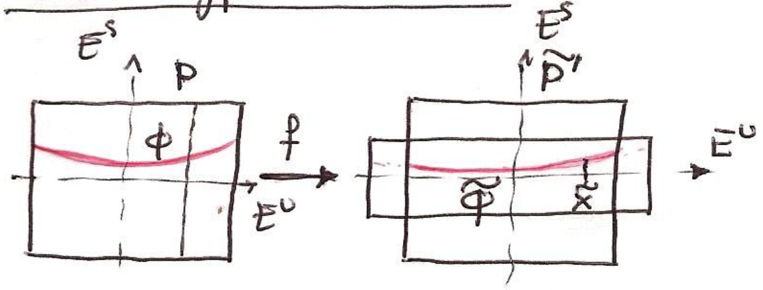
7. Unstable/stable Manifold Thm

Linear hyperbolic case



$P = D^u(R^u) \times D^s(R^s) = D^u \times D^s$

$\tilde{P} = \tilde{D}^u(\tilde{R}^u) \times \tilde{D}^s(\tilde{R}^s) = \tilde{D}^u \times \tilde{D}^s$

$E = E^u \times E^s$

e.g.

$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$

$$\left(\begin{array}{c|c} 2 & 0 \\ \hline 0 & 1/2 \end{array}\right) \qquad \underbrace{\left(\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & 1/2 & 0 \end{array}\right)}_{\Lambda_1,\ \Lambda_2}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Lambda_1 x \\ \Lambda_2 y \end{pmatrix} \qquad \Lambda_1 \in L(E^u), \quad \Lambda_2 \in L(E^s)$$

$K^u(P) = \{\phi : D^u \to D^s\}$

$y = \phi(x)$ $\quad \phi$ cont. fct. $\phi : D^u \to D^s$

Define Graph transform:

$$\begin{pmatrix} \tilde{x} \\ \tilde{\phi}(\tilde{x}) \end{pmatrix} := f\begin{pmatrix} x \\ \phi(x) \end{pmatrix} = \begin{pmatrix} \Lambda_1 x \\ \Lambda_2 \phi(x) \end{pmatrix}$$

$$x = \Lambda_1^{-1} \tilde{x}$$

$$\Gamma^u(\phi)(\tilde{x}) = \tilde{\phi}(\tilde{x}) = \Lambda_2 \phi(x) = \Lambda_2 \phi(\Lambda_1^{-1} \tilde{x})$$

Suppose $\|\Lambda_1^{-1}\| \le \theta_1 < 1$, $\quad P = \tilde{P}$

$\|\Lambda_2\| \le \theta_2 < 1$.

Then $(\Gamma^u)^{(n)}(\phi)(\tilde{x}) = \Lambda_2^n \phi(\Lambda_1^{-n} \tilde{x}) \longrightarrow 0$

uniformly in $\tilde{x}$

$\phi^*(x) \equiv 0 \quad \forall x \in D^u$

is an invariant manifold (subspace in this linear case)

$$W^u_{loc}(0) = \{(x, \phi^*(x)) : x \in D^u\} = \{(x,0) : x \in D^u\}$$

is the local unstable manifold for f at 0.

Γ^u contracts K^u

If f is invertible (Λ_2)

$$f^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Lambda_1^{-1} x \\ \Lambda_2 y \end{pmatrix}$$

one considers

$K^s = \{\phi : D^s \to D^u\}$

$\Gamma^s(\phi)(\tilde{y}) = \Lambda_1^{-1} \phi(\Lambda_2 y)$

which contracts K^s

$$W^s_{loc}(0) = \{(\phi^*(y), y) : y \in D^s\} = \{(0, y) : y \in D\}$$

local stable mfld at 0 for f.

Non-linear hyperbolic case

E Banach space direct prod

$E = E^u \times E^s$

$|(x,y)|_E = |x|_{E^u} + |y|_{E^s}$

Given $\xi_0 = (x_0, y_0)$ and $R > 0$ we consider the product of R-balls centered at ξ_0

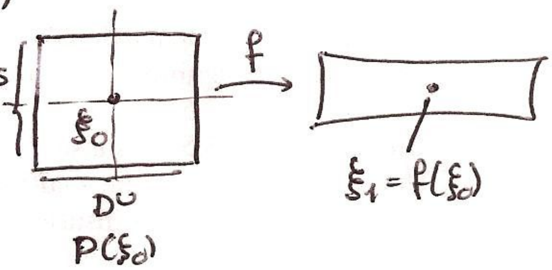
$$\boxed{P_R(\xi_0) = \underbrace{B_{E^u}(x_0, R)}_{D^u} \times \underbrace{B_{E^s}(y_0, R)}_{D^s}}$$

Def 7.1 Fix $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and $0 < \alpha < \min\{\frac{1-\theta_1}{4}, \frac{1-\theta_2}{4}\}$. We consider the class of maps

$\boxed{f \in \mathcal{F} = \mathcal{F}(\xi_0, \theta_1, \theta_2, \alpha)} \subset C^1(P_R(\xi_0), E)$

such that for $\xi = \xi_0 + (x,y)$

$$f(\xi) = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \delta f(x,y) + f(\xi_0)$$



where

$\Lambda_1 \in GL(E^u)$ $\quad \|\Lambda_1^{-1}\| < \theta_1$

$\Lambda_2 \in L(E^s)$ $\quad \|\Lambda_2\| < \theta_2$

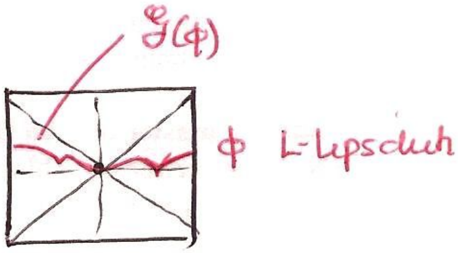
$\delta f(0,0) = 0$

and $\|\delta f'\|_{P(\xi_0)} = \sup_{\xi \in P(\xi_0)} \|\delta f'(\xi)\|_{L(E)} \leq \alpha$.

