3 Cone contraction and ~~invariant~~ Gibbs measure

(Ω, d) compact metric sp.

$C^0 = C^0(\Omega, d) \quad C^0_+ = \{\varphi \in C^0 : \varphi \geq 0\}$

X : function space over (Ω, d) which is a Banach space and densely embedded in C^0, $X \hookrightarrow C^0$

ex : $X = Lip(\Omega, d)$, $\|\phi\| = |\phi|_0 + Lip\,\phi$

K : a cone in $X \cap C^0_+$ which is inner + outer regular in X. We assume $\mathbb{1} \in K$ (const fct.)

$\mathcal{L} : C^0 \to C^0$ bd lin operator which preserves C^0_+, X and K^*.

(so in particular $\varphi \geq 0 \Rightarrow \mathcal{L}\varphi \geq 0$)

Theorem 3.1 Suppose $\mathcal{L} : K^* \to K^*$ is a uniform contraction for the Hilbert metric on K^*

$$\Delta = \operatorname{diam}_{K^*} \mathcal{L} K^* < +\infty$$

Then $\mathcal{L}$ has a spectral gap in X with the following properties

$$\exists!\ \lambda > 0,\ \nu \in M^1_+(\Omega, \mathcal{B}) \ \text{(proba on } \Omega\text{)}$$
$$h \in K^* \text{ with } \int h\, d\nu = 1 \text{ s.t.}$$

$$\forall \phi \in X : |\lambda^{-n}\mathcal{L}^n\phi - h\int \phi\, d\nu|_X \leq C\eta^{n-1}|\phi|_X$$
$$\forall f \in C^0 : |\lambda^{-n}\mathcal{L}^n f - h\int f\, d\nu|_{C^0} \xrightarrow[n]{} 0$$

$\eta = \tanh \frac{\Delta}{4} < 1$

Proof: From the spectral gap thm:
$\exists! \ell \in X'$ (Banach dual of X)
$\ell(\mathbb{1}) = 1$ ($\ell \geq 0$ on K)
$\langle \ell, h \rangle = 1$ ($\ell > 0$ on K^*)
$h \in K^*$, $\lambda > 0$ such that for $\phi \in X$:
$$|\lambda^{-n}\mathcal{L}^n\phi - h\langle \ell, \phi\rangle|_X \leq C\eta^{n-1}|\phi|_X$$

Want to show that ℓ extends to a linear functional on C^0

First let $\phi \in X \cap C^0_+$. Then
$\varphi \geq 0 \Rightarrow \lambda^{-n}\mathcal{L}^n\phi \geq 0$ $(n \to \infty)$
$\Rightarrow h\langle \ell, \phi\rangle \geq 0$
$\Rightarrow \langle \ell, \phi \rangle \geq 0$
(since $h > 0$)

Now let $\phi \in X$. $\|\phi\|_{C^0} \leq 1$.
Then
$\mathbb{1} \pm \phi \geq 0 \Rightarrow$
$\langle \ell, \mathbb{1} \pm \phi \rangle \geq 0 \Rightarrow$
$|\langle \ell, \phi \rangle| \leq \langle \ell, \mathbb{1} \rangle = 1$
so for every $\phi \in X \cap C^0$
$|\langle \ell, \phi \rangle| \leq \|\phi\|_{C^0}$
As X is dense in C^0, ℓ unif bd wrt $\|\cdot\|_{C^0}$ extends uniquely to a pos bd linear functional
$\nu \in (C^0)'$, $\nu \geq 0$
$|\nu(f)| \leq \|f\|_{C^0}$, $\forall f \in C^0$
and $\nu(\mathbb{1}) = 1$. By Riesz $\exists!$ Borel proba $d\nu$ st
$$\nu(f) = \int_\Omega f\, d\nu$$
When $-\mathbb{1} \leq f \leq \mathbb{1}$ then
$$\underbrace{-\lambda^{-n}\mathcal{L}^n\mathbb{1}}_{\to -h} \leq \lambda^{-n}\mathcal{L}^n f \leq \underbrace{\lambda^{-n}\mathcal{L}^n\mathbb{1}}_{\to h}$$
so $\|\lambda^{-n}\mathcal{L}^n f\|_{C^0} \leq M \cdot \|f\|_{C^0}$
Then use density of X //

<u>Shifts of finite type</u>
Ruelle-Perron-Frobenius thm.

