8. Erg of geod. flows on the Poincaré disc

Thm 8.1 [Birkhoff invertible]
Let $(X,\mathcal{B},\mu)$ be a proba measure space. and $T: X\to X$ an invertible measure preserving transf.

Let $f\in L^1(X,\mu)$. Then there is a subset $\Omega\subset X$ of full measure. such that $\forall x\in\Omega$:

$$f^+(x) = \lim_n \frac{1}{n}\sum_0^{n-1} f\circ T^k(x)$$

$$f^-(x) = \lim_n \frac{1}{n}\sum_1^{n} f\circ T^{-k}(x)$$

exists and $f^+(x) = f^-(x)$.

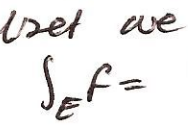
proof: f^+ and f^- exists by the usual Birkhoff. For $E = T^{-1}(E) = T(E)$ (mod 0) an invariant subset we have also

$$\int_E f = \int_E f^+ = \int_E f^-$$

Let $E = E_{\alpha,\beta} = \{ f^- \le \alpha < \beta \le f^+\}$ for rational $\alpha<\beta$. Then E is T-invariant and

$$\beta\mu(E) \le \int_E f^+ = \int_E f = \int_E f^- \le \alpha\mu(E)$$

shows that $\mu(E)=0$. So $f^- \ge f^+$ a.e. and by symmetry the same is true for $f^+ \ge f^-$ a.e. //



$\circlearrowleft \begin{pmatrix}1&1\\1&0\end{pmatrix}$ preserves Lebesgue

$f^+ = f^-$ a.e.

Flow $\varphi: \mathbb{R}\times M \to M$

$\varphi(t,x) := \varphi_t(x)$

preserves $\mu \iff \forall f\in L^1(M,\mu)$

$$\int_M f\circ\varphi_t(x)\,d\mu(x) = \int_M f(y)\,d\mu(y)$$

Thm 8.2 [Birkhoff flows]
Let $(X,\mathcal{B},\mu)$ be a proba space and $\varphi_t: X\to X$, $t\in\mathbb{R}$ a measure preserving flow.

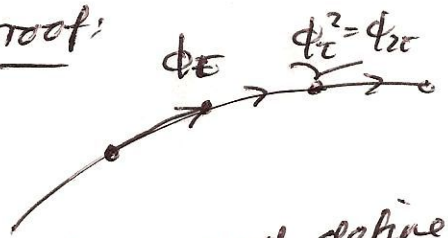
Let $f\in L^1(X,\mu)$ then for a subset $\Omega\subset X$ of full measure:

$$f^+(x) = \lim_T \frac{1}{T}\int_0^T f(\varphi_t x)\,dt$$

$$f^-(x) = \lim_T \frac{1}{T}\int_0^T f(\varphi_{-t} x)\,dt$$

exist and are equal.

proof:



Let $\tau>0$ and define

$T = \varphi_\tau$,

$F(x) = \frac{1}{\tau}\int_0^\tau f(\varphi_t x)\,dt$

Then $(X,\mathcal{B},\mu)$ is T-invariant and we may apply the previous thm to F:
By Fubini

$$\int F\,d\mu = \frac{1}{\tau}\int_0^\tau \int f\circ\varphi_t(x)\,d\mu(x)\,dt = \frac{1}{\tau}\int_0^\tau\int f(y)\,d\mu(y)\,dt = \int f(y)\,d\mu(y)$$

so $\int|F| \le \int|f|$.

and Birkhoff states that

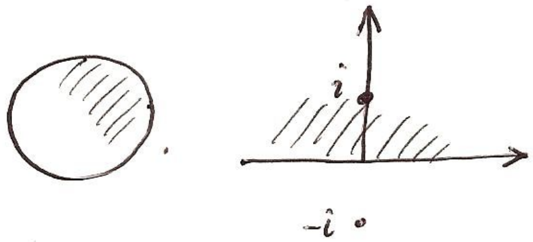
$F^+ = \lim \frac{1}{n}\sum_{k=0}^{n-1} F\circ T^k$ $\quad F^- = \lim\frac{1}{n}\sum_1^n F\circ T^{-k}$

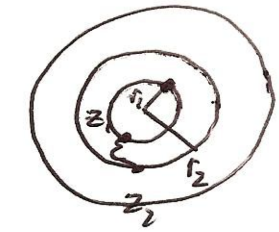
exists a.e. and are equal

The Hyperbolic disk & plane

§3

$\mathbb{D} = \{|z|<1\}$ $\mathbb{H}_+ = \{\mathrm{Re}\, z>0\}$





$0<r_1<r_2<1$

$\gamma: [0,1] \xrightarrow[C^1]{}$ path z_1 to z_2

$j: \mathbb{H}_+ \xrightarrow{\sim} \mathbb{D}$, $j(\omega) = \dfrac{\omega - i}{\omega + i}$

