TD 02 : Dynamical Systems and invariant measures

The doubling map

Let $I = [0,1]$, $\mathcal{B}$ the associated Borel-σ-algebra and $d\lambda$ Lebesque measure on I.
We consider the doubling map (discontinuous) on I: $T(x) = 2x \bmod 1$, $x \in I$. We recall that $\lambda(T^{-1}A) = \lambda(A)$ for every $A \in \mathcal{B}$ (i.e. $d\lambda$ is T-invariant). Below we consider two ways of showing strong mixing (whence ergodicity).

Composing f with T^n

1. Let $f = \mathbf{1}_J$ with $J = [a,b]$, $0 \le a < b \le 1$. Write $f \circ T^n$ as a sum of characteristic functions.
2. Show that for $g \in C^0([0,1])$ (Hint: Reformulate the first integral as a suitable Riemann sum):

$$\lim_{n\to+\infty} \int_X f \circ T^n \; g \; d\lambda = \int_X f \; d\lambda \int_X g \; d\lambda.$$

3. Show that this limit holds, first for $f \in L^\infty$ and $g \in C^0$, then for any $f \in L^\infty$ and $g \in L^1$.
4. Fix $a \in [0,1]$ and write (δ_ξ meaning Dirac delta):

$$f_n = \frac{1}{2^n} \sum_{\xi : T^n \xi = a} \delta_\xi.$$

Show that for every $g \in C^0(I)$: $\lim_n \int f_n \; g = \int g \; d\lambda$ but that this may fail if $g \in L^1(I)$.

Why is this situation different from the previous where we integrated against $f \circ T^n$?

Composing by pre-images of T

5. Let $X = [0,1]$ and define the transfer operator

$$L\phi(y) = \frac{1}{2}\left(\phi\left(\frac{y}{2}\right) + \phi\left(\frac{1+y}{2}\right)\right).$$

6. Show that for $\phi \in C^1(X)$: $\left\| L^n\phi(x) - \int_0^1 \phi(t) \; dt \right\|_\infty \le C(\phi)\frac{1}{2^n}$. (Hint: The MVT).
7. Use the previous result to show that for $f \in L^\infty(X)$ and, first for $g \in C^1(X)$, then $g \in L^1(X)$:

$$\lim_{n\to+\infty} \int_X f \circ T^n \; g \; d\lambda = \lim_{n\to+\infty} \int_X f \; (L^n g) \; d\lambda = \int_X f \; d\lambda \int_X g \; d\lambda$$