← NB δf is then α-Lipschitz on $P(\xi_0)$

Def 7.2 For $L < \infty$ and a product ball $P(\xi_0) = \underbrace{B_{E^u}(x_0, R)}_{D^u} \times \underbrace{B_{E^s}(y_0, R)}_{D^s}$ we define

$$K_L(\xi_0) = \{\phi : D^u \to D^s \text{ such that } \mathrm{Lip}(\phi) \leq L,\ \phi(x_0) = y_0\}$$



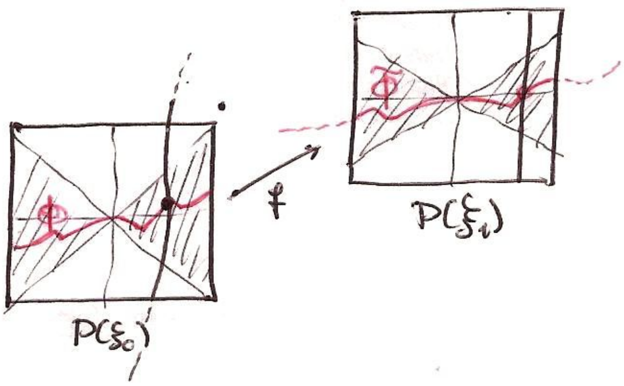
We call

$$\mathcal{G}_{P(\xi_0)}(\phi) = \{(x, \phi(x)) : x \in D^u\} \subset P(\xi_0)$$

the graph of $\phi \in K_L(\xi_0)$

~~Def Prop~~ Theorem 7.3

A. Let $f \in \mathcal{F}(\xi_0, \theta_1, \theta_2, \alpha)$ (Def) and Set $\xi_1 = f(\xi_0)$. Then for every $\phi \in K_{L=1}(\xi_0)$ the image of the graph of ϕ, $\mathcal{G}(\phi)$ under f intersects $P_1 = P(\xi_1)$ in a set which is itself a graph of a function $\Phi = \Gamma(\phi) \in K_{\theta_1}(\xi_1)$ ← recall $\theta_1 < 1$



Thus
$P(\xi_1) \cap f(\mathcal{G}_{P_{\xi_0}}(\phi)) = \mathcal{G}_{P_{\xi_1}}(\Phi)$

We call $\Gamma : K_1(\xi_0) \to K_{\theta_1}(\xi_1)$ the graph transform induced by f.

B. The graph transform is a contraction in the uniform norm: $\exists \eta < 1$:

$\forall \phi_1, \phi_2 \in K_{L=1}(\xi_0)$:

$\|\Gamma(\phi_1) - \Gamma(\phi_2)\|_{\infty} \leq \eta \|\phi_1 - \phi_2\|_\infty$

* L1

$\mathrm{Lip}\,\phi \le L \le 3$ here

1. Solving $\tilde{x} = f_1(x, \phi(x))$ $\qquad \tilde{x} \in D_0(\tfrac{R}{8})$

$f_1(x,\phi(x)) = \Lambda_1 x + \delta f_1(x,\phi(x))$ $\qquad \delta f_1$ is α-Lipschitz. $\quad \alpha \le \frac{1}{4}(1-\theta_1)$

so $\tilde{x} = f_1(x,\phi(x)) \Leftrightarrow$

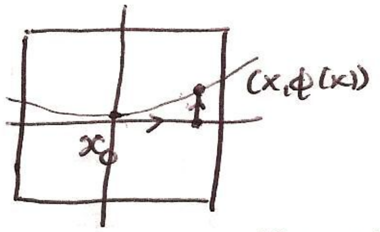
$x = \Lambda_1^{-1}\tilde{x} - \Lambda_1^{-1}\delta f_1(x,\phi(x))$ $\qquad \delta f_1(0,0) = 0$

$=: g(x) \qquad (\tilde{x},\phi)$ fixed.

$\mathrm{Lip}\, g \le \theta_1 \cdot \alpha \cdot (1 + \mathrm{Lip}\,\phi)$

$\le \theta_1 \cdot \alpha \cdot 4 \le \theta_1(1-\theta_1) < 1$

so a contraction



$|\delta f_1(x,\phi(x)) - \delta f_1(0,0)|$
$\le \alpha \cdot (R + LR)$

$|g(x)| \le \theta_1 \cdot R + \theta_1 \cdot \alpha(1+L)R$

$\le [\theta_1 + \theta_1(1-\theta_1)]R$

$= (1 - (1-\theta_1)^2)R < R$

so preserves $D_0(x_0, R)$

Contracting map principle

$\Rightarrow$ unique fixed point

$x = I(\tilde{x}, \phi)$

verifying

$x = \Lambda_1^{-1}\tilde{x} - \Lambda_1^{-1}\delta f_1(x,\phi(x))$

2. Calculating the graph transform

Given $\tilde{x} \in D_u(\xi_1, R_\varphi)$ compute $x = I(\tilde{x}, \varphi)$ and then

$$\tilde{y} = f_2(x, \phi(x)) = \Lambda_2 \phi(x) + \delta f_2(x, \phi(x))$$

We have

$$
\begin{aligned}
|\tilde{y}| &\le \theta_2 |\phi(x)| + \alpha \cdot (1+L) R \\
&\le \theta_2 L R + \alpha (1+L) R \\
&\le (\theta_2 + \alpha \cdot 4) R \qquad \leftarrow \text{assuming } L \le 3 \\
&\le \left(\theta_2 + \tfrac{1}{4}(1-\theta_2)\right) R \\
&= \tfrac{1}{2} \text{[illegible]} R
\end{aligned}
$$

so $\tilde{y} \in D_s(\xi_1)$

We define $\tilde{\Phi}(\tilde{x}) := \tilde{y} = f_2(x, \phi(x))$

$\underline{L = 1}$

3. Lipschitz cont of $\tilde{\Phi}$:

Let $\tilde{x}_1, \tilde{x}_2 \in D_u$, $x_1 = I(\tilde{x}_1, \varphi)$, $x_2 = I(\tilde{x}_2, \varphi)$

