TD 01 : Introduction to Dynamical Systems

Toral Endo- and automorphisms

Write $\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ for the canonical projection. Let $[x] = x + \mathbb{Z}^n$ be the equivalence class of $x \in \mathbb{R}^n$. Write $\mathrm{L}_n := M_n(\mathbb{Z})$ ($n \times n$ matrices with integer entries). We say that $A \in \mathrm{GL}_n(\mathbb{Z})$ iff $A \in \mathrm{L}_n$ is invertible and $A^{-1} \in \mathrm{L}_n$.

1. Assume $A \in L_n$. Show that $A \in \mathrm{GL}_n(\mathbb{Z})$ iff $|\det(A)| = 1$. We assume from now on that this is the case.
2. Show that when $A \in \mathrm{L}_n$ then there is a unique map $f_A : \mathbb{T}^n \to \mathbb{T}^n$ making the diagram to the right commutative. Show also that if $A \in \mathrm{GL}_n(\mathbb{Z})$ then f_A is invertible.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^n & \xrightarrow[f_A]{} & \mathbb{T}^n \end{array}.$$

3. Show that if $[x] \in \mathrm{Per}_p(f_A)$ then $(\mathbf{1} - A^p)x \in \mathbb{Z}^n$.
4. Show that $\mathrm{Per}_p(f_A)$ has infinite cardinality iff 1 is an eigenvalue of A^p.

We assume from now on that every eigenvalue of A has modulus different from 1.

5. Show that if $[x] \in \mathrm{Per}_p(f_A)$ then $x = \frac{1}{q}\overrightarrow{m}$ where $q = \det(\mathbf{1} - A^p)$ and $\overrightarrow{m} \in \mathbb{Z}^n$.
6. Show that if $x = \frac{1}{q}\overrightarrow{m}$ with $q \in \mathbb{Z}^*$ and $\overrightarrow{m} \in \mathbb{Z}^n$ then $[x]$ is a periodic point for f_A.
7. Show that periodic points are dense in $\mathbb{T}^n$.

From now on consider the case $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

8. Show that there are one dimensional A-invariant subspaces $E^u \subset \mathbb{R}^2$ and $E^s \subset \mathbb{R}^2$ so that

$$\forall x \in E^s : \lim_{k \to +\infty} A^k x = 0 \quad \text{and} \quad \forall x \in E^u : \lim_{k \to -\infty} A^k x = 0.$$

9. Show that $W^s(0) = \pi(E^s) \subset \mathbb{T}^2$ and $W^u(0) = \pi(E^u) \subset \mathbb{T}^2$ are f_A-invariant and that

$$\xi \in W^s(0) \text{ iff } \lim_{k \to +\infty} f_A^k(\xi) = 0 \quad \text{and} \quad \xi \in W^u(0) \text{ iff } \lim_{k \to -\infty} f_A^k(\xi) = 0$$

10. Show that $W^s(0)$ and $W^u(0)$ are both dense in $\mathbb{T}^2$.
11. Describe the (forward and backward) orbit of $\xi \in W^s(0) \cap W^u(0)$.