9. Conjugacy and the "small divisor" problem.

Intro: $R_\theta : S^1 \to S^1$, $\theta \in \mathbb{R}$ (mod $\mathbb{Z}$)
$z \mapsto z+\theta \mod \mathbb{Z}$
is the rigid rotation.
Its orbit is dense for $\theta \in \mathbb{R}\setminus\mathbb{Q}$

Let $f : S^1 \to S^1$ be an orient. preserving homeo, $f \in \mathrm{homeo}_+(S^1)$.

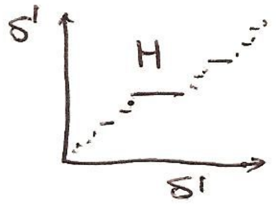
Def[9.1]: f is said to be

A. semi conjugated to R_θ ($\theta \in \mathbb{R}\setminus\mathbb{Q}$) iff $\exists H : S^1 \to S^1$ a continuous surjective map (orient pres.) such that
$$H \circ f = R_\theta \circ H$$

B. conjugated to R_θ if in addition $H \in \mathrm{homeo}_+(S^1)$

C. smoothly conjugated to R_θ if in addition $H \in \mathrm{Diff}^k_+(S^1)$ $(k \geq 1)$ or $H \in \mathrm{Diff}^\omega_+(S^1)$ (real-analytique).

Rem: In A. the map H need not be injective. It may be a "devil staircase"



Denjoy "counter example" $f \in \mathrm{Diff}^1(S^1)$

B. If $f \in \mathrm{Diff}^2(S^1)$ then (Denjoy) $H \in \mathrm{homeo}_+(S^1)$

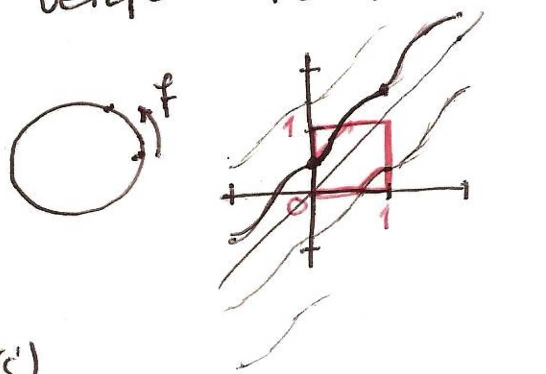
C. If $f \in \mathrm{Diff}^\omega(S^1)$ is close enough to R_θ then they are smoothly conjugated (Arnold)

Rem:
C. is an example of the "small divisor" problem which also occurs in the
- K.A.M. - theory,
- Siegel's thm (cmplx dyn.)
- twist map

FACT (topology)
$S^1 = \mathbb{R}/\mathbb{Z}$. $\mathbb{R} \xrightarrow[\text{nat.}]{\pi} \mathbb{R}/\mathbb{Z}$

Any $f \in \mathrm{homeo}_+(S^1)$ admits a lift to $\mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$ with $F \in \mathrm{homeo}_+(\mathbb{R})$
$$\pi \circ F = f \circ \pi.$$

F is unique up to an additive integer constant and verifies $F(x+1) = F(x) + 1$.

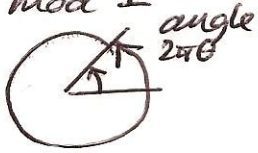


Circle homeomorphisms

$S^1 = \mathbb{R}/\mathbb{Z}$

Recall: $R_\theta(x) = x+\theta \mod 1$

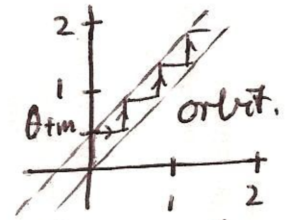
$R_\theta : S^1 \circlearrowleft$



Lifting to $\mathbb{R}$:

$\hat{R}_\theta(x) = x+\theta+m$ ($m \in \mathbb{Z}$ arbitr.)

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\hat{R}_\theta"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "R_\theta"'] & S^1 \end{tikzcd}$$



$\hat{R}_\theta(x+1) = \hat{R}_\theta(x)+1$

More generally

Let $f: S^1 \to S^1$ be a homeo$^+$, i.e. an orientation preserving homeomorphism of S^1,

We may lift f to a homeo$^+$ of the real line

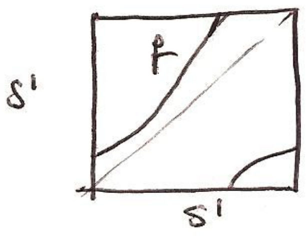
$F: \mathbb{R} \to \mathbb{R}$ ($\text{homeo}^+(\mathbb{R})$)

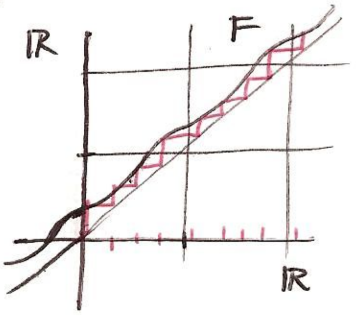
such that

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$F(x+1) = F(x)+1$ ← degree of the map f.