Write $\Delta x = |x_1 - x_2|$, $\Delta \tilde{x} = |\tilde{x}_1 - \tilde{x}_2|$. Then

$$
\begin{aligned}
\Delta x &\le |\Lambda_1^{-1}(\tilde{x}_1 - \tilde{x}_2)| + |\Lambda_1^{-1}(\delta f(x_1, \varphi(x_1)) - \delta f(x_2, \varphi(x_2)))| \\
&\le \theta_1 \Delta\tilde{x} + \theta_1 \alpha \cdot (1+L) \Delta x
\end{aligned}
$$

Or $\Delta x \le (1 - \theta_1 \alpha (1+L))^{-1} \theta_1 \Delta \tilde{x}$ $\quad$ const $\left(\le \frac{2\theta_1}{1 - 2\theta_1 \frac{1-\theta_1}{4}} \le \frac{2\theta_1}{1+\theta+(1-\theta)^2} \le \frac{1+\theta_1}{2} \right)$

Then

$$
\begin{aligned}
|\tilde{\varphi}(\tilde{x}_1) - \tilde{\varphi}(\tilde{x}_2)| &= |f_2(x_1, \varphi(x_1)) - f_2(x_2, \varphi(x_2))| \\
&\le \theta_2 |\varphi(x_1) - \varphi(x_2)| + \alpha(1+L)\Delta x \\
&\le (\theta_2 \cdot L + \theta_1 \alpha (1+L))(1 - \theta_1 \alpha(1+L))^{-1} \theta_1 \Delta\tilde{x} \le
\end{aligned}
$$

$$L(\theta_2 + 2\alpha)\frac{1+\theta_1}{2}\Delta\tilde{x} \le \underbrace{\frac{1+\theta_2}{2} \cdot \frac{1+\theta_1}{2}}_{<1=L} \Delta\tilde{x}$$

so

$$
\begin{aligned}
\operatorname{Lip}\tilde{\varphi} &\le L \cdot \frac{\theta_1\theta_2 + \alpha(\frac{1}{L}+1)\theta_1}{1 - \theta_1 \alpha (1+L)} \\
&\le L \cdot \frac{\theta_1\theta_2 + \theta_1 \alpha (\frac{1}{L}+1)}{1 - \theta_1 \alpha (1+L)} \le L
\end{aligned}
$$

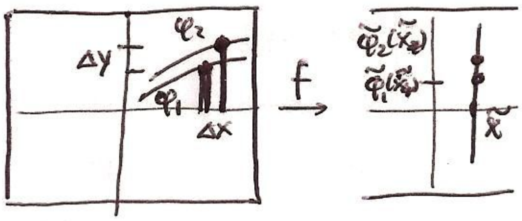
~~When~~ provided

$$
\begin{aligned}
& \theta_1\theta_2 + \theta_1\alpha\left(\tfrac{1}{L}+1\right) \le 1 - \theta_1\alpha(1+L) \\
\Leftrightarrow\ & \theta_1\theta_2 + \theta_1\alpha\left(\tfrac{1}{\sqrt{L}} + \sqrt{L}\right)^2 \le 1
\end{aligned}
$$

when $L = 1$: $\quad \theta_1\theta_2 + \theta_1 \alpha \cdot 4 \le \theta_1\theta_2 + (1-\theta_2)\theta_1 = \theta_1$

so $K_1 \to K_{\tilde{\theta}, 1} \subset K_1$ $\quad \tilde{\theta} = \frac{1+\theta_2}{2}\frac{1+\theta_1}{2} < 1$

4. Contraction in $\|\cdot\|_\infty$



Fix $\tilde{x} \in D_0$, $\varphi_1, \varphi_2 \in K_L$.
Write $\Delta\varphi = \|\varphi_1 - \varphi_2\|_\infty$ and set

$x_1 = I(\tilde{x}, \varphi_1)$, $x_2 = I(\tilde{x}, \varphi_2)$ $\quad \Delta x = |x_1 - x_2|$

$y_1 = \varphi_1(x_1)$ $\quad y_2 = \varphi_2(x_2)$ $\quad \Delta y = |y_1 - y_2|$

$$x_1 = \Lambda_1^{-1}(\tilde{x} - \delta f_1(x_1, y_1)) \qquad \Delta y \le |\varphi_1(x_1) - \varphi_1(x_2)| + |\varphi_1(x_2) - \varphi_2(x_2)|$$
$$x_2 = \Lambda_1^{-1}(\tilde{x} - \delta f_1(x_2, y_2)) \qquad \le L\Delta x + \Delta\varphi$$

$$\Delta x \le \theta\alpha(\Delta x + \Delta y)$$
$$\le \theta\alpha(1+L)\Delta x + \theta\alpha\Delta\varphi \text{ so}$$
$$\Delta x \le (1 - \theta\alpha(1+L))^{-1}\theta\alpha\Delta\varphi = \frac{\theta\alpha}{1-\theta\alpha(1+L)}\Delta\varphi$$
$$\Delta y \le ((1-\theta\alpha(1+L))^{-1}\theta\alpha L + 1)\Delta\varphi = \frac{1-\theta\alpha}{1-\theta\alpha(1+L)}\Delta\varphi$$

$$|\tilde{\Phi}_1(\tilde{x}) - \tilde{\Phi}_2(\tilde{x})| = |f_2(x_1, \varphi_1(x_1)) - f_2(x_2, \varphi_2(x_2))|$$
$$\le \theta_2\Delta y + \alpha(\Delta x + \Delta y)$$
$$\le \left(\theta_2 \frac{1-\theta_1\alpha}{1-\theta_1\alpha(1+L)} + \frac{\alpha}{1-\theta_1\alpha(1+L)}\right)\Delta\varphi$$
$$= \frac{\theta_2(1-\theta_1\alpha)+\alpha}{1-\theta_1\alpha(1+L)}\Delta\varphi$$