$\gamma = r(t)e^{i\theta(t)}$

$\dot\gamma = \dot r e^{i\theta} + i r\dot\theta e^{i\theta}$

$|\dot\gamma|^2 = |\dot r + i r\dot\theta|^2 = \dot r^2 + r^2\dot\theta^2 \geq \dot r^2$

Poincaré metric on $\mathbb{D}$

$$ds^2 = \frac{4\,dz\,d\bar z}{(1-z\bar z)^2} \qquad ds = \frac{2|dz|}{1-|z|^2}$$

$$= g(x,y)(dx^2+dy^2)$$

$$g(x,y) = \frac{4}{(1-(x^2+y^2))^2} =: \text{"}g(x+iy)\text{"}$$

$ds = \sqrt{g(\gamma(t))}\,|\dot\gamma|\,dt$ so

$$\mathrm{len}(\gamma) = \int_0^1 \frac{2}{1-|\gamma|^2}\cdot|\dot\gamma|\,dt$$

$$\geq \int_{r_1}^{r_2} \frac{2}{1-r^2}\,dr$$

$$= \cancel{\tfrac12}\left|\ln \frac{1+r_2}{1+r_1}\cdot\frac{1-r_1}{1-r_2}\right|$$

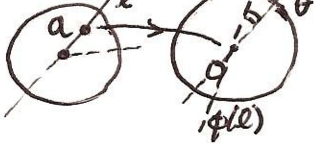
Orientation preserving (shown by max principle) automorphisms:

equality iff $0, z_1, z_2$ are aligned (and γ straight path). //

$$\mathrm{Aut}_{+}(\mathbb{D}) = \{\phi: \mathbb{D} \xrightarrow[+]{\sim} \mathbb{D}\} \qquad \frac{az+b}{cz+d} \ \det=1,\ |b|<|a|,\ |c|<|d|$$

$$= \left\{\phi(z) = e^{i\theta}\frac{z-a}{1-\bar a z} : |a|<1,\ \theta\in\mathbb{R}/2\pi\mathbb{Z}\right\}$$

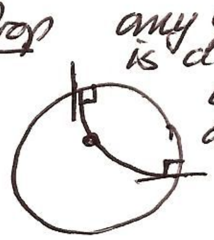
$$\simeq \mathbb{D}\times S^1$$



$\mathrm{Aut}_+(\mathbb{D}) = \text{Möb}(\mathbb{D})$ maps straight lines to circles preserving angles. so

Lemma: $\phi^* g = g$

Prop any geodesic in $\mathbb{D}$ is a circular arc whose limit directions are normal to $\partial\mathbb{D}$



proof: $\phi' = \dfrac{1-|a|^2}{(1-\bar a z)^2}$ $\quad \underset{\theta=0}{g\circ\phi} = \dfrac{4|1-\bar a z|^2}{(1-|a|^2)(1-|z|^2)}$

so $g(z) = g(\phi(z))\,|\phi'(z)|^2$ $\quad w=\phi(z)$

$g(z)|dz|^2 = g(w)|dw|^2$

true for geodesics through 0 and preserved by autom.

Lemma Geodesics through 0 are straight lines.

Lemma $\forall z_1, z_2 \in \mathbb{D}$

$$d_{\mathbb{D}}(z_1,z_2) \geq \left|\ln \frac{1+r_2}{1+r_1}\,\frac{1-r_1}{1-r_2}\right| \cancel{\tfrac12}$$

with $r_1=|z_1|$, $r_2=|z_2|$ and equality iff $t_2 z_2 = t_1 z_1$ for some $(t_1,t_2)\in\mathbb{R}\times\mathbb{R}\setminus\{(0,0)\}$

$\mathcal{U}^1 = \mathbb{D}\times S^1$ describes the unit tangent bundle of $\mathbb{D}$, normalized s.t. for $(z,v)\in\mathcal{U}^1$ $\|v_z\|_{\mathbb{D}} = 1$

Def The geodesic flow on a Riemannian manifold (complete) is ~~a~~ the unique C^1 map $\phi : \mathbb{R}\times SM \to SM$

$(t, \xi) \mapsto \phi_t \xi$

such that

$\phi_t(x_0, v_0) = (x_t, v_t)$

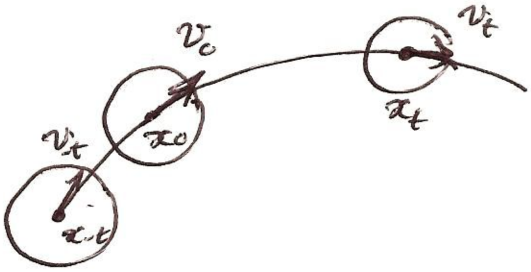
verifies $v_t = \frac{d}{dt} x_t$, $\|v_t\| = 1$.

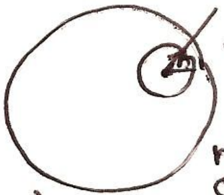
and $\forall x \in M$, $v \in S_x M$ $\exists \delta > 0$ s.t.