F is unique up to an additive integer const





Thm 9.2 [Rotation number]
Let $f \in \text{homeo}^+(S^1)$ and $F: \mathbb{R} \to \mathbb{R}$ a lift of f.
Then

$$\tau(F) := \lim_{n\to\infty} \frac{1}{n}(F^n(x) - x)$$

exists for all $x \in \mathbb{R}$ and is indep of x.
It is unique mod $\mathbb{Z}$.
The value $\tau(F) \mod \mathbb{Z}$ is called the rotation number of f.

ex: $\theta \pmod 1$ is the rotation number of R_θ

$R_\theta^n(x) = x+\theta n$ so

$\frac{1}{n}R_\theta^n(x) = \frac{x}{n}+\theta \xrightarrow[n\to+\infty]{} \theta$

Proof:

We look at the "Increment"

$$\Delta(x) = F(x) - x \quad \text{and}$$
$$\Delta_n(x) = F^n(x) - x, \quad n \in \mathbb{Z}.$$

Since
$$\Delta_n(x+1) = F^n(x+1) - (x+1)$$
$$= F^n(x) + 1 - x - 1$$
$$= \Delta_n(x)$$

Δ_n is 1-periodic and takes its max and min values

Def: $a_n = \min_{S^1} \Delta_n, \quad b_n = \max_{S^1} \Delta_n$

By monotonicity of F (and F^n)

$x \le y < x+1 \Rightarrow F^n(x) \le F^n(y) < F^n(x)+1$

so $\Delta_n(x) - 1 < \Delta_n(y) < \Delta_n(x) + 1$

Thus,

$$\boxed{\begin{array}{l} \forall x, y \in \mathbb{R} \quad |\Delta_n(x) - \Delta_n(y)| < 1. \\ \text{in particular} \quad 0 \le b_n - a_n < 1 \end{array}}$$

By continuity of Δ_n:

$$\boxed{\Delta_n(S^1) = [a_n, b_n]}$$

(surjectivity)

Co-cycle property:

$$\Delta_{n+m}(x) = F^n(F^m x) - x$$
$$= F^n(F^m x) - F^m x + F^m x - x$$
$$= \Delta_n \circ F^m(x) + \Delta_m(x)$$

$$\boxed{\Delta_{n+m} = \Delta_n \circ F^m + \Delta_m}$$

Taking min and max over S^1:

$$b_{n+m} \le b_n + b_m \quad \text{(subadd)}$$
$$a_{n+m} \ge a_n + a_m \quad \text{(superadd)}$$

so by sub/super additivity

$$\lim_{n \to +\infty} \frac{1}{n} b_n = \inf_{n \ge 1} \frac{1}{n} b_n \quad \text{exists}$$
$$\lim_{n \to +\infty} \frac{1}{n} a_n = \sup \frac{1}{n} a_n \quad \text{exists}$$

and $0 \le \frac{1}{n}(b_n - a_n) \le \frac{1}{n} \xrightarrow[n \to +\infty]{} 0$

implies the existence of a unique number (rotation #)

$$\boxed{\begin{array}{l} \tau(F) = \lim \frac{1}{n} b_n = \inf \frac{1}{n} b_n \\ = \lim \frac{1}{n} a_n = \sup \frac{1}{n} a_n \end{array}}$$

which also verifies

$$\boxed{a_n \le n\tau(F) \le b_n}$$

so in particular

$$\boxed{n\tau(F) \in \Delta_n(S^1)}$$

Since $a_n \le \Delta_n(x) \le b_n$ we get $\forall x \in S^1$:

$$\lim \frac{1}{n} \Delta_n(x) = \lim \frac{1}{n} F^n(x)$$
$$= \tau(F).$$

If $\tilde{F}$ is another lift off then

$\tilde{F}(x) = F(x) + r$ (some $r \in \mathbb{Z}$)

so

$\tilde{F}^n(x) = F^n(x) + nr$

and $\tilde{\Delta}_n = \Delta_n + nr$ whence

$$\tau(\tilde{F}) = \tau(F) + r$$

so unique mod $\mathbb{Z}$ //

Remark:

Since
$$\Delta_n(x) = F^n(x) - x$$
$$= F^n(x) - F^{\pm n}(F^{-n} x)$$
$$= -\Delta_n(F^{-n} x)$$

we have

$$\Delta_{-n}(S^1) = -\Delta_n(S^1)$$

so $[a_{-n}, b_{-n}] = [-b_n, -a_n]$

Thm 9.3

(1) $\tau(F) \in \mathbb{Q}$ iff

(2) $\exists q \in \mathbb{Z}^+ : \Delta_q(S^1) \cap \mathbb{Z} \neq \emptyset$ iff

(3) f admits a periodic pt.