A : a $d\times d$ transition matrix topologically mixing
$\exists n : (A^n_{ij}) \geq 1 \ \forall i,j$.

(ex $A = \begin{pmatrix}1&1\\1&0\end{pmatrix}$, $A^2 = \begin{pmatrix}2&1\\1&1\end{pmatrix} \geq 1_{ij}$)

Σ_A^+ : shift space with transit. matrix A : $\xi = (i_0 i_1 \ldots) \in \Sigma_A^+$ iff $A_{i_k i_{k+1}} = 1$. $\forall k \geq 0$ (T=shift)

$d_\theta(\xi, \eta) = \sup \{\theta^k : \xi_k \neq \eta_k\}$ $0<\theta<1$.

C^0 : cont fct on (Σ_A^+, d_θ)
X : Lipsch. fcts on (Σ_A^+, d_θ).

For $g \in X$ define the (Ruelle) transfer operator: $\phi \in X$ or C^0:

$$\mathcal{L}_g \phi\,(y) = \sum_{x : Tx = y} e^{g(x)} \phi(x)$$

for all $y \in \Sigma_A^+$.

Thm [Ruelle] 3.2
$\exists!\ P(g) \in \mathbb{R}$, $h = h_g \in X^+$, $\nu = \nu_g \in \mathcal{M}^1_+(\Sigma_A^+)$ with $\int h\, d\nu = 1$ s.t.: $\forall \phi \in X$:

$$\| e^{-nP(g)} \mathcal{L}_g^n \phi - h \int \phi\, d\nu \|_X \leq C \eta^{n-1} \|\phi\|_X$$

with $0 < \eta < 1$ and $C < +\infty$.
Also for $f \in C^0$

$$\| e^{-nP(g)} \mathcal{L}_g^n f - h \int f\, d\nu \|_{C^0} \xrightarrow[n]{} 0$$

Proof: Because of A possibly having zero entrees it is technically easier to work with the following cone family: For $a>0$, $0 \leq b < 1$:

$$K_{a,b} = \{\phi \in X : \phi \geq 0 \text{ and :}$$
within the same 1-cylinder $\rightarrow$ $\forall 1 \leq j \leq d;\ x, y \in [j] : \phi(x) \leq e^{a d(x,y)} \phi(y)$
global $\rightarrow$ $\forall x, y \in \Sigma_A^+ : b\,\phi(x) \leq \phi(y)\}$

There is $p \geq 1$ st $A^p_{ij} \geq 1 \ \forall ij$
Then (exercise):

$\exists a > 0$, $0 < b < 1$, $0 < \sigma < 1$:

$$\mathcal{L}^p : K^*_{a,0} \longrightarrow K^*_{a\sigma, b}$$

Furthermore:
*) $\operatorname{diam}_{K_{a,0}} K^*_{a\sigma,b} < +\infty$.
*) $K_{a,0}$ is inter + ext reg.

This implies that $\mathcal{L}^p$ has a spectral gap.
*) and then that $\mathcal{L}$ has a spectral gap (with $\lambda_g > 0$ as $\mathcal{L}$ is positive)
(use that $\mathcal{L}^n = \mathcal{L}^{(pk+j)}$ with $0 \leq j < p$.

$$\| \lambda_g^{-n} \mathcal{L}_g^n - h_g \otimes \ell_g \|_X \leq C \eta^{n-1}$$

Then use the C^0_+ embedding, $\mathbb{1} \in K_{a,0}$ and write

$$P(g) = \log \lambda_g$$

Note that in particular

$$\mathcal{L}_g h_g = e^{P(g)} h_g$$
$$\nu_g \mathcal{L}_g = e^{P(g)} \nu_g$$

Notation:

for $M \geq 1$, $u, v > 0$ we write
$$u \underset{M}{\asymp} v \Leftrightarrow \frac{1}{M} \leq \frac{u}{v} \leq M$$

u and v are M-comparable.

