TD 04 : Cone contraction and spectral gap

The cross ratio

Let x, y, u, v be four vectors, no two being parallel. Let $\ell = \{\xi(t) = tw + z : t \in \mathbb{R}\}$ define an affine line (not containing the origin) and cutting the 1-D vectorspaces $\mathbb{R}x, \mathbb{R}y, \mathbb{R}u, \mathbb{R}v$ in the four points $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$ and $\xi(t_4)$. Express the cross-ratio (as given in the lectures) in terms of t_1, t_2, t_3, t_4.

Postive matrices.

1. Let $K_a = \{x \in \mathbb{R}^n_+ : x_i \geq a x_j\},\ 0 \leq a \leq 1$. Let $M = (m_{ij})$ be a strictly positive $n \times n$ matrix for which there is $b > 0$ so that $m_{ij} \geq b\ m_{kl}$ (for all indices). Let the eigenvalues be ordered as: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

   (a) Let $0 < a \leq 1$. Give a (finite) bound for $\operatorname{diam}_{K_0} K_a$.

   (b) Show that K_a is outer and inner regular.

   (c) Show that: $\dfrac{|\lambda_2|}{\lambda_1} \leq \max\limits_{i,j,k,l} \dfrac{m_{ij} - m_{kl}}{m_{ij} + m_{kl}}$. Optimal example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

2. Collatz-Wielandt formula: Let M be a strictly positive matrix.
   Define a partial order on $\mathbb{R}^n$ by: $x \preceq y \quad \Leftrightarrow \quad y - x \in \mathbb{R}^n_+ \quad \Leftrightarrow \quad x_i \leq y_i\ ,\ \ \forall\, i.$

   (a) Show that M preserves the partial order on $\mathbb{R}^n_+$.

   (b) Show that if $x \in (\mathbb{R}^n_+)^*$, $r > 0$ are such that $Mx \preceq rx$ (respectively, $Mx \succeq rx$) then $\lambda_1 \leq r$ (respectively, $\lambda_1 \geq r$).

   (c) Show that $\lambda_1 = \inf\limits_{x \in K_0^*} \max\limits_i \dfrac{(Mx)_i}{x_i} = \sup\limits_{x \in K_0^*} \min\limits_i \dfrac{(Mx)_i}{x_i}$.

   (d) Give a bound for λ_1 with $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ using the above and the vector $x = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$.

A family of operators associated with the Gauss map.

Let $I = [0, 1]$, equipped with the metric $d(x, x') = \left|\log \frac{1+x}{1+x'}\right|$.

Denote $I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right]$ and $\psi_k(y) = \frac{1}{y+k}$, $k \geq 1$, a diffeo of I obto I_k. The collection of $\psi_k, k \geq 1$ defines the inverse branches of the socalled Gauss map: $f(x) = \frac{1}{x} - \left\lfloor\frac{1}{x}\right\rfloor$. Let $X = \mathrm{Lip}\,(I, d)$. Let $s \in \ell^\infty_+(\mathbb{N})$ be a family of non-negative 'weights'. We define the transfer operator:

$$L_s\phi(y) = \sum_{k\geq 1} s_k \frac{1}{(k+y)^2} \phi\left(\frac{1}{k+y}\right) \qquad \phi \in X, \;\; y \in I.$$

1. Show that L_s is a bounded operator when acting upon X.
2. For $a > 0$, let $K_a = \left\{\phi \geq 0 : \phi(x) \leq \phi(x')e^{ad(x,x')}, \; \forall\, x, x' \in I\right\}$.
3. Suppose that some s_k's are strictly positive. Show that there are $a > 0$, $0 < \sigma < 1$ such that $L_s(K^*_a) \subset K^*_{\sigma a}$.
4. Estimate the contraction rate in the Hilbert metric of L_s and show that L_s admits a spectral gap.
5. In the case when $s_k \equiv 1$, $k \geq 1$ show that $h(x) = \frac{1}{\log(2)} \frac{1}{1+x}$ is an eigenvector of $L_{s\equiv 1}$.
6. Show that the Gauss map admits a unique invariant probability measure which is abs cont w.r.t. Lebesque and that this measure is strongly mixing (whence ergodic).