Proof: $(1) \Rightarrow (2)$ If $\tau(F) = \frac{p}{q} \in \mathbb{Q}$ then $p = q\tau(F) \in \Delta_q(S^1)$

$(2) \Rightarrow (3)$ If $p \in \Delta_q(S^1)$ then $\exists x : p = \Delta_q(x) = F^q x - x$ so $f^q(x) = x \mod \mathbb{Z}$.

$(3) \Rightarrow (1)$ $f^q(x) = x \mod \mathbb{Z} \Rightarrow$
$\Delta_q(x) = F^q(x) - x = p \in \mathbb{Z} \Rightarrow$
$\Delta_{kq}(x) = kp$ so
$\tau(F) = \lim_{k \to \infty} \frac{kp}{kq} = \frac{p}{q} \in \mathbb{Q}$

Remark: By contreposition:

$\tau(F)$ is irrational iff $\forall q \in \mathbb{Z}^+ : \Delta_q(S^1) \cap \mathbb{Z} = \emptyset$ iff f has no periodic points

Coroll 9.4

Suppose $\tau = \tau(F)$ is irrational. Then $\forall n, m \in \mathbb{Z}$:

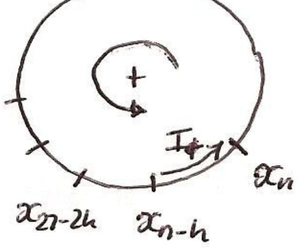
$n\tau > m$ iff $\forall x \in S^1 : \Delta_n(x) > m$

proof "$\Leftarrow$" $n\tau \in \Delta_n(S^1) > m$
"$\Rightarrow$" $m < n\tau \in \Delta_n(S^1)$ but $\Delta_n(S^1)$ is connected and disjoint from $\mathbb{Z}$ (since τ is supposed irrational) so $\forall x \in S^1 : \Delta_n(x) > m$.

$\tau(f) \in \mathbb{R}\setminus\mathbb{Q}$

Lemma 9.5 Let $x \in S^1$ and $I_+ = [x_m, x_n]_+$ for $m \neq n \in \mathbb{Z}$. Then $\forall y \in S^1$: $O_+(y)$ and $O_-(y)$ both meet I_+ ~~infinitely many times~~ (in fact infinitely often)

proof: Assume $n = m+h$, $h \geq 1$.



$I_1 = f^{-h} I = [x_{n-2h}, x_{n-h}]_+$ is adjacent to I_0 and

$I_{-k} = f^{-kh} I = [x_{n-(k+1)h}, x_{n-kh}]_+$ is adjacent to I_{k+1}

We claim that $\bigcup_{k \geq 1} I_{-k} = S^1$ (which shows ~~so~~ that $O_+(y)$ meet I_0).

If not x_{n-kh} must converge to some $z \in S^1$ with $f^h z = z$ which contradicts $\tau(f)$ irrational. //

Recall $\omega_+(x) = \bigcap_{n \geq 0} Cl\{f^k(x) : k \geq n\}$ $\omega_-(x)$ similarly.

Proposition 9.6

~~Let $x \in S^1$~~

For $x, y \in S^1$ we have $\omega_+(x) = \omega_-(x) = \omega_+(y) = \omega_-(y)$

The set $E = \omega_\pm(x)$ is either S^1 or a perfect nowhere dense subset of S^1.

Let $z \in \omega_+(x)$, i.e. $\exists\, n_i \to +\infty$ with $f^{n_i}(x) \to z$. Let $I_i = [x_{n_i}, x_{n_i+1}]_+$ be a shrinking seq to $\{z\}$

$O_-(f^{-i}y) \cap I_i \neq \emptyset$

By Lemma $\Rightarrow \exists$ $m_i \leq -i$ with $f^{m_i}(y) \in I_i$ so that $f^{m_i}(y) \to z$ as well. and $z \in \omega_-(y)$.

E is invariant and the only ~~a~~ minimal such set. since ~~it must contain~~ any minimal set must contain the closure of an orbit whence E. Thus, $\emptyset$ and E are the only closed invariant subsets of E. ∂E is a closed invariant subset of E whence $\partial E = \emptyset$ or $\partial E = E$

Let $x \in E$. Then as $E = \omega(x)$ there is a seq. $k_n \to +\infty$ s.t. $f^{k_n}(x) \to x$ and $f^{k_n}(x) \neq x$ (since $\tau(f)$ is irrational) So x is accumulation point of $E \setminus \{x\}$

Lemma Suppose $\tau = \rho(F) \in \mathbb{R}\setminus\mathbb{Q}$ Then for any $x \in S^1$, $m_1, m_2, n_1, n_2 \in \mathbb{Z}$:

$n_1\tau + m_1 < n_2\tau + m_2 \Leftrightarrow F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$

proof: writing $n = n_2 - n_1$, $m = m_2 - m_1$, $y = F^{n_1}(x)$ we have equivalently

$n\tau - m > 0 \Leftrightarrow$

$F^n(y) - m > y \Leftrightarrow$

$\Delta_n(y) - m > 0$ or

$\tau > \frac{m}{n} \Leftrightarrow \frac{1}{n}\Delta_n(y) > \frac{m}{n}$

(next page)