With $L \le 2$
$$\le \frac{\theta_2(1-\theta_1\alpha)+\alpha}{1-\theta_1\alpha\cdot 3}\Delta\varphi$$

and $\theta_2(1-\theta_1\alpha)+\alpha < 1 - \theta_1 3\alpha \Leftrightarrow$

$3\theta_1\alpha + \alpha + \theta_2(1-\theta_1\alpha) < 1$ or $\theta_2 + \alpha(1+3\theta_1 - \theta_1\theta_2) < 1$

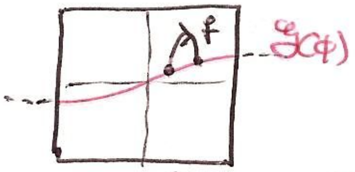
it suffices that
$$4\alpha + \theta_2 < 1$$
which is the case.

So $\|\Delta\tilde{\Phi}\| = \sup_{\tilde{x}\in D_0} |\tilde{\Phi}_1(\tilde{x}) - \tilde{\Phi}_2(\tilde{x})| \le \eta\,\Delta\varphi$
with $\eta < 1$.

Theorem 7.4

~~Let $f \in C^1$~~

Let $f \in \mathcal{F}(\xi_0, \theta_1, \theta_2, \alpha)$ as in Def and suppose that $f(\xi_0) = \xi_0$. Then $\exists!\ \phi^* \in C^1(D^u; D^s)$ ~~whose~~ with $\phi^*(0) = 0$ whose $\mathcal{G}_p(\phi)$ graph is f-invariant and such that f is uniformly expanding on $\mathcal{G}_p(\phi)$:



We call $\mathcal{G}_p(\phi) = W^u_{loc}(\xi_0)$ the local unstable manifold of f at ξ_0

Proof: As $\xi_0 = f(\xi_0)$ the graph transform maps $K_{L=1}(\xi_0)$ into $K_{L=\theta_1}(\xi_0)$ and is contracting in the $\|\cdot\|_\infty$ norm.

Let $\phi_{n+1} = \Gamma(\phi_n)$, $\phi_0 \in K_1(\xi_0)$ then ϕ_n OCD converges point wise $\forall x \in D^u$: $\phi_*(x) = \lim \phi_n(x)$

Now for $x, y \in D^u$: $\phi_*(0) = \lim \phi_n(0) = 0$,

$$|\phi_*(x) - \phi_*(y)| = \lim_n |\phi_n(x) - \phi_n(y)| \le \theta_1 |x - y|$$

Shows that $\phi_* \in K_{\theta_1}$ as well, and

$$f\, \mathcal{G}_p(\phi_*) \cap P = \mathcal{G}_p(\phi_*)$$

$$\boxed{|I(\tilde{x}, \phi_*)| \le \frac{1+\theta_1}{2} |\tilde{x}|}$$

Given $\tilde{x} \in D^u$ let $x = I(\tilde{x}, \phi_*)$. Then

$$|x| = |\Lambda_1^{-1}(\tilde{x} - f(x, \phi_*(x))| \quad L = 1$$

$$\le \theta_1 |\tilde{x}| + \alpha(1+\theta_1)\theta_1 |x| \text{ or}$$

$$|x| \le \frac{\theta_1}{1 - \alpha\theta_1 \cancel{(1+\theta_1)}2} |\tilde{x}| \le \frac{2\theta_1}{1+\theta_1} |\tilde{x}|$$

(when $\alpha < \frac{1-\theta_1}{4}$) and $\frac{2\theta_1}{1+\theta_1} < 1$

so $f(x, \phi_*(x)) = (\tilde{x}, \phi_*(\tilde{x}))$ implies $|\tilde{x}| \ge \frac{1+\theta_1}{2\theta_1} |x|$ uniform expansion

$$\left(1 - \frac{1-\theta_1}{1+\theta_1}\right) |\tilde{x}|$$

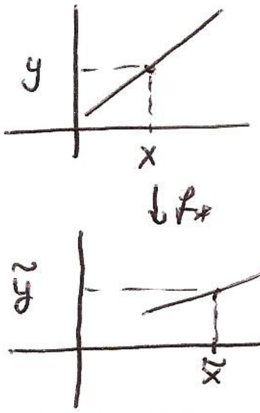
$$\frac{\theta_1}{1 - \theta_1 \frac{1-\theta_1}{2}} = \frac{2\theta_1}{2 - \theta_1 + \theta_1^2} = \frac{2\theta_1}{1 + \theta + (\theta - 1)^2} \le \frac{1+\theta_1}{2}$$

Write $Df(x,y) = \begin{pmatrix}F_{11}&F_{12}\\F_{21}&F_{22}\end{pmatrix} = \begin{pmatrix}\Lambda_1&0\\0&\Lambda_2\end{pmatrix} + \underbrace{\begin{pmatrix}\alpha_{11}&\alpha_{12}\\\alpha_{21}&\alpha_{22}\end{pmatrix}}_{\|\cdot\|\le\alpha<\min\left\{\frac{1-\theta_1}{4},\frac{1-\theta_2}{4}\right\}}$

Tangent plane map.

$DK_L = \{A\in L(E^u;E^s) : \|A\|\le L\}$ (tangent cone)

$A\in DK_L$ describes a tangent plane at (any) $(x,y)\in P$ given by $h\mapsto Ah$.