Thm 3.3 [Gibbs measure]

a) $d\mu_g = h_g \, d\nu_g$ defines a T-invariant probability measure which is mixing, whence ergodic.

b) $d\mu_g$ is the unique T-invar proba measure such that $\exists M < \infty$ for which:

$\forall n \geq 1$, $\forall C \in Z_n$ (n-cylinder) and $\forall x \in C$
$$\mu_g(C) \underset{M}{\asymp} e^{S_n g(x) - nP(g)}$$

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proof: Using the "transfer property"
$$\mathcal{L}_g(f \circ T \, \phi)(y) = \sum_{x: Tx = y} e^g f \circ T \cdot \phi = f(y) (\mathcal{L}_g \phi)(y)$$

$$\begin{aligned}\mu_g(f \circ T) &= \nu_g(\mathcal{L}_g f \circ T \cdot h_g) \\ &= e^{-P(g)} \nu_g(\mathcal{L}_g(f \circ T h_g)) \\ &= e^{-P(g)} \nu_g(f \cdot \mathcal{L}_g h_g) \\ &= \nu_g(f \cdot h_g) \\ &= \mu_g(f)\end{aligned}$$

For mixing use that $\mathcal{X}$ is dense in $L^1(\Sigma_A^+, \mu_g)$ since C^0 is.

Let $n \geq 1$, $C = [i_0 i_1 \dots i_{n-1}] \in Z_n$.
We have
$$T^k C = [i_k \dots i_{n-1}] \in Z_{n-k}$$
and then with $L = \mathrm{Lip}(g)$
$$\underset{C}{\mathrm{var}} \, g \circ T^k \leq L \cdot \mathrm{diam} \, T^k C \leq L \cdot \theta^{n-k-1}$$
for $0 \leq k < n$.

whence
$$\underset{C}{\mathrm{var}} \, S_n g \leq L(1 + \theta + \dots + \theta^{n-1}) \leq \frac{L}{1-\theta}$$

We have: $\overline{e^{nP} \nu_g(\mathcal{L}_g^n \mathbb{1}_{[j]})} \geq \text{const} \cdot \mathbb{1}$
$$1 \geq m = \min_{1 \leq j \leq d} \nu_g([j]) > 0$$
also since $h_g \in K^*_{a,b}$
$$h_g(x) \underset{1/b}{\asymp} h_g(x') \quad \forall x, x' \in \Sigma_A^+$$

Let $x \in C = [i_0 i_1 \dots i_{n-1}] \in Z_{n-1}$
Then
$$\begin{aligned}\mu_g(C) &= \nu_g(\mathbb{1}_C h_g) \\ &= \nu_g(e^{-nP(g)} \mathcal{L}_g^n(\mathbb{1}_C h_g)) \\ &= \cancel{e^{-nP} \nu_g(\mathcal{L}_0^n(e^{S_n g} \mathbb{1}_C h_g))} \\ &\underset{1/b}{\asymp} \nu_g(e^{-nP} \mathcal{L}_g^n(\mathbb{1}_C)) \\ &= e^{-nP} \nu_g(\mathcal{L}_0^n(e^{S_n g} \mathbb{1}_C)) \\ &\underset{\exp \frac{L}{1-\theta}}{\asymp} e^{S_n g(x) - nP(g)} \nu_g(\mathcal{L}_0^n \mathbb{1}_C) \\ &= \text{---} \text{''} \text{---} \; \nu_g([i_{n-1}]) \\ &\underset{1/m}{\asymp} e^{S_n g(x) - nP(g)}\end{aligned}$$
as wanted

We used: $\mathcal{L}_g = \mathcal{L}_0 e^g$
$$\begin{aligned}\mathcal{L}_g^n &= \mathcal{L}_0 e^g (\mathcal{L}_0 e^g \dots) \\ &= \mathcal{L}_0^2 e^{g \circ T + g} \dots \\ &= \mathcal{L}_0^n e^{S_n g}\end{aligned}$$

Finally if $\tilde{\mu}$ is another T-invar proba verifying the b) then $\tilde{\mu} \ll \mu \Rightarrow \tilde{\mu} = \mu$.