Lemma 9.7: Suppose $\tau(F) \in \mathbb{R}\setminus\mathbb{Q}$
Then $\forall n_1, n_2, m_1, m_2 \in \mathbb{Z}$:

$$n_1\tau + m_1 > n_2\tau + m_2 \iff$$
$$F^{n_1}(x) + m_1 > F^{n_2}(x) + m_2 \quad \forall x \in \mathbb{R}.$$

proof Setting $n = n_1 - n_2$, $m = m_1 - m_2$ the statement is equiv to

$$n\tau + m > 0 \iff$$
$$F^n(F^{n_2}x) - F^{n_2}x + m > 0 \quad \forall x \in \mathbb{R}$$

or $$n\tau + m > 0 \iff \forall y \in S^1: \Delta_n(y) + m > 0.$$

Since $\tau \notin \mathbb{Q}$ we have $\Delta_n(S^1) \cap \mathbb{Z} = \emptyset$ for $n \neq 0$. Also $\Delta_n(S^1)$ is connected and contains τ. So

$$\begin{aligned} n\tau + m > 0 &\Rightarrow \exists y \in S^1: \Delta_n(y) + m > 0 \\ &\iff \forall y \in S^1: \Delta_n(y) + m > 0 \\ &\Rightarrow n\tau + m > 0 \quad // \end{aligned}$$

9.8 Let $\theta = \rho(f) \in \mathbb{R}\setminus\mathbb{Q}$

Prop: Let E ~~[illegible]~~ be the minimal invariant subset of (S^1, F). ~~Then~~ There is a monotone continuous surjective map H: $\mathbb{R} \to \mathbb{R}$ such that
$$H \circ f = R_\theta \circ H$$

proof: Pick $x_0 \in E$. ~~and then Define~~

~~Φ [illegible]~~

Let $A = \{F^n(x_0) + m : n, m \in \mathbb{Z}\}$
The map
$$\Phi: A \longrightarrow \mathbb{R}$$
given by $\phi(F^n x + m) = n\theta + m$ is monotone (Lemma) and has dense image $\Rightarrow$ extends to $\bar\Phi: \bar{A} \to \mathbb{R}$ surjective.

On each open interval $I \subset \bar{A}^c$ we set ~~Φ [illegible]~~

$x \in$ ~~[illegible]~~ $H(x) =$
~~[illegible]~~ $= \sup_{\substack{u \in A \\ u < x}} \phi(u) = \inf_{\substack{v \in A \\ x < v}} \phi(v)$

$\Phi(F(F^n x + m)) =$
$\Phi(F^{n+1}x + m) =$
$\theta(n+1) + m =$
$\phi(F^n x + m) + \theta$
$\phi(F(y)) = \phi(y) + \theta$
$y = F^n x + m$

Smooth conjugation and small divisors

~~Assume: $f: S^1 \to S^1$ is real analytic with irrational rotation number $\theta \in \mathbb{R}\setminus\mathbb{Q}$.~~

Def 2.9 We say that θ is Diophantine of type (γ, α) if

$$(*) \quad \forall\, p, q \in \mathbb{Z}: \ |q\theta - p| \geq \frac{\gamma}{q^{2+\alpha}} > 0$$

Rem necessarily $\alpha \geq 0$ and $0 < \gamma \leq 1/2$.

θ is Diophantine if it is Dioph of type (γ, α) for some $0 < \gamma < 1/2$, $\alpha > 0$

Lemma 2.10: $(*)$ implies

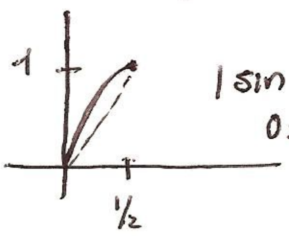
$$\forall\, q \in \mathbb{Z}: \ \frac{1}{|e^{2\pi i \theta q} - 1|} \leq \frac{q^{\alpha+2}}{4\gamma}.$$

"the small divisor" polynomial growth.

proof: Pick $p \in \mathbb{Z}$ with $|p - q\theta| < 1/2$.
Then

$$|e^{2\pi i \theta q} - 1| = |e^{2\pi i (\theta q - p)} - 1| =$$

$$2 \cdot |\sin(\pi(\theta q - p))| \geq 2 \cdot 2 |\theta q - p|$$

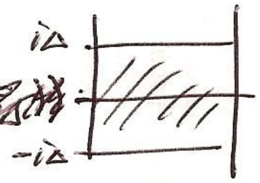


$|\sin \pi t| \geq 2t$ for $0 \leq t \leq 1/2$.

Theorem 9.11

Let f be a real analytic diffeo of the circle with rotation number $\theta = \tau(f)$ of Diophantine type (γ, α).

