3. Topological Entropy

$(X,\mathcal{T})$ compact topol sp.
$T: X \to X$ a contin. map.

Def 3.1 For open covers α, β of X, we define their join $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. Similarly $\bigvee_{i=1}^{n} \alpha_i = \{A_1 \cap \dots \cap A_n : A_1 \in \alpha_1, \dots, A_n \in \alpha_n\}$.
An open cover β is a refinement of α, $\alpha < \beta$ if $\forall B \in \beta \ \exists A \in \alpha : B \subset A$

Remarks: β, α open covers:
1) $T^{-1}\alpha$ is also an open cover of X.
$\alpha_N = \alpha \vee T^{-1}\alpha \vee \dots \vee T^{-(N-1)}\alpha$ open cover.
2) $\alpha < \alpha \vee \beta$, $\alpha < \beta \Rightarrow T^{-1}\alpha < T^{-1}\beta$
$T^{-1}(\alpha \vee \beta) = T^{-1}\alpha \vee T^{-1}\beta$.

Def 3.2 For an open cover α let $N(\alpha)$ = min Card of a finite subcover of α. We call $H(\alpha) = \log N(\alpha)$ the (topol) entropy of α. (autom. finite if X is compact)

Remarks: $\alpha < \beta \Rightarrow H(\alpha) \leq H(\beta)$

Prop 3.3 $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$

- If $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_m\}$ ~~cove~~ both covers X then so does $\{A_i \cap B_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ so $N(\alpha \vee \beta) \leq N(\alpha) + N(\beta)$.

Prop 3.4 $H(T^{-1}\alpha) \leq H(\alpha)$. If T is surjective then $H(T^{-1}\alpha) = H(\alpha)$

- If $\{A_1, \dots, A_n\}$ covers X then so does $\{T^{-1}A_1, \dots, T^{-1}A_n\}$. so $N(T^{-1}\alpha) \leq N(\alpha)$
If T is surjective and $\{T^{-1}A_1, \dots, T^{-1}A_n\}$ covers X then so does $\{A_1, \dots, A_n\}$ and $N(T^{-1}\alpha) = N(\alpha)$

Thm 3.5 The limit $h(T,\alpha) := \lim \frac{1}{n} H(\bigvee_0^{n-1} T^{-i}\alpha)$ exists for any open cover α. We call it the topol entropy of T relative to α.

$a_n = H(\bigvee_0^{n-1} T^{-i}\alpha)$
$a_{n+m} = H(\bigvee_0^{n+m-2} T^{-i}\alpha) = H(\bigvee_0^{n-1} T^{-i}\alpha \vee (T^{-n} \bigvee_0^{m-1} T^{-i}\alpha)) \leq H(\bigvee_0^{n-1} T^{-i}\alpha) + H(\bigvee_0^{m-1} T^{-i}\alpha) = a_n + a_m$
so (a_n) is subadditive //

Remarks:
1) $h(T,\alpha) \geq 0$, $h(T,\alpha) \leq H(\alpha)$
2) $\alpha < \beta \Rightarrow h(T,\alpha) \leq h(T,\beta)$

Def 3.6 We define the topological entropy of T:
$h_{top}(T) = \sup_\alpha h_{top}(T,\alpha) \geq 0$
where α ranges over open covers of X.

Thm 3.7 If $T: X \to X$ is a homeo then $h_{top}(T) = h_{top}(T^{-1})$

- For any open cover α
$h_{top}(T,\alpha) = \lim \frac{1}{n} H(\bigvee_0^{n-1} T^{-i}\alpha)$
$= \lim \frac{1}{n} H(T^{n-1} \bigvee_0^{n-1} T^{-i}\alpha)$
$= \lim \frac{1}{n} H(\bigvee_0^{n-1} T^{i}\alpha)$
$= h_{top}(T^{-1},\alpha)$ //

Lemma 3.8 For α an open cover and $T: X \to X$ contin:
$\forall p \geq 1 : h_{top}(T,\alpha) = h_{top}(T, \bigvee_0^{p-1} T^{-i}\alpha)$
If T is a homeo
$\forall p \geq 1 : h_{top}(T,\alpha) = h_{top}(T, \bigvee_{-n}^{n} T^{-i}\alpha)$

- proof: $a_n(\alpha) = H(\bigvee_0^{n-1} T^{-k}\alpha)$.
Then $a_n(\bigvee_0^{p-1} T^{-i}\alpha) = H(\bigvee_0^{n-1} T^{-k} \bigvee_0^{p-1} T^{-i}\alpha) = H(\bigvee_0^{n+p-2} T^{-k}\alpha) = a_{n+p-1}(\alpha)$ and
$\lim \frac{1}{n} a_n = \lim \frac{1}{n} a_{n+p-1}$ for fixed p. Similarly for the case of a homeo. //

(X,d) compact topol. space. $T: X \to X$ contin.