$\gamma_\delta = \{\phi_t(x) : -\delta \le t \le \delta\}$

is the unique geodesic from $\phi_{-\delta}(x)$ to $\phi_\delta(x)$. We have

len $\gamma_t = |t|$ $\forall\, 0 \le t \le \delta$.





$\xi = \dfrac{e^{i\theta}(1-|z|^2)}{2}$

v_t

natural coordinates: (z, θ)

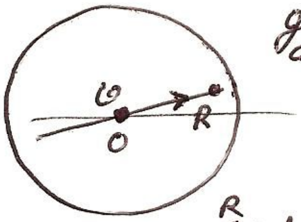
geodesic (z_t, θ_t)

$\dfrac{dz_t}{dt} = e^{i\theta_t}\dfrac{(1-|z_t|^2)}{2}$

$\|\dot z_t\|_{\mathbb{D}} = 1$

Ex : Poincaré disc

Let $z_0 = 0 \in \mathbb{D}$ (the center)



geodesics through 0 are straight lines

$s = \int_0^R \frac{2\,ds}{1-s^2} = \ln \frac{1+R}{1-R}$

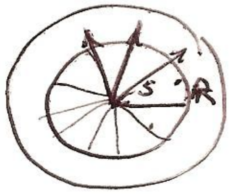
or $R = \frac{e^s - 1}{e^s + 1} = \operatorname{th} \frac{s}{2}$

$v_0 = e^{i\theta} \in S^1$

$\phi_t(0, e^{i\theta}) = \left(e^{i\theta} \operatorname{th} \frac{t}{2},\ e^{i\theta}\frac{\left(1 - \operatorname{th}^2 \frac{t}{2}\right)}{2}\right)$

$(z_t, \theta_t) = \left(e^{i\theta} \operatorname{th} \frac{t}{2},\ \theta\right)$

Horocycles and stable / unstable mflds.



$s \to \int ds = \int \frac{2|dz|}{1-|z|^2} = \int_0^R \frac{2dr}{1-r^2} = \ln \frac{1+R}{1-R}$

$\Leftrightarrow R = \frac{e^s - 1}{e^s + 1} = \operatorname{th} \frac{s}{2}$

$B_{\mathbb{D}}(0,s) = B_{\mathbb{C}}(0, R(s))$

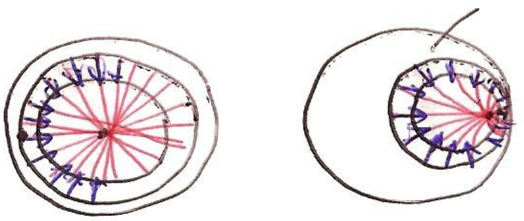
- Equal time (s) circle centered at the origin "horo-circle"
- Geodesics are normal to the circle

The Möbius transformation

$\Phi_a(z) = \frac{z+a}{1-\bar{a}z}$ $\quad a = \operatorname{th} s_0/2$

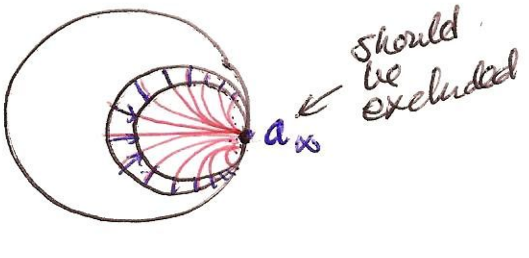
maps 0 to a
If $s = s_0 + t$ then $r_0 = \operatorname{th} s_0/2$

Φ $a = e^{i\alpha} \operatorname{th} \frac{s_0}{2} \to \partial\mathbb{D}$ equal time circles



$B_{\mathbb{D}}(a, s_0) = \Phi_a(B_{\mathbb{D}}(0, s_0))$

Letting $a_{s_0} \to \partial\mathbb{D}$ and keeping $s = s_0 + t$ we get the socalled "Horocycles" $H_t^{a}(a_\infty)$



Let $\hat{H}^s_t(a_\infty)$ denote the "inward" normal bundle of $H_t(a_\infty) \setminus \{a_\infty\}$

$\boxed{\text{Prop: The flow } \phi_t \text{ maps } \phi_t : \hat{H}^s_\tau(a_\infty) \to \hat{H}^s_{\tau+t}(a_\infty)}$ $t \in \mathbb{R}$

Proof true for $\hat{B}^s_{\mathbb{D}}(0, s_0)$ (normal bundle)

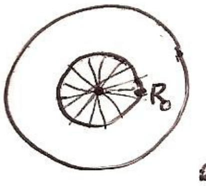
$\phi_t : \hat{B}^s_{\mathbb{D}}(0, s_0) \to \hat{B}^s_{\mathbb{D}}(0, s_0 + t)$ $t \geq s_0$

and preserved by Möbius and taking limits $s_0 \to +\infty$

Def

Let $\xi_0 = (z_0, v_0) \in S_1\mathbb{D}$ and
Let $a_+ = \lim_{t \to \infty} \pi(\phi_t(\xi_0)) \in \partial\mathbb{D}$ $\quad a_- = \lim_{-\infty} \pi\, \phi_t(\xi)$
be the socalled "ideal" limit points