Suppose



$f : \mathcal{C}_\Delta \longrightarrow \mathbb{C}/\mathbb{Z}$

is the analytic extension of f to $\mathcal{C}_\Delta$ and

$$f(z) = z + \delta f(z)$$

Then there is $\varepsilon = \varepsilon(\gamma, \alpha, \Delta) > 0$ s.t. if

$$\| \delta f - \theta \|_\Delta \leq \varepsilon$$

then $\exists\, H \in \mathrm{Diff}^\omega_+(S^1)$ s.t.

$$(*) \quad H^{-1} \circ f \circ H = R_\theta : z \mapsto z + \theta$$

Steps in proof (Iterative) start $n=0$, $\Delta_0 = \Delta$ $f_0 = f$

I. Linearize equation $(*)$ on the domain $\mathcal{C}_{\Delta_n}$ and

II. Solve the linearized (cohomological) eqn on a smaller domain $\mathcal{C}_{\Delta_{n+1}}$

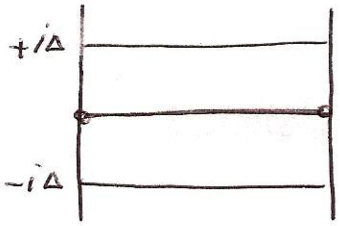
III. Conjugate f by the approximate solution

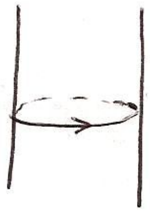
$$f_{n+1} = H_{n+1}^{-1} \circ f_n \circ H_{n+1}$$

IV. Show convergence of $\Delta_n \to \Delta' > 0$ and of $f_n \longrightarrow R_\theta$ on $\mathcal{C}_{\Delta'}$ and finally $H_n \circ \dots \circ H_1$ on $\mathcal{C}_{\Delta'}$

$$H = \lim H_n \circ \dots \circ H_1$$

solving $(*)$





$$\varphi(z) = \sum c_k e^{2\pi i k z}$$

$$A(\Delta) = \{ C^\omega(\mathcal{E}_\Delta) \cap C^0(\overline{\mathcal{E}}_\Delta)$$

$$\|\varphi\|_\Delta = \sup_{z \in \mathcal{E}_\Delta} |\varphi(z)|$$

<u>Lemma A1</u> $\forall z \in \mathcal{E}_\Delta$, $\alpha \in \mathbb{R}$

$$\varphi(z) = \sum c_k e^{2\pi i k z}$$

$$|c_k| \le \|\varphi\|_\Delta e^{-2\pi |k| \Delta}$$

$k \neq 0$ $\quad \|\varphi - \alpha\|_\Delta e^{-2\pi|k|\Delta}$

We even have for $k \neq 0$ and any $\alpha \in \mathbb{R}$

$$|c_k(\varphi)| \le \|\varphi - \alpha\|_\Delta e^{-2\pi |k| \Delta}$$

<u>Proof</u> φ admits a Fourier expansion on S^1, and

$$c_k = \oint_{S^1} \varphi(z) e^{-2\pi i k z} dz$$

(using Cauchy Theory) $= \oint_{\pm i\Delta + S^1} \varphi(z) e^{-2\pi i k z} dz$

choose $+i\Delta$ for $k < 0$,
$-i\Delta$ for $k > 0$.

$$|c_k| \le \|\varphi\|_\Delta \cdot e^{-2\pi |k| \Delta} \quad //$$

Lemma A2 For $0<\lambda<1, \beta\geq 0$:

$$\sum_{n\geq 0} n^\beta \lambda^n \leq \Gamma(\beta+1)\left(\frac{1}{1-\lambda}\right)^{\beta+1}$$

proof

$$\left(\frac{1}{1-\lambda}\right)^{\beta+1} = \sum_{n\geq 0} \binom{n+\beta}{n} \lambda^n = \sum \frac{\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)} \lambda^n$$

$$\geq \frac{1}{\Gamma(\beta+1)} \sum n^\beta \lambda^n$$

Lemma A3: For $\beta\geq 0$, $\delta>0$

$$\sum_{k\in\mathbb{Z}} |k|^\beta \exp(-2\pi|k|\delta) \leq 2\left(1+\frac{1}{2\pi\delta}\right)^{\beta+1}\Gamma(\beta+1)$$

For $t>0$ one has $e^t \geq 1+t \Leftrightarrow \frac{1}{1-e^{-t}} \leq \frac{1+t}{t}$

proof: $\text{LHS} \leq 2\sum_{k\geq 0} k^\beta \left(\frac{1}{e^{2\pi\delta}}\right)^k$

$$\leq 2\left(\frac{1}{1-e^{-2\pi\delta}}\right)^{\beta+1}\Gamma(\beta+1) \leq 2\cdot\left(1+\frac{1}{2\pi\delta}\right)^{\beta+1}\Gamma(\beta+1)$$

$$(1-\lambda)^{-(\beta+1)} = 1 + (\beta+1)\lambda + \frac{(\beta+1)(\beta+2)}{2}\lambda^2 + \dots$$

$$= 1 + \frac{(\beta+1)!}{\beta!\,1!}\lambda + \frac{(\beta+2)!}{\beta!\cdot 2!}\lambda^2 + \frac{(\beta+3)!}{\beta!\,3!}\lambda^3 + \dots$$