Def 3.9 An open cover α of X is said to be T-fine iff

For every $x \in X$, $U \subset X$ open neighborhood of x $\exists n \geq 1$:

$\forall B \in \bigvee_0^{n-1} T^{-i}\alpha$, $x \in \overline{B} \Rightarrow B \subset U$.

Prop 3.10 If α is a T-fine cover of X and β is any cover then there is $N \geq 1$ so that $\beta < \alpha_N = \bigvee_0^{N-1} T^{-i}\alpha$. In particular $h_{TOP}(T,\beta) \leq h_{TOP}(T,\alpha_N) = h_{TOP}(T,\alpha)$

proof: Let $\alpha = \{A_1, \dots, A_d\}$ and let $\beta = \{B_1, \dots, B_m\}$. Define for each n a map ~~[illegible]~~

$$\phi_n : \Sigma_d^+ \to \alpha_n$$

by $\phi_n((i_0 i_1 \dots)) = A_{i_0} \cap T^{-1}A_{i_1} \cap \dots \cap T^{-(n-1)}A_{i_{n-1}}$

Let $E_n \subset \Sigma_d^+$ be the collection of $\xi = (i_0 i_1 \dots) \in \Sigma_d^+$ with the property that

$$\phi_n(\xi) \not\subset B_j \quad \forall 1 \leq j \leq m.$$

If $E_N = \emptyset$ for some N then every $A \in \alpha_N$ is included is some B_j (since $A = \phi_n(\xi)$ for some ξ) and we are done.

So suppose $E_N \neq \emptyset$. $\forall N$. E_N is a union of N-cylinders (when compact in Σ_d^+).

Also $E_N \supset E_{N+1} \supset \dots$

since $\phi_{n+1}(\xi) = \phi_n(\xi) \cap T^{-n}A_{i_n} \subset \phi_n(\xi)$)

When $E_N \neq \emptyset$, $\forall N$ their intesection is non-empty So let $\xi = (i_0 i_1 \dots) \in \bigcap_N E_N$

Now $\emptyset \neq \overline{\phi_{n+1}(\xi)} \subset \overline{\phi_n(\xi)}$ so $\overline{\phi_n(\xi)}$ is a decreasing sequence of ~~non~~ compact non-empty sets in X so there is

$$x \in \bigcap_{n \geq 1} \overline{\phi_n(\xi)}$$

But β is an open cover so $x \in B_j$ for some j. Since $\phi_n(\xi) \in \alpha_n$ and $x \in \overline{\phi_n(\xi)}$ T-fineness implies that $\phi_n(\xi) \subset B_j$ for some n. This contradicts the construction of ξ. //

Thm 3.11 If α is a T-fine cover of X then

$$h_{TOP}(T) = h_{TOP}(T,\alpha)$$

Coroll 3.12 When (X,T) is a metric space (compact) and α is an open cover such that:

$$\sup\{\operatorname{diam} A : A \in \bigvee_0^{n-1} T^{-i}\alpha\}$$

tends to zero as $n \to \infty$ then α is T-fine.

$A \in d \times d$ matrix

Example: $\Sigma_A^+ = \{ \xi = (b_n)_{n \geq 0} : A_{b_n b_{n+1}} = 1 \} \circlearrowleft T$ shift

$\xi_A = \{ [0]_{\Sigma_A^+}, \ldots, [d-1]_{\Sigma_A^+} \}$ generates the topology and is T-fine (exo).

$\xi_{A_n} =$ set of n-cylinders

$\mathrm{Card}\, \xi_{A_n} = \sum_{i=1}^{d} \sum_{j=1}^{d} (A^n)_{ij}$ (at least if topol. trans.)

With the sup-norm on $\mathbb{R}^d$:

$\|A^n\| = \sup_i \sum_j (A^n)_{ij}$

$\leq \mathrm{Card}\, \xi_{A_n} \leq d \cdot \|A^n\|$ so

$\rho_{sp}(A) = \lim_n \|A^n\|^{1/n} = \lim_n |\mathrm{Card}\, \xi_{A_n}|^{1/n} = e^{h_{top}}$

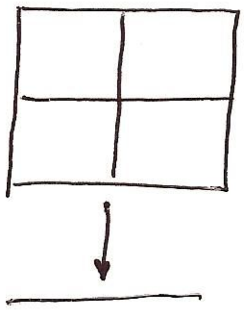
A positive $\Rightarrow$ A has a largest positive eval λ_1 and $\lambda_1 = e^{h_{top}}$

P-F (exo) cone

(NB: need not be strictly pos.)

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\lambda_1 = \frac{1+\sqrt{5}}{2} \Rightarrow h_{top} = \log \frac{1+\sqrt{5}}{2}$

Entropy is an invariant of topological conjugacy.



$\mathbb{T}^2 \circlearrowleft 2x$

$h_{top} = \log 4$

$\mathbb{T}^1 \circlearrowleft 2x$

$h_{top} = \log 2$

Thm $^{3.13}$ X_1, X_2 compact spaces. $T_i : X_i \circlearrowleft$ contin. $i = 1, 2$ and $\phi : X_1 \to X_2$ cont ~~ant~~ surjective and $\phi \circ T_1 = T_2 \circ \phi$ then $h(T_1) \geq h(T_2)$. If ϕ is a homeo then $h(T_1) = h(T_2)$.