Prop: $\xi_0 \in H_{\tau_0}$



$0<R_0<1$

$s_0 = d_{\mathbb{D}}(0,R_0) = \int_0^{R_0} \frac{2\,dr}{1-r^2} = \ln\frac{1+R_0}{1-R_0}$ or

$R_0 = R(s_0) = \tanh s_0/2 = \mathrm{th}\, s_0/2$

so $B_{\mathbb{D}}(0,s_0) = B_{\mathbb{C}}(0,R(s_0))$

Equal time circle centered at 0. "horo-circle"

geodesics are normal to (through 0) the circle

Let $s = s_0 + t$ and $r = \mathrm{th}\, s/2$

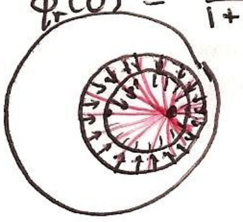
The Möbius transf.

$\phi_r(z) = \frac{z+r}{1+rz}$ maps

$\phi_r(-r_0) = \frac{r-r_0}{1-r\cdot r_0} = \mathrm{th}\left(\frac{s-s_0}{2}\right) = \mathrm{th}\left(\frac{t}{2}\right)$

$\phi_r(r_0) = \frac{r+r_0}{1+rr_0} = \mathrm{th}\left(\frac{s+s_0}{2}\right) = \mathrm{th}\left[\frac{2r_0+t}{2}\right]$

$\phi_r(0) = \frac{r}{1+0} = r = \mathrm{th}\left(\frac{s_0+t}{2}\right)$

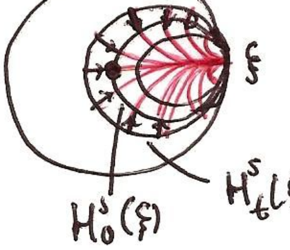


$B_{\mathbb{D}}(r_t, s_0) = \phi_r(B_{\mathbb{D}}(0,s_0))$

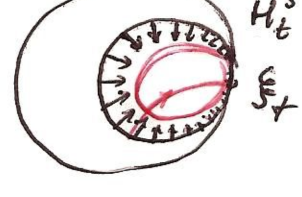
describes the geodesic disk centered at $r_t = \mathrm{th}\left(\frac{s_0+t}{2}\right)$ of radius (geod) s_0

Letting $s_0 \to +\infty$ we obtain the family of horocycles

$r_0 \to \xi \in \partial\mathbb{D}$



Let $\hat{H}_t^s(\xi)$ be the forward normal bundle of $H_t(\xi)$



$\hat{H}_t^s \xrightarrow{\Phi^u} H_{t+u}^s$

Then for $(z,v) \in \hat{H}_t^s(\xi_+)$

$\Phi^u(z,v) \in H_{t+u}^s(\xi)$

Theorem ~~Which E~~

Let $(z_0,v_0) \in \mathcal{U}^1(\mathbb{D})$ and

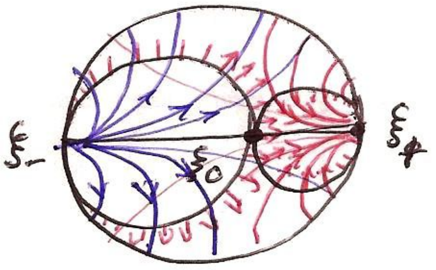
$\xi_+ = \lim_{t\to+\infty} \Phi^t(z_0,v_0) \in \partial\mathbb{D}$ be the ideal limit point.

Then $(z_0,v_0) \in \hat{H}_{t_0}^s(\xi_+)$ for some t_0 and

$\hat{H}_{t_0}^s(\xi_0)$ is the stable manifold of (z_0,v_0)

Similarly $(z_0,v_0) \in \hat{H}_{t_-}^{-}(\xi_-)$ for some t_- and

$\hat{H}_{t_-}^{-}(\xi_-)$ is the unstable manifold of (z_0,v_0)



$\forall \eta \in \hat{H}^s(\xi_0)$, $x \in \hat{H}^u(\xi_0)$

$d_{\mathbb{D}}(\Phi^t(\eta), \Phi^t(\xi_0)) \xrightarrow[t\to+\infty]{} 0$

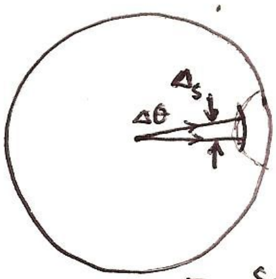
$d_{\mathbb{D}}(\Phi^t(x), \Phi^t(\xi_0)) \xrightarrow[t\to-\infty]{} 0$