Analytic
Perturbation theory.

[Kato: Perturbation theory for Linear operators Chap VII §1]

X : Banach algebra i.e.

$\forall \varphi, \psi \in X$: $\varphi \cdot \psi \in X$ and
$\|\varphi\psi\| \le \|\varphi\| \cdot \|\psi\|$.

$X = Lip(\Sigma_A^+, d)$ is a Banach alg.

$T : \Sigma_A^+ \to \Sigma_A^+$ the shift.

Given $g, A \in X$ we consider for $t \in \mathbb{C}$ the operator:

$$\mathcal{L}_t \varphi(y) = \sum_{x : Tx = y} e^{g + tA(x)} \varphi(x)$$

The map

$t \in \mathbb{C} \mapsto \mathcal{L}_t \in L(X)$

is analytic

$$\mathcal{L}_t \varphi = \sum \left(\sum \frac{t^k}{k!} A(x)^k\right) e^{g} \varphi(x)$$

$$= \sum \frac{t^k}{k!} \sum e^{g(x)} A(x)^k \varphi(x)$$

$$= \sum \frac{1}{k!} t^k \underbrace{\mathcal{L}_g(A^k \varphi)}_{\|\cdot\| \le C \cdot \|A\|^k}$$

converges uniformly in $L(X)$ $\forall t \in \mathbb{C}$

Thm 3.4 [Kato Chap VII §1]
Let $\mathcal{L}_t \in L(X)$ be an analytic family of bd lin operators for $t \in$ neighborhood of 0 in $\mathbb{C}$

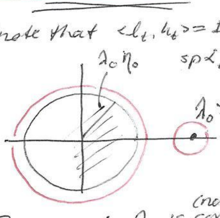
Suppose λ_0 is a simple isolated eval for $\mathcal{L}_{t=0}$ with 1-dim projection $h_0 \otimes \ell_0$
$h_0 \in X$, $\ell_0 \in X'$

Then there is $\delta > 0$ ~~such~~ and ~~that~~ λ_t, h_t, ℓ_t ~~that~~ are analytic in $|t| < \delta$ such that λ_t is a simple isolated eval for $\mathcal{L}_t$ with projection $h_t \otimes \ell_t$

Furthermore for δ small enough a spectral gap for $\mathcal{L}_0$ persists for $|t| < \delta$, i.e: $\exists\, \eta < 1$, $C < \infty$
$\forall |t| < \delta$; $\varphi \in X$:

$$\|\lambda_t^{-n} \mathcal{L}_t^n \varphi - h_t \langle \ell_t, \varphi \rangle\|_X \le C \eta^n \|\varphi\|_X$$

(note that $\langle \ell_t, h_t \rangle = 1$ $\forall |t| < \delta$)



In general λ_t is complex (non-real) for non-real t.

$P(t) = \log \lambda_t$ is also analytic for $|t| < \delta$. (unique continuation of log)

The pressure function 1st derivative

Let $g \in \mathrm{Lip}\,\Sigma_A^+$ and $A \in \mathrm{Lip}\,\Sigma_A^+$

We consider the Ruelle transfer operator on $C(\Sigma_A^+)$ with exponential weight $g+tA$ for $t\in\mathbb{C}$

By thm 3.4 we have $\exists\delta>0$ for which

$$\mathcal{L}_t := \mathcal{L}_{g+tA} = \mathcal{L}_g e^{tA}$$

has a spectral gap, i.e.

$$\lambda_t^{-n}\mathcal{L}_t^n = e^{-nP(t)}\mathcal{L}_t^n = h_t\otimes\ell_t + R_t^n$$

where $\|R_t^n\| \le C\eta^{n-1}$, $\eta<1$, and h_t, ℓ_t, λ_t, $P(t)$ are analytic in $|t|<\delta$.