When $\beta\in\mathbb{N}$

$$\frac{(\beta+n)!}{n!} = (n+1)(n+2)\cdots(n+\beta) \geq n^\beta \quad \text{more generally} \quad \frac{\Gamma(\beta+n+1)}{\Gamma(n+1)} \geq n^\beta$$

$$\beta!\,(1-\lambda)^{-(\beta+1)} = \sum_{n\geq 0}\frac{(\beta+n)!}{n!}\lambda^n \geq \sum_{n\geq 0} n^\beta \lambda^n$$

So $\varphi' = \sum 2\pi i k c_k e^{2\pi i k z}$ on $\mathcal{C}_{\Delta-\delta}$

$$|\varphi'(z)| \leq \sum 2\pi|k|\,|c_k|\, e^{2\pi|k|(\Delta-\delta)}$$

(any $\alpha\in\mathbb{C}$)

$$\leq \|\varphi-\alpha\|_\Delta \sum 2\pi|k|\, e^{-2\pi|k|\delta}$$

$$\leq \|\varphi-\alpha\|_\Delta\, 4\pi\left(1+\frac{1}{2\pi\delta}\right)^2\cdot 1$$

$$\|\delta f'\|_{\Delta-\delta} \leq \|\delta f - 0\|_\Delta \cdot 4\pi\left(1+\frac{1}{2\pi\delta}\right)^2 \leq \varepsilon\cdot \delta^{-2}$$

I $\quad H(z) = z + h(z) \qquad f(z) = z + \delta f(z) \qquad H\circ R_\theta = f\circ H = H + \delta f\circ H$

$z+\theta+h(z+\theta) = z + h(z) + \delta f\circ H(z)$

Non-linear cohom eqn:

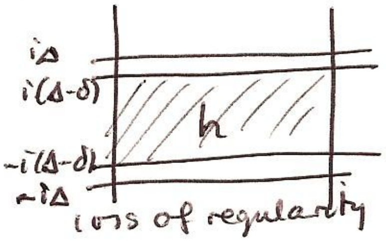
$h(z+\theta) - h(z) = -\theta + \delta f\circ H(z) \qquad 0 = \oint h(z+\theta) - h(z)\, dz$

$= -\theta + \oint \delta f\circ H(z)\, dz$

Linearized cohomological eqn: ↙ to ensure that $\oint RHS = 0$ (since $\oint LHS = 0$)

$h(z+\theta) - h(z) = \delta f(z) - \oint \delta f(z)\, dz$

$k \neq 0: \quad c_k(h)\,(e^{2\pi i k\theta} - 1) = c_k(\delta f)$ ~~solut~~ so



II solution $\boxed{c_k(h) = \dfrac{c_k(\delta f)}{e^{2\pi i k\theta} - 1}}$ ← small divisor

(Lemma)

$|c_k(h)| \leq |c_k(\delta f)| \dfrac{|k|^{2+\alpha}}{4\gamma} \leq 2\epsilon \dfrac{|k|^{2+\alpha}}{4\gamma} \cdot e^{-2\pi|k|\Delta}$

A.4

Recover regularity by shrinking the domain

For $z \in \mathcal{C}(\Delta-\delta)$, $0<\delta<\Delta$:

$|h(z)| \leq \sum |c_k(h)|\, e^{2\pi|k|(\Delta-\delta)} \leq \sum \frac{2\epsilon}{4\gamma} |k|^{2+\alpha} e^{-2\pi|k|\delta}$

$|h'(z)| \leq \frac{4\pi\epsilon}{4\gamma} \sum |k|^{3+\alpha} e^{-2\pi|k|\delta}$

(Lemma A3) $\leq \frac{4\pi\epsilon}{4\gamma} \cdot 2 \cdot \Gamma(4+\alpha) \cdot \left(1 + \frac{1}{2\pi\delta}\right)^{4+\alpha} \qquad \leq \left(\frac{2}{2\pi\delta}\right)^{4+\alpha} (2\pi\delta < 1)$

$(2\pi\delta<1) \quad \leq \frac{2\pi}{4\gamma}\Gamma(4+\alpha)\left(\frac{1}{\pi}\right)^{4+\alpha} \epsilon \cdot \delta^{-(4+\alpha)}$

$= C_{\gamma,\alpha} \cdot \varepsilon \cdot \delta^{-(4+\alpha)} \qquad (1)$

To simplify: $|h(z)| \leq$ ~~same bound~~ $C_{\gamma,\alpha} \cdot \varepsilon\, \delta^{-(3+\alpha)} \qquad (2)$

Geometry

Need *1 $|h'(z)| < \frac{1}{\beta}$ (<1) for $H(z) = z + h(z)$ to be injective

*2 $|h(z)| < \delta$ so that $H(\mathcal{C}_{\Delta-\delta}) \supset \mathcal{C}_{\Delta-2\delta}$

and $H^{-1}: \mathcal{C}_{\Delta-2\delta} \to \mathcal{C}_{\Delta-\delta}$ is well-defined.