$$ds = \frac{2|dz|}{1-|z|^2} \qquad s = \int_0^r \frac{2\,dr}{1-r^2} = \ln\frac{1+r}{1-r} \Leftrightarrow r = th\frac{s}{2}$$

$$d_{\mathbb{D}}(0,r) \qquad B_{\mathbb{D}}(0,s) = B_{\mathbb{C}}(0, th\tfrac{s}{2})$$

$$\ell_s = |\partial B| = \oint \frac{2r\,d\theta}{1-r^2} = 2\pi\frac{2r}{1-r^2} = 2\pi \cdot 2\,sh\tfrac{s}{2}\,ch\tfrac{s}{2} = 2\pi\,sh\,s.$$

or $\ell_s = \pi(e^s - e^{-s}) =$



dist between radial trajectories

$$\Delta_s \approx \Delta\theta \cdot sh\,s \approx \Delta\theta\cdot\tfrac{1}{2}e^s \quad (s \text{ large}) \qquad \frac{\Delta\theta}{2\pi}\cdot\ell_s \approx \Delta\theta\cdot\frac{e^s}{2}$$

time s_0 - ~~ball~~ circle centered at 0.

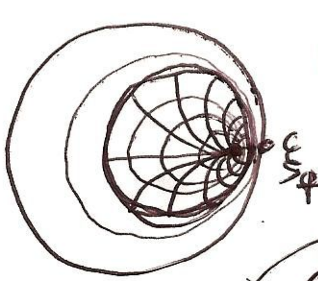
radial flow

$B_{\mathbb{D}}(0,s_0)$

$$\Delta_{s_0+t} \approx \Delta\theta\cdot sh(s_0+t)$$
$$= \Delta\theta\cdot\tfrac{1}{2}(e^{s_0+t} - e^{-s_0-t})$$
$$\simeq \Delta_{s_0}\cdot e^t \qquad (s_0 \gg 1)$$

nearby trajectories $\dfrac{\Delta_{s_0+t}}{\Delta_{s_0}} \approx e^t$

valid when $s_0 \gg 1$, $|t| \ll s_0$

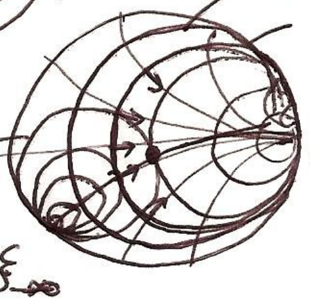


$\varphi_{a_0}(B_{\mathbb{D}}(0, s_0+t)) \qquad |a_0| = th\frac{s_0}{2}$

$a_0 \to \xi_\varphi \in \partial\mathbb{D}$

$H_t(\xi_\varphi)$

$H_0(\xi_\varphi)$



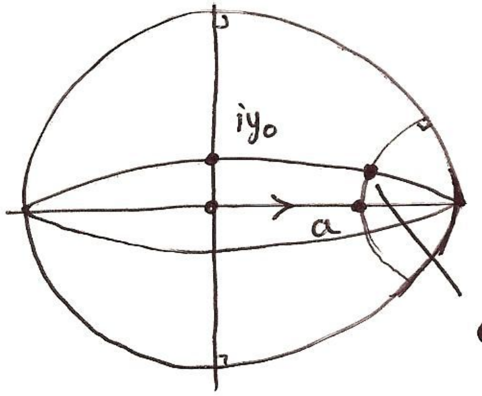
$H^t(\xi_{-\infty})$

$$H_t(\xi_\varphi) = \lim_{\substack{a_0\to\xi_0 \\ |a_0| = th\frac{s_0}{2}}} \varphi_{a_0}(B_{\mathbb{D}}(0, s_0+t)) \qquad \text{limit exists}$$

$\hat{H}_t^-(\xi_{\varphi\infty})$ inward normal ~~...~~ bundle

---

More precise



$y_0 = th\frac{s_0}{2} \qquad d_{\mathbb{D}}(0, iy_0) = s_0$

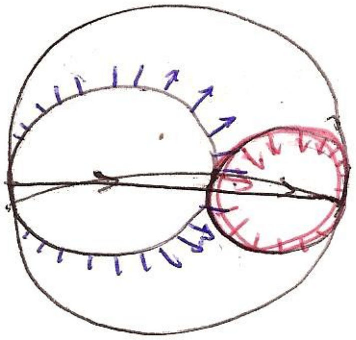
$d_{\mathbb{D}}(-iy_0, iy_0) = 2s_0$

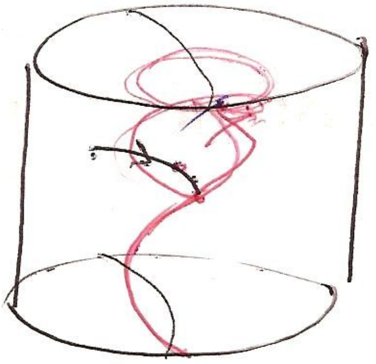
$$\varphi_a(z) = \frac{z+a}{1+az}$$

$$d_{\mathbb{D}}(\varphi_a(iy_0), \varphi_a(-iy_0)) = 2s_0 \quad (\text{isometry } \varphi_a)$$