The image of this plane by $f_*(x,y)$ is $k\mapsto \tilde A k = R(A)k$ tangent plane at $(\tilde x,\tilde y)=f(x,y)$

We have $\begin{pmatrix}k\\\tilde A k\end{pmatrix} = Df\begin{pmatrix}h\\Ah\end{pmatrix} = \begin{pmatrix}F_{11}&F_{12}\\F_{21}&F_{22}\end{pmatrix}\begin{pmatrix}1\\A\end{pmatrix}h = \begin{pmatrix}U_1\\U_2\end{pmatrix}h$

where $U_1 = F_{11}+F_{12}A$
$U_2 = F_{21}+F_{22}A$

then $\tilde A = R(A) = U_2U_1^{-1}$

$\boxed{L=1}$ $\boxed{\|A\|\le 1}$

(L=1)

Now $k = U_1 h = \Lambda_1 h + (\alpha_{11}+\alpha_{12}A)h$

or $|h| = |\Lambda_1^{-1}(k-(\alpha_{11}+\alpha_{12}A))h| \le \theta_1|k| + \theta_1\alpha(1+L)|h| \le \theta_1|k|+2\alpha\theta_1|h|$

so $|h|\le (1-2\alpha\theta_1)^{-1}\theta_1|k|$ Since $\alpha<\frac{1-\theta_1}{4}$, $1-2\alpha\theta_1 \ge (1+\theta_1+(\theta_1-1)^2)/2$

so $|h|\le \frac{2\theta_1}{1+\theta_1}|k|$ $\le\frac{1+\theta_1}{2}$ or $\|U_1^{-1}\|\le \frac{2\theta_1}{1+\theta_1} = \left(1-\frac{1-\theta_1}{1+\theta_1}\right) < 1$ $\le\frac{1+\theta_1}{2}$

We have $\|U_2\|\le \theta_2\|A\|+\alpha(1+L) \le \theta_2\|A\| + \frac{1-\theta_2}{2} \le \frac{1+\theta_2}{2} < 1$

Thus $\|R(A)\|\le \frac{1+\theta_2}{2}\cdot\frac{2\theta_1}{1+\theta_1} =: \eta < 1$ $R: DK_{L=1}\to DK_{L=\eta}$

$\frac{1+\theta_1}{2}$

Pick $A,B\in DK_{L=1}$ $R(A)=U_2U_1^{-1}$, $R(B)=V_2V_1^{-1}$

$R(B)-R(A) = V_2V_1^{-1}(U_1-V_1)U_1^{-1} + (V_2-U_2)U_1^{-1}$

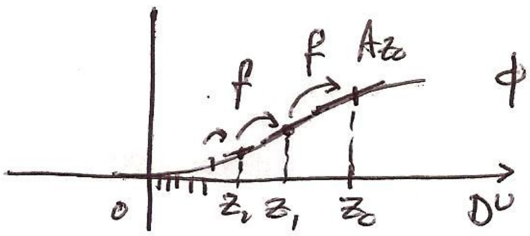
$\|R(B)-R(A)\| \le \eta\cdot\alpha\cdot\frac{2\theta_1}{1+\theta_1} + (\theta_2+\cancel{2}\alpha)\frac{2\theta_1}{1+\theta_1} \le (\theta_2+2\alpha)\frac{2\theta_1}{1+\theta_1}\le\frac{1+\theta_2}{2}\frac{2\theta_1}{1+\theta_1}=\eta$

so $R: DK_{L=1}\to DK_{L=\eta}$ is an η-Lipschitz contraction

$\eta = \frac{1+\theta_2}{2}\cdot\frac{1+\theta_1}{2} < 1$

Given $z \in D^u$, ϕ the unstable graph
we construct a sequence
of preimages of $(z, \phi_*(z)) =: \xi(z)$

$z_k = I(z_{k-1}, \phi)$, $z_0 = z$.



Then $(z_k)_{k \geq 0}$ tends to zero (x_0)
as $k \to \infty$. $|z_k| \leq \frac{1+\epsilon_1}{2} |z_{k-1}|$
The tangent map sequence

$R_k = R_{\xi(z_k)}$

is contracting so there
is a unique seq of $(A_k)_{k \geq 0}$, $A_k \in DK_{L=1}$
A_0 s.t. $A_k = R_{k+1}(A_{k+1})$

which defines $A_{z_0} = A_0$.
Since f is C^1 the contracting
maps $R.$ depends continuously
on z_0 so the
limit

$A_{\phi z_0} = \lim_k R_{\xi(z_0)} \circ R_{\xi(z_1)} \circ \dots \circ R_{\xi(z_k)}(0)$

also depends cont upon z_0.

As f is C^1 we have
$\sup_{\xi \in B(\xi_0, r)} \|f'(\xi) - f'(\xi_0)\| \leq \varepsilon_f(r) \xrightarrow[r \to 0]{} 0$
By the MVT:

$$\boxed{\begin{gathered} |f(\xi_1) - f(\xi_2) - f'(\xi_0)\cdot(\xi_1 - \xi_2)| \leq \\ \varepsilon_f(r)\, |\xi_1 - \xi_2| \\ \text{whenever } \xi_1, \xi_2 \in B(\xi_0, r) \end{gathered}}$$

We write

$A : D^u \longrightarrow DK_{L=1}$

for the map that assigns to
z the fixed point A_z.
When $f(\xi(z)) = \xi(w)$ then

$$f_{*\xi_z} \begin{pmatrix} 1 \\ A_z \end{pmatrix} = \begin{pmatrix} 1 \\ A_w \end{pmatrix} u_w$$

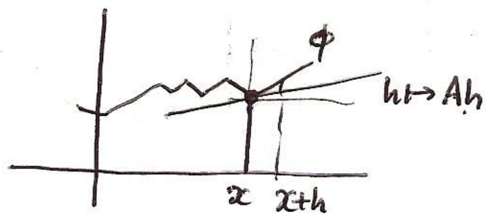
where $u_w = Df_{11} + Df_{12} A_z$

Proving that ϕ_* is C^1.