Suffices that $\boxed{C_{\gamma,\alpha}\, \varepsilon\, \delta^{-(4+\alpha)} \leq \frac{1}{\beta}}$

then $|h'| < 1/\beta$ and $|h| < \delta/\beta$

Choose $\beta \geq 2\alpha$ (for ex)

$|h'(z)| \leq 1/2$ for $z \in \mathcal{E}_{\Delta-\delta}$

provided $\quad C_{\delta,\alpha}\, \varepsilon\, \delta^{-(4+\alpha)} \leq \frac{1}{4}\beta \leq \frac{1}{2}$

(gives lower bd on δ as a fct of ε, β)

Then $\quad H(z) = z + h(z)$, $\frac{3}{2} \geq |H'| \geq 1/2$

and the local inverse verifies $\frac{2}{3} \leq |(H^{-1})'| \leq 2$

Step III $\quad$ ~~$H(\mathcal{E}_{\Delta-\delta}) \supset \mathcal{E}_{\Delta-2\delta}$~~

$$\mathcal{E}_{\Delta-2\delta} \subset H(\mathcal{E}_{\Delta-\delta}) \subset \mathcal{E}_{\Delta}$$

$$H(\mathcal{E}_{\Delta-2\delta}) \subset \mathcal{E}_{\Delta-\delta}$$

$$\tilde{f} : \mathcal{E}_{\Delta-4\delta} \xrightarrow{H} \mathcal{E}_{\Delta-3\delta} \xrightarrow{f} \mathcal{E}_{\Delta-2\delta} \xrightarrow{H^{-1}} \mathcal{E}_{\Delta-\delta}$$

$\tilde{f} = H^{-1} \circ f \circ H \; : \; \mathcal{E}_{\Delta-4\delta} \to \mathcal{E}_{\Delta-\delta}$ is well-defined

$H \circ \tilde{f} = f \circ H$ is valid $\forall z \in \mathcal{E}_{\Delta-4\delta}$

$\quad = H + \delta f \circ H$

$$\begin{aligned} H\circ\tilde{f}(z) &= z + h(z) + \delta f(z) + (\delta f \circ H - \delta f) \\ &= \underline{z + h(z) + \delta f(z) - \oint \delta f} + (\delta f \circ H - \delta f) + \oint \delta f \\ &= z + h(z+\theta) + (\delta f \circ H - \delta f) + \oint \delta f \\ &= \underbrace{z+\theta + h(z+\theta)}_{H\circ R_\theta} + \underbrace{(\delta f \circ H - \delta f)}_{\text{small}} + \underbrace{(\oint \delta f - \theta)}_{\text{const}} \end{aligned}$$

so

$$H\circ(\tilde{f}(z)) = H(z+\theta) + \underbrace{(\delta f(z+h(z)) - \delta f(z))}_{\text{small}} + \underbrace{(\oint \delta f - \theta)}_{\text{const} = c}$$

$\leq$

(a priori not that small) $\leq \varepsilon$

A 13

$H(z) - H(\omega) = z - \omega + h(z) - h(\omega)$

~~H(z~~

$H(\omega+u) - H(\omega) = \cancel{z} u + h(\omega+u) - h(\omega)$

$|H(\omega+u) - H(\omega) \cancel{z} - u| \le \|h'\| \cdot \|u\| \le \frac{1}{\beta}|u|$

$H(\tilde{f}(z)) = H(z+\theta) + (\Delta(z)+c)$

$\tilde{f}(z) = z + \cancel{\theta} \, \delta\tilde{f}(z)$

$|\delta\tilde{f}(z) - (\cancel{\theta}\Delta(z)+c)| \le \frac{1}{\beta}(\|\Delta\| + |c|)$

$\frac{1}{\beta}(\varepsilon.\delta + \varepsilon)$

Here using rot number

~~Now~~ So $|\delta\tilde{f}(z) - (\theta+c)| \le (1+\frac{1}{\beta})\|\Delta\| + \frac{1}{\beta}|c|$

$\boxed{\text{But } \tau(f) = \theta = \tau(\tilde{f})}$ since f and $\tilde{f}$ are conjugated $\exists z: \delta\tilde{f}(z) = \theta \Rightarrow$

$|c| \le (1+\frac{1}{\beta})\|\Delta\| + \frac{1}{\beta}|c|$

$(1-\frac{1}{\beta})|c| \le (1+1/\beta)\|\Delta\| \Rightarrow$

$|c| \le \frac{1+1/\beta}{1-1/\beta}\|\Delta\| \le 3\|\Delta\|$

So $|\delta\tilde{f}(z) - \theta| \le (1+\frac{1}{\beta})(\|\Delta\|+|c|) \le \frac{3}{2} \cdot 4\|\Delta\| = 6\|\Delta\|$