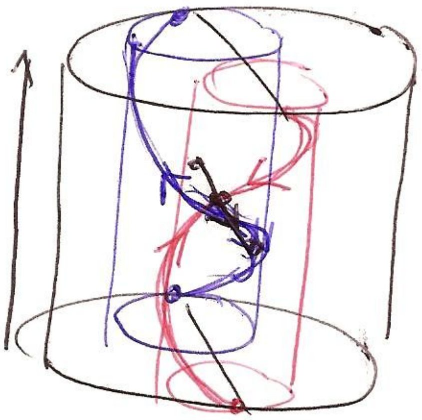
$$a\frac{1+y_0^2}{1+a^2y_0^2} + iy_0\frac{1-a^2}{1+a^2y_0^2}$$

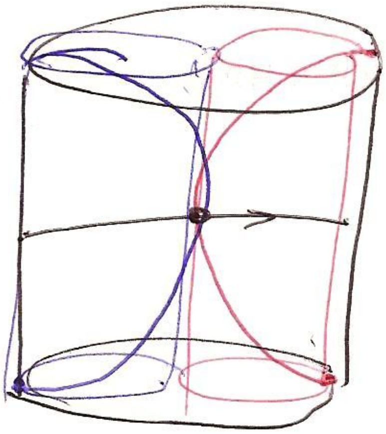
$\varphi_a(0) = a \qquad \varphi_a(\infty) = \frac{1}{a}$





θ





Lagrange + Hamiltonian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot z} = \frac{\partial L}{\partial z}$$

$$L = \frac{1}{2} g_x(v,v) = \frac{1}{2} g_z(\dot z, \dot z)$$

$$L(z,\dot z) = \frac{1}{2} g_z(\dot z,\dot z) = \frac{1}{2}\frac{1}{(1-|z|^2)^2}|\dot z|^2$$

$$= \frac{1}{2}\frac{1}{(1-(x^2+y^2))^2}(\dot x^2+\dot y^2) \qquad L = \frac{1}{2} g_{\alpha\beta}v^\alpha v^\beta$$

$$p_x = \frac{\partial L}{\partial \dot x} \quad p_y = \frac{\partial L}{\partial \dot y} = \frac{1}{(1-|z|^2)^2}\dot y \qquad H(z,p) = (\dot z p - L(\dot z))\Big|_{p=\frac{\partial L}{\partial \dot z}}$$

$$= \frac{1}{\cancel{2}(1-|z|^2)^2}\dot x$$

$$p_\alpha = g_{\alpha\beta}v^\beta$$

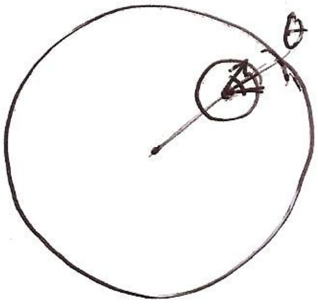
$$H(z,p) = \frac{1}{2}(1-|z|^2)^2(p_x^2+p_y^2) \qquad v^\beta = g^{\beta\alpha}p_\alpha$$

inverse matrix

$$\begin{cases} \dot x = \frac{\partial H}{\partial p_x} & \dot y = \frac{\partial H}{\partial p_y} \\ \dot p_x = -\frac{\partial H}{\partial x} & \dot p_y = -\frac{\partial H}{\partial y} \end{cases} \qquad H = \frac{1}{2} g^{\beta\alpha}p_\alpha p_\beta$$

preserves H and $dx\wedge dp_x + dy\wedge dp_y = \omega$
whence also $\omega\wedge\omega = dx\,dy\,dp_x\,dp_y = \Omega$
and $\mu = \Omega\,\delta(H-1)$ unit tangent bundle

$$(p_x,p_y) = |p|(\cos\theta,\sin\theta)$$



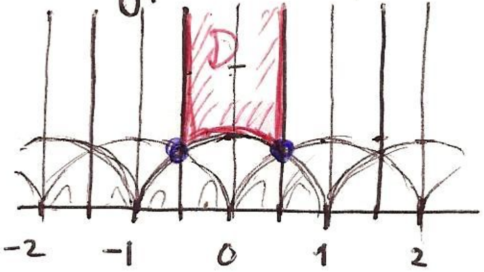
$(x,y),\theta$

$$d\mu = dx\,dy\,d\theta\cdot\frac{1}{|p|} = dx\,dy\,d\theta\cdot\frac{1}{(1-|z|^2)}$$

Aire$(\mathbb{D}\times S^1) = \infty$.

To get finite area we need to take a quotient

Quotient of The hyperbolic plane / disc.