When $t\in\mathbb{R}$, $|t|<\delta$ the operator $\mathcal{L}_t$ is real positive and Ruelle's thm 3.2 + 3.3 shows that we have a mixing (ergodic) ~~g mean~~ Gibbs measure (proba):

$$d\mu_t = h_t\, d\ell_t$$

For $\phi\in C^0(\Sigma_A^+)$

$$\mathbb{E}_t(\phi) := \mu_t(\phi) = \int_{\Sigma_A^+}\phi\, h_t\, d\ell_t$$

is the corresponding expectation.

Taking derivatives of

$$1 = \ell_t(h_t) \qquad |t|<\delta$$

we get

$$0 = \ell_t'(h_t) + \ell_t(h_t')$$

Also $(\mathcal{L}_t)' = \mathcal{L}_g e^{tA}A = \mathcal{L}_t A$.

We get by taking the derivative of

$$e^{P(t)} = \ell_t(\mathcal{L}_t h_t)$$

$$\begin{aligned} P'(t)e^{P(t)} &= \ell_t'(\mathcal{L}_t h_t) + \ell_t(\mathcal{L}_t h_t') + \ell_t(\mathcal{L}_t A h_t) \\ &= e^P\big(\ell_t'(h_t) + \ell_t(h_t') + \ell_t(A h_t)\big) \\ &= e^P\cdot \ell_t(A h_t) = e^P \mu_t(A h_t) \text{ so} \end{aligned}$$

Theorem 3.5 For $|t|<\delta$

$$\begin{aligned} P'(t) &= \cancel{\mathbb{E}_t(A)}\ \ell_t(A h_t) \\ &= \mathbb{E}_t(A) \quad (\text{for } t\in\mathbb{R}) \end{aligned}$$

~~2nd derivative~~ Correlation Function

$\hat{\mathcal{L}}_t = e^{-P(t)}\mathcal{L}_t$

Let $m_A^{(t)} = P'(t) = \mu_t(A h_t)$ $|t|<\delta$

Define the correlation fct: for $k \geq 0$:

$$C_A^t(k) = \mu_t((A-m_A)\circ T^k \cdot (A-m_A))$$

$$= \mu_t((A-m_A)\circ T^k \cdot (A-m_A) h_t)$$

$$= \mu_t(\hat{\mathcal{L}}_t^k (A-m_A)\circ T^k (A-m_A) h_t)$$

(transfer property) $$= \mu_t((A-m_A)\cdot \hat{\mathcal{L}}_t^k ((A-m_A) h_t))$$

$$= \mu_t((A-m_A)\cdot \underbrace{R_t^k ((A-m_A) h_t)}_{\|\cdot\| \leq C\eta^k})$$

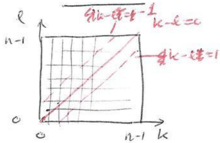
since $\mu_t((A-m_A)h_t) \equiv 0$

So we have

$$|C_A^t(k)| \leq \text{Const } \eta^k$$

By T-invariance of ν_t: $\forall k, \ell \geq 0$

$$\mu_t((A-m_A)\circ T^k \cdot (A-m_A)\circ T^\ell) = C_A^t(|k-\ell|)$$



$$\frac{1}{n}\mu_t\Big(\big(\sum_{k=0}^{n-1}(A-m_A)\circ T^k\big)^2\Big) =$$

$$\frac{1}{n}\mu_t\Big(\sum_k (A-m_A)\circ T^k \sum_\ell (A-m_A)\circ T^\ell\Big) =$$

$$\frac{1}{n}\sum_k\sum_\ell C_A^t(|k-\ell|) =$$

$$\frac{1}{n}\sum_{j=-n}^{n}(n-|j|)\, C_A^t(|j|) =$$

$$\sum_{|j|\leq n}\Big(1-\frac{|j|}{n}\Big) C_A^t(|j|)$$

$$\xrightarrow[\text{Domin. Conv}]{n\to\infty} \sum_{j\in\mathbb{Z}} C_A^t(|j|)$$