$\le 6\|\delta f'\| \cdot \|h\|$

$\le 6 \cdot \varepsilon . \delta^{-2} \cdot C_{\gamma\alpha} \cdot \varepsilon . \delta^{-(4+\alpha)}$

$= 6C_{\gamma\alpha} \cdot \varepsilon^2 \delta^{-(6+\alpha)}$

$\beta = 1/\sqrt{\varepsilon}$ $(\cancel{\le 1/2}) \ge 2$

<u>choice</u> ~~$\le \frac{1}{2}\varepsilon$~~ ~~for ε small enough and $\beta \le 1/\sqrt{\varepsilon}$~~

$\le \text{const}\, \varepsilon^{1+1/8 4}$ $\quad \delta = \left(\frac{1}{\beta C_{\gamma,\alpha}}\varepsilon^{1/2}\right)^{\frac{1}{4+\alpha}}$

$\le 1/2\, \varepsilon$ for ε small enough. $\quad = C_{\gamma,\alpha}^{\frac{1}{4+\alpha}} \text{const}\, \varepsilon^{\frac{1}{8+2\alpha}}$

Iteration : $f_0 = f \quad \Delta > 0 \quad \varepsilon_n = \frac{1}{2^n}\varepsilon_0 \quad \varepsilon_0 = \|\delta f - \theta\|_{\Delta_0}$

$\cancel{\varepsilon}\ \cancel{\delta}\ \delta_n = c_{\delta,\alpha}^{\frac{1}{4+\alpha}} \cdot \varepsilon_n^{\frac{1}{8+\alpha}} = \delta_0 \cdot 2^{-n/8+\alpha}$

$$\sum_{n\geq 0} \delta_n = \delta_0 \sum 2^{\frac{-n}{8+\alpha}} = C_\alpha \cdot \delta_0(\varepsilon) \quad \Delta_n = \Delta_{n-1} - 4\delta_n$$

$$H_n(z) \cancel{k} = z + h_n(z)$$

$$f_{n+1}(z) = H_n^{-1} \circ f_n \circ H_n(z)$$

$$= (H_n \circ \dots \circ H_1)^{-1} \circ f_0 \circ H_1$$

$$\boxed{\text{Need } \Delta_\infty = \lim \Delta_n = \Delta_0 - 4C_\alpha \delta_0(\varepsilon) > 0}$$

True for ε small enough

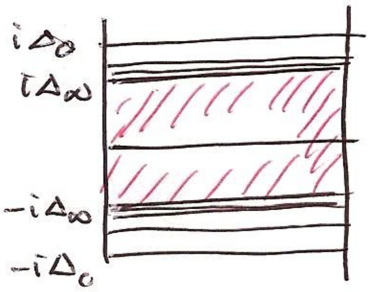
$$f_1 = H_1^{-1} \circ f_0 \circ H_1$$

$$f_2 = H_2^{-1} \circ f_1 \circ H_2 = H_2^{-1} H_1^{-1} \circ f_0 \circ H_1 \circ H_2$$

$$\vdots$$

$$f_n = (H_1 \circ \dots \circ H_n)^{-1} \circ f_0 \circ (H_1 \circ \dots \circ H_n)$$

exo : Show that $H = \lim(H_1 \circ \dots \circ H_n) : \mathcal{E}_{\Delta_\infty} \to \mathbb{C}/\mathbb{Z}$ exists and is analytic



Then as ~~$\|f_n(z) - z(z+\theta)\|_{\Delta_n}$~~

~~$\|f_n(z) - z(z+\theta)\|$~~

$$\|f_n - R_\theta\|_{\Delta_\infty} \leq \|f_n - R_\theta\|_{\Delta_n} \xrightarrow{n} 0$$

we get

$$R_\theta = H^{-1} \circ f_0 \circ H \text{ on } \mathcal{E}(\Delta_\infty) \text{ as desired}$$

Better:
$$H_1 \circ f_1 = f_0 \circ H_1 \quad \text{on } \mathcal{E}(\Delta_1)$$
$$H_2 \circ f_2 = f_1 \circ H_2 \quad \text{on } \mathcal{E}(\Delta_2)$$
$$H_3 \circ f_3 = f_2 \circ H_3 \quad \text{on } \mathcal{E}(\Delta_3)$$
$$\vdots$$

$$H_n \circ f_n = f_{n-1} \circ H_n \quad \text{on } \mathcal{E}(\Delta_n)$$

$$H_{n-1} \circ H_n \circ f_n = f_{n-2} \circ H_{n-1} \circ H_n \quad \text{on } \mathcal{E}(\Delta_n)$$

$(H_1 \circ \dots \circ H_n) \circ f_n = f_0 \circ (H_1 \circ \dots \circ H_n)$ on $\mathcal{E}(\Delta_n)$ whence also $\mathcal{E}(\Delta_\infty)$

$$\downarrow$$

$$H \circ R_\theta = f_0 \circ H \quad \text{on } \mathcal{E}(\Delta_\infty)$$