

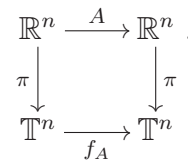
TD 01 : Introduction to Dynamical Systems

### Toral Endo- and automorphisms

Write  $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  for the canonical projection. Let  $[x] = x + \mathbb{Z}^n$  be the equivalence class of  $x \in \mathbb{R}^n$ . Write  $L_n := M_n(\mathbb{Z})$  ( $n \times n$  matrices with integer entries). We say that  $A \in GL_n(\mathbb{Z})$  iff  $A \in L_n$  is invertible and  $A^{-1} \in L_n$ .

1. Assume  $A \in L_n$ . Show that  $A \in GL_n(\mathbb{Z})$  iff  $|\det(A)| = 1$ . We assume from now on that this is the case.

2. Show that when  $A \in L_n$  then there is a unique map  $f_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$  making the diagram to the right commutative. Show also that if  $A \in GL_n(\mathbb{Z})$  then  $f_A$  is invertible.



3. Show that if  $[x] \in \text{Per}_p(f_A)$  then  $(1 - A^p)x \in \mathbb{Z}^n$ .

4. Show that  $\text{Per}_p(f_A)$  has infinite cardinality iff 1 is an eigenvalue of  $A^p$ .

We assume from now on that every eigenvalue of  $A$  has modulus different from 1.

5. Show that if  $[x] \in \text{Per}_p(f_A)$  then  $x = \frac{1}{q} \vec{m}$  where  $q = \det(1 - A^p)$  and  $\vec{m} \in \mathbb{Z}^n$ .

6. Show that if  $x = \frac{1}{q} \vec{m}$  with  $q \in \mathbb{Z}^*$  and  $\vec{m} \in \mathbb{Z}^n$  then  $[x]$  is a periodic point for  $f_A$ .

7. Show that periodic points are dense in  $\mathbb{T}^n$ .

From now on consider the case  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

8. Show that there are one dimensional  $A$ -invariant subspaces  $E^u \subset \mathbb{R}^2$  and  $E^s \subset \mathbb{R}^2$  so that

$$\forall x \in E^s : \lim_{k \rightarrow +\infty} A^k x = 0 \quad \text{and} \quad \forall x \in E^u : \lim_{k \rightarrow -\infty} A^k x = 0.$$

9. Show that  $W^s(0) = \pi(E^s) \subset \mathbb{T}^2$  and  $W^u(0) = \pi(E^u) \subset \mathbb{T}^2$  are  $f_A$ -invariant and that

$$\xi \in W^s(0) \text{ iff } \lim_{k \rightarrow +\infty} f_A^k(\xi) = 0 \quad \text{and} \quad \xi \in W^u(0) \text{ iff } \lim_{k \rightarrow -\infty} f_A^k(\xi) = 0$$

10. Show that  $W^s(0)$  and  $W^u(0)$  are both dense in  $\mathbb{T}^2$ .

11. Describe the (forward and backward) orbit of  $\xi \in W^s(0) \cap W^u(0)$ .