Prop 3.6 We have

$$\lim_{n\to\infty} \frac{1}{n}\mu_t\big((S_n(A-m_A^t))^2\big) = \sum_{j\in\mathbb{Z}} C_A^t(|j|)$$

2^{nd} derivative

$$\mathcal{L}_t^n = \mathcal{L}^n e^{tS_nA} \quad n \geq 1$$

we have

$$e^{nP(t)} = \ell_t(\mathcal{L}_t^n h_t)$$

so taking the 1^{st} deriv:

$$(*) \quad nP'(t) = \ell_t(\mathcal{L}_t^n S_nA\, h_t)\, e^{-nP(t)}$$

$$= \ell_t(\hat{\mathcal{L}}_t^n S_nA\, h_t)$$

Where $\hat{\mathcal{L}}_t = \mathcal{L}_t e^{-P(t)}$. One has:

$$(\hat{\mathcal{L}}_t^n)'(\cdot) = \hat{\mathcal{L}}_t^n(S_n(A - m_A^t)\,\cdot)$$

$$= \hat{\mathcal{L}}_t^n((S_nA - nP'(t))\,\cdot)$$

Differentiation of $(*)$ yields

$$nP''(t) = \ell_t(\hat{\mathcal{L}}_t^n(S_n(A-m_A^t))\, S_nA\, h_t))$$

$$\left.\begin{array}{l} + \ell_t'(\hat{\mathcal{L}}_t^n(S_nA)\, h_t) \\ + \ell_t(\hat{\mathcal{L}}_t^n(S_nA)\, h_t') \end{array}\right]$$

For the two last terms we use $\ell_t' h_t + \ell_t h_t' \equiv 0$ to subtract m_A^t and get

$$\ell_t'(\hat{\mathcal{L}}_t^n(S_n(A-m_A)\, h_t)) + \ell_t(\hat{\mathcal{L}}_t^n(S_n(A-m_A)\, h_t')$$

Note that $\hat{\mathcal{L}}_t^n\, A \circ T^k = \hat{\mathcal{L}}_t^{n-k} A\, \hat{\mathcal{L}}_t^k$ to obtain

$$\sum_{k=0}^{n-1} \ell_t'(\hat{\mathcal{L}}_t^{n-k}(A-m_A)\, \hat{\mathcal{L}}_t^k h_t) + \ell_t(\hat{\mathcal{L}}_t^{n-k}(A-m_A)\hat{\mathcal{L}}_t^k h_t')$$

$$= \sum \ell_t'(\mathcal{L}_t^{n-k}(A-m_A)\, h_t) + \ell_t(A-m_A)\, \hat{\mathcal{L}}_t^k h_t')$$

$$= \sum_{k=0}^{n-1} \ell_t'(\underbrace{R_t^{n-k}(A-m_A)\, h_t}_{\leq C\eta^{n-k}}) + \ell_t(A-m_A)\underbrace{R_t^k h_t'}_{\leq C\eta^k})$$

$$|\cdot| \leq \sum C(\eta^{n-k} + \eta^k) \leq \frac{2C}{1-\eta} = O(n^0)$$

For the first term we get

$$\ell_t(S_n(A-m_A)\cdot S_nA\, h_t) =$$

$$\ell_t(S_n(A-m_A)\cdot S_n(A-m_A)\, h_t) =$$

$$\mu_t((S_n(A-m_A^t))^2 h_t) \text{ (when } t \text{ real)}$$

Dividing by n and letting $n \to \infty$:

Thm 3.7 $\forall |t| < \delta$

$$P''(t) = \mu_t \sum_{j \in \mathbb{Z}} C_A(1,j)\ (S_n(A-m_A^t))^2$$

with

$$\sum_{j \in \mathbb{Z}} C_A(1,j) = \lim \frac{1}{n} \ell_t((S_nA)^2 h_t)$$

(when t real) $$= \lim \frac{1}{n} \mu_t((S_nA)^2)$$

~~(for real t)~~

(≥ 0 for real t)

$$P''(t) = \lim_{n \to \infty} \frac{1}{n} \mu_t((S_n(A-m_A^t))^2 h_t)$$

$$= \lim_n \frac{1}{n} \mu_t((S_n(A-m_A^t))^2)$$

for t real

$$= \lim \frac{1}{n} \mathbb{E}_t \frac{1}{n} (S_n(A-m_A^t))^2$$

Central limit theorem

Levy's characterization:

A sequence of random vars (Y_n) converges in law to $\mathcal{N}(0,\sigma^2)$ iff $\forall t \in \mathbb{R}$

$$\mathbb{E}(e^{itY_n}) \xrightarrow[n\to\infty]{} e^{-t^2\sigma^2/2}$$

Consider in the following a shift of finite type ~~Σ_A^+~~

$T : \Sigma_A^+ \circlearrowleft$ metric d_θ

which is topol mixing $(\exists p : A_{ij}^p \geq 1)$

$X = \mathrm{Lip}(\Sigma_A^+, d_\theta)$.

For $g \in X$ let μ_g denote the associated g-measure (which is mixing, whence ergodic).

$\mathbb{E}_g$ = expectation

Theorem [Parry-Pollicott]

Let $A \in X$ with $\mathbb{E}_g(A) = 0$. ~~Then~~ Then the ~~The limit~~ sequence $(\frac{1}{\sqrt{n}} S_n A)_{n \geq 1}$ converges in law to $\mathcal{N}(0,\sigma^2)$ with (the limit exists)

$$\sigma^2 = \lim_{n\to\infty} \mathbb{E}_g \frac{1}{n}(S_n A)^2$$

Proof: For $t \in \mathbb{C}$ ~~and~~ consider

$$\mathcal{L}_t := \mathcal{L}_{g+tA} \qquad h_t = h_{(g+tA)} \qquad \mu_t = \mu_{(g+tA)}$$

When $|t| < \delta$, $\mathcal{L}_t$ has a spectral gap and analytic leading eigenvalue and associated eigenvector.

$$\underbrace{|\lambda_t^{-n}\mathcal{L}_t^n \phi - h_t \langle \nu_t, \phi\rangle}_{\lambda_t^{-n} R_t^n \phi}|_X \leq C \eta^{n-1} |\phi|_X$$

By the pressure formulae

$$P(t) = \log \lambda(t)$$

verifies

$$P'(0) = \mathbb{E}_g(A)$$

$$P''(0) = \lim \mathbb{E}_g \frac{1}{n}(S_n A)^2 =: \sigma^2$$

For fixed $t \in \mathbb{R}$ consider

$$\mathbb{E}_g(e^{i\frac{t}{\sqrt{n}} S_n A}) =$$

$$\mu_0(e^{i\frac{t}{\sqrt{n}} S_n A} h_0) =$$

$$e^{-nP(0)} \mu_0(\mathcal{L}_g^n e^{i\frac{t}{\sqrt{n}} S_n A} h_0) =$$

$$e^{-nP(0)} \mu_0(\mathcal{L}^n_{g+\frac{it}{\sqrt{n}}A} h_0) =$$

$$e^{-nP(0)} [\nu_0(h_{it/\sqrt{n}}) \nu_{it/\sqrt{n}}(h_0) \cdot e^{nP(\frac{it}{\sqrt{n}})} +$$

$$\nu_0(R^n_{\frac{it}{\sqrt{n}}} h_0)]$$

~~For θ_0 small enough~~

$$\| R^n_{\frac{it}{\sqrt{n}}} \| e^{-nP(0)} \xrightarrow[n\to\infty]{} 0$$

$$e^{nP(\frac{it}{\sqrt{n}}) - nP(0)} =$$

$$e^{nP'(0)\frac{it}{\sqrt{n}} + nP''(0)(\frac{it}{\sqrt{n}})^2/2 + n\, o(\frac{t^2}{n})}$$