Def 7.5 For $\phi \in K_L$ we define its non linearity relative to $A \in L(E^u, E^s)$ at $x \in D^u$:

$$N_\phi(A, x, r) = \sup_{0<|h|<r} \frac{1}{|h|} |\Delta_h \phi(x) - Ah|$$

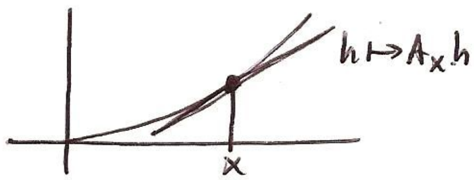
where $\Delta_h \phi(x) = \phi(x+h) - \phi(x)$ (and $x+h \in D^u$).



Remarks

1. With $A \equiv 0$: $N_\phi(0, x, r) \leq L$ $\forall x \in D^u$, $0<r$ since ϕ is ~~Lipsch~~ L-Lipschitz

2. ϕ is diff at x with derivative A_x iff $N_\phi(A_x, x, r) \xrightarrow[r \to 0^+]{} 0$



3. ϕ is C^1 iff A_x is cont.

As f is diff at each $\xi \in P$

$$f(\xi + V) = f(\xi) + f'(\xi).V + \underbrace{o(V)}_{= |V| \cdot \varepsilon_\xi(V)}$$

where

$\varepsilon_\xi(V) \xrightarrow[|V| \to 0]{} 0$ (pointwise)

As f is C^1 (at ξ_0) we also have

$$|f(\xi_1) - f(\xi_2) - f'(\xi_0).(\xi_1 - \xi_2)| \leq \varepsilon(r) |\xi_1 - \xi_2|$$

for $\xi_1, \xi_2 \in B(\xi_0, r)$ and where $\varepsilon(r) \xrightarrow[r \to 0]{} 0$

Solving

$f(\xi_z)=\xi_w$

$f(\xi_{z+h})=\xi_{w+k}$

$\xi_x=(x,\phi(x))$

$\xi_{x+h}=(x+h,\phi(x+h))$

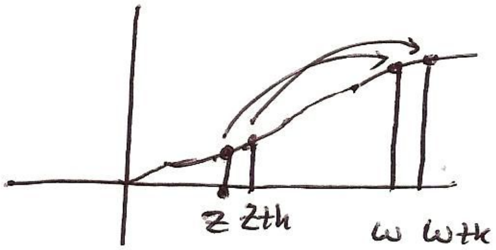
$=:(x+h,\phi(x)+\Delta_h\phi(x))$

Since $|\Delta_h\phi|\le L|h|\le|h|$ $(L=1)$

we have $O(\Delta\xi)=O(h)$ $\quad\Delta\xi=\xi_{x+h}-\xi_h$

$f(z,\phi(z))=(w,\varphi(w))$

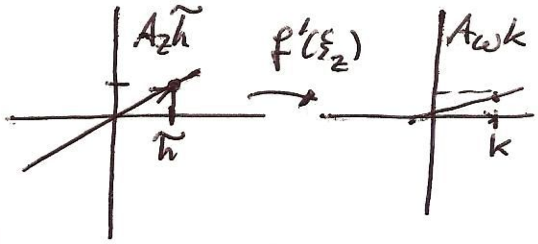
$f(z+h,\varphi(z+h))=(w+k,\varphi(w+k))$



Since f is diff at ξ_z:

$$\begin{pmatrix}k\\ \Delta_k\phi(w)\end{pmatrix} = f'(\xi_z)\begin{pmatrix}h\\ \Delta_h\phi(z)\end{pmatrix} + o_z(h)$$ depends upon z.

We compare with the invariant tangent map.
There is $\tilde h$ (different from h in general):



$$\begin{pmatrix}k\\ A_w k\end{pmatrix} = f'(\xi_z)\begin{pmatrix}\tilde h\\ A_z\tilde h\end{pmatrix} \quad \text{(exact)}$$

$$h=\Lambda_1^{-1}(k-(\alpha_{11}h+\alpha_{12}\Delta_h\phi))+o(h)$$

$$\tilde h=\Lambda_1^{-1}(k-(\alpha_{11}+\alpha_{12}A_z)\tilde h)$$

NB: h here

$$|h-\tilde h|\le\theta_1\,2\alpha\,|h-\tilde h|+\theta_1\alpha\,|\Delta_h\phi-A_z\cdot h|+o_z(h)$$

$$\Rightarrow\ |h-\tilde h|\le\frac{\theta_1}{1-2\alpha\theta_1}\,\alpha N_\phi(\phi,A,|h|)|h|+o_z(h)$$

$$\le\underbrace{\frac{1+\theta_1}{2}}_{<1}\alpha\cdot N_\phi(\phi,A,|h|)|h|+|h|\cdot\underbrace{\varepsilon_z(h)}_{\text{can be made small for } |h| \text{ small enough.}}$$

rather

$$|h-\tilde h|\le\frac{1+\theta_1}{2}\,\alpha\cdot|\Delta_h\phi-A_z\cdot h|+o_z(h)$$

also recall: $|h|\le\frac{1+\theta_1}{2}|k|$

$$\Delta_k \phi(\omega) - A_\omega \cdot k = \alpha_{21} h + (\lambda_2 + \alpha_{22}) \Delta_h \phi(z) + o(h)$$
$$- \alpha_{21} \tilde{h} - (\lambda_2 + \alpha_{22}) A_z \tilde{h}$$

So cost replacing $\tilde{h}$ by h

$$|\Delta_k \phi - A_\omega k| \leq \alpha |h - \tilde{h}| + (\theta_2 + \alpha) |h - \tilde{h}| + (\theta_2 + \alpha) |\Delta_h \phi(z) - A_z h| + o(h)$$
$$\leq (\theta_2 + 2\alpha) |h - \tilde{h}| + (\theta_2 + \alpha) |\Delta_h \phi(z) - A_z h| + o(h)$$
$$\leq (\theta_2 + \alpha + (\theta_2 + 2\alpha)(\tfrac{1+\theta_1}{2})\alpha) |\Delta_h \phi(z) - A_z h| + |k| \varepsilon_2(h)$$
$$\leq (\theta_2 + \alpha + \tfrac{1+\theta_2}{2} \tfrac{1+\theta_1}{2} \alpha) \quad '' + ''$$
$$\leq (\theta_2 + 2\alpha) \quad ---$$
$$\leq (\tfrac{1+\theta_2}{2}) |\Delta_h \phi(z) - A_z \cdot h| + |k| \varepsilon_2(h)$$

