that $S_n A = S_n \psi + T^n b - b$ and that for $n \geq 1$: $\|S_n A\|_H^2 = n\|\psi\|_H^2 + \mathcal{O}(1)$.

   (g) Deduce that $\sigma_A^2 = \|\phi\|_H^2$ and conclude that $\sigma_A = 0$ iff A is a co-boundary.