$$= e^{-\sigma^2 t^2/2 + \underbrace{n\, o(\frac{1}{n})}_{\to 0}}$$

and

$$\nu_0(h_{it/\sqrt{n}}) \nu_{it/\sqrt{n}}(h_0) \to 1$$

by continuity

Entropy of Gibbs measures

Recall: $g \in Lip(\Sigma_A^+, d_\theta)$,
$d\mu_g = h_g \, d\nu_g$ verifies:
- T-invar. mixing, ergodic
- $\forall n \geq 1$, $C \in Z_n$, $x \in C$:

$$\mu_g(C) \underset{M}{\asymp} e^{S_n g(x) - nP(g)}$$

($M \in [1, +\infty)$ uniformly in n)

$Z_1 = \{[1], \dots, [d]\}$ generates the Borel-σ-alg dynamically
$\sigma(\bigvee_n^\infty Z_n) = \mathcal{B}_{\Sigma_A^+}$ so by Kolmo-Sinai

$$h_{\mu_g}(T) = h_{\mu_g}(T, Z_1) = \lim_n \frac{1}{n} H_{\mu_g}(Z_n)$$

$$\begin{aligned} H_{\mu_g}(Z_n) &= -\sum_{C \in Z_n} \mu(C) \log \mu(C) \\ &= -\sum \mu(C)(S_n g(x_C) - nP(g)) + O(1) \\ &= nP(g) - \sum \int_C S_n g \, d\mu_g + O(1) \\ &= nP(g) - \int_{\Sigma_A^+} S_n g \, d\mu_g + O(1) \\ &= n\Big(P(g) - \int_{\Sigma_A^+} g \, d\mu_g\Big) + O(1) \end{aligned}$$

so

Thm 3.9

$$\boxed{h_{\mu_g}(T) = P(g) - \int_{\Sigma_A^+} g \, d\mu_g}$$

Thm 3.10 [Variational Principle]

$$\boxed{P(g) = \sup_{\mu \in M_1^T(T)} \Big(h_\mu(T) + \int_{\Sigma_A^+} g \, d\mu\Big)}$$

realized for $\mu = \mu_g$
(even uniquely, not shown here)

We will use the following

Lemma: $\forall a_1, \dots, a_k \in \mathbb{R}$, $p_1, \dots, p_k \geq 0$, $\sum p_i = 1$:

$$-\sum p_i \log p_i + \sum a_i p_i \leq \log \sum e^{a_i}$$

with equality iff $p_i = e^{a_i}/\Omega$, $\Omega = \sum e^{a_i}$

proof ($\varphi(t) = -t \log t$ concave)

$$\begin{aligned} 0 = \varphi(1) = \varphi\Big(\sum p_i\Big) &= \varphi\Big(\sum \frac{e^{a_i}}{\Omega} \frac{p_i \Omega}{e^{a_i}}\Big) \geq \\ \sum \frac{e^{a_i}}{\Omega} \varphi\Big(\frac{p_i \Omega}{e^{a_i}}\Big) &= -\sum p_i \log \frac{p_i \Omega}{e^{a_i}} = \\ &-\log \Omega + \sum(-p_i \log p_i + a_i p_i) \end{aligned}$$

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proof of Thm 3.10: For $n \geq 1$

$$\begin{aligned} H_\mu(Z_n) + n \int g \, d\mu &= \sum_C \Big(-\mu(C) \log \mu(C) + \int_C S_n g \, d\mu\Big) \leq \\ &\sum \mu(C)(-\log \mu(C) + S_n g(x_C)) \end{aligned}$$

$x_C \in C$ for which max is attained

$$\begin{aligned} &\leq \log \sum \exp(S_n g(x_C)) \\ &= \log \sum \mu(C) e^{nP(g) + O(1)} \\ &= nP(g) + O(1) \end{aligned}$$

so

$$\lim_n \frac{1}{n} H_\mu(Z_n) + \int g \, d\mu \leq P(g)$$

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