Dividing by $|h|$ we have obtained

<u>Lemma 7.6</u> $N_\phi(A, \omega, |k|) \leq \frac{1+\theta_2}{2} N_\phi(A, z, \frac{1+\theta_1}{2} |k|) + \varepsilon_z (\frac{1+\theta_1}{2} |k|)$

$$N_\phi(A, \omega, |k|) \leq \theta_2' N_\phi(A, z, \theta_1' |k|) + \varepsilon_z(\theta_1' |k|)$$

Given $z = z_0 \in D^0$ consider the sequence of $z_k = I(z_{k-1}, \phi)$ described previously.

We have $\forall k, j \;\exists r_j$: for $|k| < r_j$:

$$N_\phi(A, z_0, |k|) \leq \varepsilon_{z_1}(\theta_1' |k|) + \theta_2' \varepsilon_{z_2}(\theta_1'^2 |k|) + \theta_2'^2 \varepsilon_{z_3}(\theta_2'^3 |k|)$$
$$+ \dots + \theta_2'^j \varepsilon_{z_{j+1}}(\theta_2'^{j+1} |k|)$$
$$+ \theta_2'^{j+1} \cdot 2 \cdot L$$

so $\overline{\lim}_{k \to 0} N_\phi(A, z_0 |k|) \leq \theta_2'^{j+1} 2L \quad \forall j$

whence $\boxed{\lim_{k \to 0} N_\phi(A, z_0, |k|) = 0}$

Thm 7.6 Let $U \subset E$ be an open subset, $f \in C^k(U;E)$, $k \geq 1$ and $\xi_0 \in U$ a fixed point of f: $f(\xi_0) = \xi_0$

We say that ξ_0 is hyperbolic

Suppose that $E = E^u \oplus E^s$ and ~~f~~ $Df: E^u \to E^u$, $Df: E^s \to E^s$ with

$\| Df_{|E^u}^{-1} \| \leq \theta_1 < 1$ If finite dim: $d^u = \dim E^u$

$\| Df_{|E^s} \| \leq \theta_2 < 1$ $d^s = \dim E^s$

Then there exist a local C^k submanifold $W^u_{loc}(\xi_0)$ of dim d^u ~~such that~~ tangent to E^u ~~such~~ invariant by f and such that $f^{-1}_{|W^u_{loc}}$ is unif ~~exp.~~ contracting

If Df is invertible at ξ_0 then $\exists$ a local C^k submfld of dim d^s $W^s_{loc}(\xi_0)$ tangent to E^s f-inv and s.t. $f_{|W^s_{loc}}$ is unif contracting

When ~~f~~ $Df(\xi_0)$ is invertible (and f C^1) f is locally invertible and we may apply Thm 7.4 to f^{-1} switching E^u and E^s

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proof: [$k=1$] Let $\alpha < \min\{\frac{1-\theta_1}{4}, \frac{1-\theta_2}{4}\}$

We may assume that ξ ~~=(x,y)~~ $V = (h,k) \in E^u \times E^s$ has norm $|V|_E = |h|_{E^u} + |k|_{E^s}$

As f is differentiable at $\xi_0 = (x_0, y_0)$ we may write

$f'(\xi_0) = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$ with $\|\Lambda_1^{-1}\| \leq \theta_1$, $\|\Lambda_2\| \leq \theta_2$

as f is C^1 there is $R > 0$ s.t.

$|f'(\xi) - f'(\xi_0)| \leq \alpha$

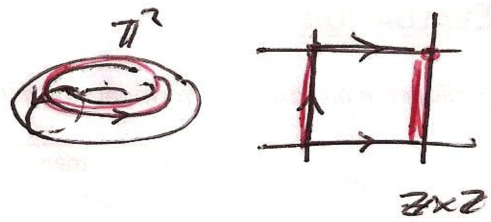
$\forall \xi \in D^u \times D^s = B_{E^u}(x_0, R) \times B_{E^s}(y_0, R)$

Thm 7.4 applies and we find $W^u_{loc}(\xi_0) = \mathcal{G}(\phi)$ with $\phi \in C^1(D^u, D^s)$. $f^{-1}_{|\mathcal{G}(\phi)}$ is unif contr.

Since $f_{*\xi_0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Lambda_1$

$W^u_{loc}(\xi_0)$ is tangent to E^u

Compact Manifolds



Let (M, g) be a Riemannian manifold of dim $n \geq 2$
(e.g. $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$)
$f \in \mathrm{Diff}^k(M)$
Let $\xi_0 \in M$ be a hyperbolic fixed point with unstable dim $d^u \geq 1$
stable dim $d^s \geq 1$

Then $\exists!$ globally defined C^k submanifolds $W^u(\xi_0)$ (dim d^u) and $W^s(\xi_0)$ (dim d^s) s.t.

$W^u(\xi_0)$ and $W^s(\xi_0)$ are tangent to E^u and E^s resp.
$f^{-1} W^u(\xi_0) = W^u(\xi_0)$, $f W^s(\xi_0) = W^s(\xi_0)$
~~a contraction~~
Both contractions
$\forall p \in W^u(\xi_0)$ $f^{-k}(p) \xrightarrow[k \to \infty]{} \xi_0$, $\forall q \in W^s(\xi_0)$ $f^k(q) \xrightarrow[k \to \infty]{} \xi_0$

<u>proof:</u> Construct $W^u_{loc}(\xi_0)$ and set $W^u_r(\xi_0) = \bigcup f^{+k} W^u_{loc}(\xi_0)$
(increasing union of C^k submflds.).