$\mathbb{H} = \{\operatorname{Im} z > 0\}$

Fundamental domain

$D = \{ z \in \mathbb{H} : |z| \geq 1, |\operatorname{Re} z| \leq \frac{1}{2} \} \cong \Gamma \backslash \mathbb{H}$ identif of bdry-pts

$\Gamma = PSL(2,\mathbb{Z}) < PSL(2,\mathbb{R})$

ΓD tesselates the plane

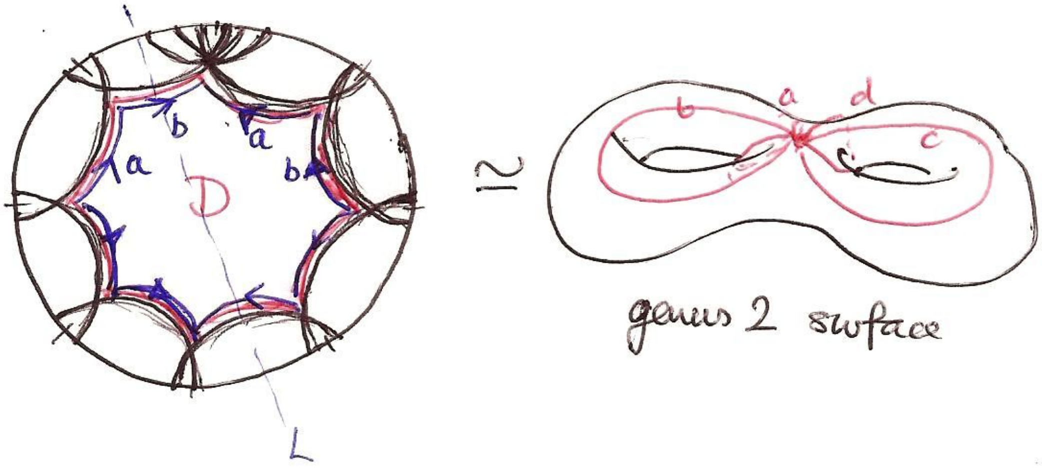
$\gamma D \cap \gamma' D \neq \emptyset \Leftrightarrow \gamma = \gamma'$

ΓD is dense in $\mathbb{H}$

Edge points: $\pm \frac{1}{2} + i\frac{\sqrt{3}}{2}, \infty$

angles $\frac{\pi}{3}, \frac{\pi}{3}, 0$.

But <u>does not yield a</u> compact mfld.



genus 2 surface

Stable and Unstable sets

(M,g) Riem. mfld.
Φ_t complete flow on M.

Def For $p\in M$

$W^s(p) = \{q\in M : d(\Phi_t p, \Phi_t q) \xrightarrow[t\to+\infty]{} 0\}$ (stable set of p)

$W^u(p) = \{ \quad \text{—} \text{''} \text{—} \quad \xrightarrow[t\to-\infty]{} 0\}$ (unstable set of p)

Lemma: Let $f\in C^0(M)$ be unif cont.
Let $p\in M$ and suppose that

$f^+(p) := \lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f\circ\Phi_t(p)\,dt$ and

$f^-(p) := \lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f\circ\Phi_{-t}(p)\,dt$ exist.

Then $\forall q\in W^s(p): f^+(p) = f^+(q)$ and
$\forall r\in W^u(p): f^-(r) = f^-(p)$.

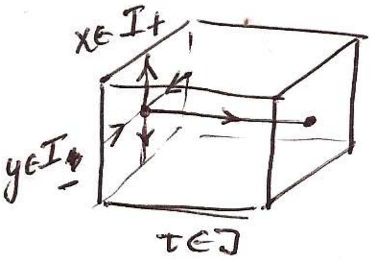
proof: By unif cont of f:

$\lim\limits_{t\to+\infty} d(\Phi_t q, \Phi_t p) \to 0 \Rightarrow$

$\lim\limits_{t\to+\infty} |f\circ\Phi_t(q) - f\circ\Phi_t(p)| = 0$

so $f^+(p) = f^+(q)$
(similarly for $f^-(r) = f^-(p)$). //

$\exists\ \Omega \subset B = I_+ \times I_- \times J$
of full measure s.t.
$f^+(\omega) = f^-(\omega),\ \forall \omega \in \Omega.$



$(1^s)\quad f^+(x,y,t) = a^+(x)$ indep of y,t

$(1^u)\quad f^-(x,y,t) = a^-(y)$ indep of x,t

Fubini $\Rightarrow$ $\exists A_+ \subset I_+$ of full measure such that $\forall x \in I_+$:
$U_x := \{(y,t) \in I_- \times J : (x,y,t) \in \Omega\}$
is of full measure