Since $f^{-1} W^u_{loc}(\xi_0) \subset W^u_{loc}(\xi_0)$ is a strict contraction
If $p \in f^{+k} W^u_{loc}(\xi_0)$ then $f^{-k}(p) \in W^u_{loc}(\xi_0)$ so $f^{-k}(p) \xrightarrow[k]{} \xi_0$

Anosov Diffeos (special cases)

~~(M,g) compact Riem~~ $M = \mathbb{T}^2$ $\quad \mathbb{R}^2 = E^u \times E^s$ $\quad f : ~~M \to~~ \mathrm{Diff}^k(M)$

Suppose there exists families of ~~tan~~ cone fields
$K_x^u(L) = \{ (v^u, v^s) \in E^u \times E^s : |v^s| \le L |v^u| \}$
(same $\forall x \in \mathbb{T}^2$) $\quad K_x^s(L) = \{ (v^u, v^s) \in E^u \times E^s : |v^u| \le L |v^s| \}$

Such that $f_* K_x^u(L) \subset K_{fx}^u(\theta_1 L)$ $\forall x \in M$ $\quad \theta < 1$

$|Df\, v| \ge \frac{1}{\theta_1} |v|$

$|Df\, u| \le \theta_2 |u|$ $\quad f_{*x}^{-1} K_x^s(L) \subset K_{f^{-1}x}^s(\theta_2 L)$ $\forall x \in M$

Then $\exists!$ continuous splittings at each $x \in M$:

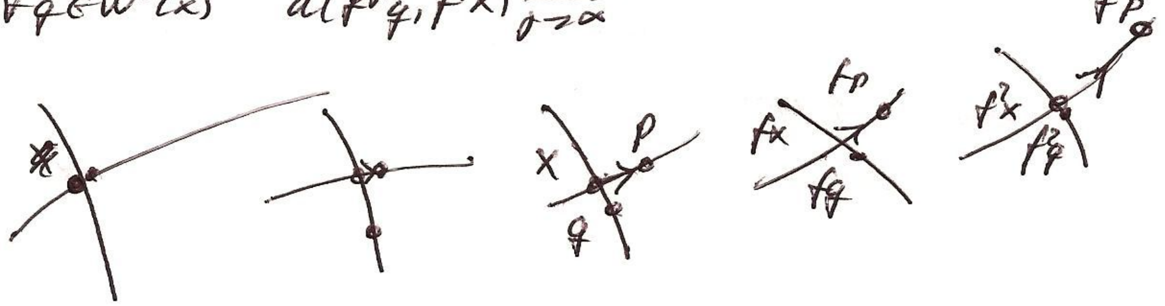
$$T_x M = E_x^u \times E_x^s$$

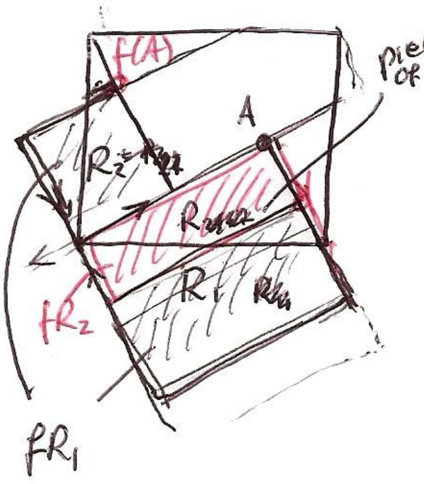
s.t. $f_*^{-1} E_x^u = E_{fx}^u$ $\quad f_*^{+1} E_x^s = E_{fx}^s$
are ~~strict~~ uniform contractions

Through any $x \in M$ $\exists!$ globally defined C^k-mflds $W^u(x)$, $W^s(x)$ ~~contracted by~~

$\forall p \in W^u(x)$ $\quad d(f^{-j} p, f^{-j} x) \xrightarrow[j \to \infty]{} 0$

$\forall q \in W^s(x)$ $\quad d(f^{j} q, f^{j} x) \xrightarrow[j \to \infty]{} 0$





$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x\end{pmatrix} = \underbrace{\begin{pmatrix}1&1\\1&0\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix}$$

$v^u = \begin{pmatrix}\gamma\\1\end{pmatrix}$ $\quad Av^u = \gamma v^u \quad \gamma = \frac{1+\sqrt{5}}{2}$

$v^s = \begin{pmatrix}-\frac{1}{\gamma}\\1\end{pmatrix}$ $\quad Av^s = -\frac{1}{\gamma}v^s$

Transitions

$R_1 \to R_1 + R_2$ $\quad A = \begin{pmatrix}1&1\\1&0\end{pmatrix}$
$R_2 \to R_1$

$\partial R_1 = \partial R_1^u + \partial R_1^s$
$\partial R_2 = \partial R_2^u + \partial R_2^s$

$\left|\begin{array}{l} f\,\partial R_i^s \subset \bigcup_{A_{ij}=1} \partial R_j^s \\ f^{-1}\partial R_j^u \subset \bigcup_{A_{ij}=1} \partial R_i^u \end{array}\right.$ Markov

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\phi} & \mathbb{T}^2 \\ \circlearrowleft & & \circlearrowleft \\ T \text{ shift} & & f \end{array}$$

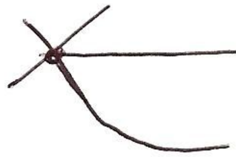
$f \circ \phi = \phi \circ T$

ambiguities when $\xi \to \underline{W^s(0) \cup W^u(0)}$
measure 0.

Gibbs meas $\to$ Leb.

(Bowen)

$\begin{matrix}4 & 1\\ -1 & -\frac{3}{4}\end{matrix}$ $\quad e$



Stay the same under C^1-perturbation