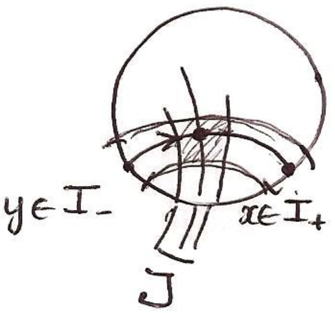
__Lemma__ $\forall x, x' \in A_+ : a^+(x) = a^+(x')$

proof Since U_x and $U_{x'}$ have full measure they intersect so
(*) $\exists\ (\tilde y, \tilde t) \in U_x \cap U_{x'}$. Then

$$
\begin{aligned}
a^+(x) &= f^+(x,\tilde y,\tilde t) && \text{since } (x,\tilde y,\tilde t) \in \Omega\\
&= f^-(x,\tilde y,\tilde t) \\
&= f^-(x',\tilde y,\tilde t) && \text{by } (1^u)\\
&= f^+(x',\tilde y,\tilde t) && (x',\tilde y,\tilde t) \in \Omega\\
&= a^+(x')
\end{aligned}
$$

__Prop__ $f^+(x,y,t) = f^-(x,y,t) = \text{const}$ a.e. on B.

__proof__: On $A_+ \times I_- \times J$: f^+ is const. Similarly there is $A_- \subset I_-$ of full meas so f^- is const on $A_+ \times A_- \times J$ and they are equal on a full measure set.



"Good local coordinates"

$a^+(x)$ depends only upon unstable coor.

~~This area~~ ~~On the compact surface~~

__Theorem__ ~~g_t is~~
g_t is ergodic with respect to Liouville measure on S_1M. (when M is a compact quotient of $\mathbb{D}$ (finite vol suffices))

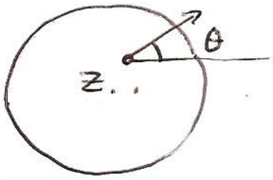
__proof__ [Hopf, 39]
Given $f \in L^1(S_1M, \mu)$
~~Let~~ approximate f by a cont. fct (whence unif cont fct) on S_1M.
Birkhoff $\Rightarrow$ f^+ and f^- $\exists$ a.e. and are equal.
By coroll ___
f^+ and f^- are const along stable and unst. mflds respectively.
By prop ___ f^+ and f^- are locally a.e const whence a.e const on M.
(the const $= \int_M f\,d\mu$)
So g_t is ergodic.

Complement

Unit tangent bundle

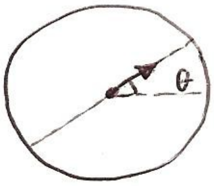
$S_1\mathbb{D} \cong \{(z,\theta) : z \in \mathbb{D},\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$

we associate to (z,θ) a tangent vector in $T_z\mathbb{D}$ $\quad v = v_{z,\theta}$



$v = e^{i\theta}\,\frac{1-|z|^2}{2}$ which makes $|v|_{\mathbb{D}} = 1$

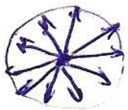
$\pi : S_1\mathbb{D} \to \mathbb{D}$
$(z,\theta) \mapsto z$ natural projection



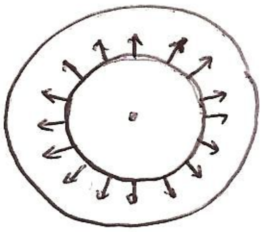
geodesic flow at 0

$g_t(0,\theta) = (e^{i\theta}\,\mathrm{th}\,\frac{t}{2},\ \theta)$

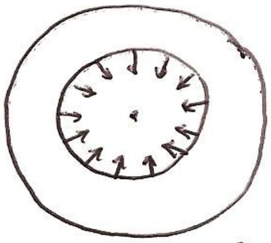
Horocircles: Let $\hat{0} = \pi^{-1}(0) = \{(0,\theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ be the unit vectors based at the origin in $\mathbb{D}$.



$\pi^{-1}(0)$



$g_t(\hat{0})$ for $t>0$



$g_t(\hat{0})$ for $t<0$

Hypebolic set. Complement

~~Def~~ M smooth manifold $U \subset M$ open subset, $T : U \to M$ a C^1-diffeomorphism onto its image and $\Lambda \subset U$ a compact T-invariant set.

$$T(\Lambda) = \Lambda.$$

Def Λ is called a

1) hyperbolic set for T if there is a Riemannian metric on U (open nghd of Λ) and $\theta < 1$ such that DT admits a hyperbolic θ-splitting, i.e. $\forall x \in \Lambda$ we have $T_x \cancel{E}_x = E^+_{x} \oplus E^-_x$ ~~and~~ with

$$\| DT_{|E^-_x} \| \le \theta$$

$$\| DT^{-1}_{|E^+_x} \| \le \theta$$

and $DT_x E^\pm_x = E^\pm_{T(x)}$

~~and E^+_x, E^-_x depends continuously upon x~~

2) The hyperbolic set is locally maximal or basic if $\exists$ open nghd U of Λ s.t.

$$\bigcap_{n \in \mathbb{Z}} f^n U = \Lambda$$

3) The diffeo T is called ANOSOV if $\Lambda = M$.

Lemma: E^+_x and E^-_x depends continuously on x. ~~and~~

$\dim E^+_x$ and $\dim E^-_x$ are locally constant.