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \phi \downarrow & & \downarrow \phi \\ X_2 & \xrightarrow[T_2]{} & X_2 \end{array}$$

Proof: α open cover of X_2 $\Rightarrow \phi^{-1}\alpha$ is an open cover of X_1 and $H_{X_1}(\phi^{-1}\alpha) = H_{X_2}(\alpha)$. Also for a refinement

$\alpha \vee T_2^{-1}\alpha \vee \ldots \vee T_2^{-k+1}\alpha$

we have

$$\phi^{-1}(\alpha \vee T_2^{-1}\alpha \vee \ldots \vee T_2^{-k+1}\alpha) = \phi^{-1}(\alpha) \vee T_1^{-1}\phi^{-1}(\alpha) \vee \ldots \vee T_1^{-k+1}\phi^{-1}(\alpha)$$

so $h(T_1, \phi^{-1}(\alpha)) = h(T_2, \alpha)$

and
$$h(T_1) = \sup_\beta h(T_1, \beta) \geq \sup_\alpha h(T_1, \phi^{-1}(\alpha)) = \sup_\alpha h(T_2, \alpha) = h(T_2)$$

When ϕ is a homeom. $\beta = \phi^{-1}\alpha$ covers X_1 iff $\phi\beta = \alpha$ covers X_2.

//

Thm $^{4.19}$ If $T : X \to X$ is a homeom of a compact space X then $h(T) = h(T^{-1})$

proof

$$h(T, \alpha) = \lim_n \frac{1}{n} H\left(\bigvee_0^{n-1} T^{-i}\alpha\right)$$
$$(\text{prop}) = \lim_n \frac{1}{n} H\left(T^{n-1} \bigvee_0^{n-1} T^{-i}\alpha\right)$$
$$= \lim_n \frac{1}{n} H\left(\bigvee_0^{n-1} T^{k}\alpha\right)$$
$$= h(T^{-1}, \alpha)$$

The Varational principle

(X,d) metric space,
$T: X \to X$ contin.

Def 3.14 Dynamical Metric [Bowen]

$d_n(x,y) = \max\{d(T^k x, T^k y) : 0 \le k \le n\}$

$B_n(x,r) = \{y : d_n(x,y) < r\}$

$= \bigcap_{0 \le k \le n} T^{-k} B(T^k x, r)$

Def 3.15 $E \subset X$ is said to (n,ϵ) separated if $\forall x, y \in E$
$x \neq y \Rightarrow d_n(x,y) \ge \epsilon$

It is maximally (n,ϵ)-sep if $\bigcup_{x \in E} B_n(x,\epsilon) \supseteq X$

~~We can~~ note

$s_n(\epsilon, X) = \inf\{\text{Card } E :$
E is maximally (n,ϵ) separated in $X\}$

Thm [Bowen] 3.16

$$\lim_{\epsilon \to 0} \overline{\lim_n} \frac{1}{n} \log s_n(\epsilon, X) = h_{TOP}(X,T)$$

(admitted)

The Varational Principle

3.17

Lemma X compact metric sp, $\mu \in M_T^1(X)$

(1) $\forall x \in X, r>0\ \exists\ 0<\delta<r : \mu(\partial B(x,\delta)) = 0$

(2) $\forall \delta > 0\ \exists$ a finite meas partition $\xi = \{C_1, \dots, C_k\}$ with $\mu(\partial\xi) = 0$ and $\operatorname{diam} C_i < \delta$.

Proof: $\bigcup_{0<\delta<r} \partial B(x,\delta)$ is an uncountable disjoint union with finite measure

(2) Let $B_1, \dots, B_k$ be a cover of X by balls of radius $<\delta/2$ and $\mu(\partial B_i) = 0\ \forall i$. Set

$C_1 = \bar{B}_1,\ C_i = \bar{B}_i \setminus \bigcup_{j<i} \bar{B}_j$.

This is a δ-partition and $\partial\xi \subset \bigcup \partial B_i$ so has zero measure.

3.18

Lemma: If ξ is a finite partition with $\mu(\partial\xi) = 0$ ~~thereabsand~~ μ is T-invariant then ~~also~~:

1) ~~$\mu(\partial T^{-1}\xi) = 0$~~

$$\mu\left(\partial \bigvee_{i=0}^{n-1} T^{-i}\xi\right) = 0$$

2) Also if $\mu_n \xrightarrow{w^*} \mu$ then:

$H_{\mu_n}(\xi) \to H_\mu(\xi)$

proof: $T^{-i}\partial\xi$ has zero measure

1) and $\partial(A \cap B) \subset \partial A \cup \partial B$

$\partial(T^{-1}A) \subset T^{-1}\partial A$ (exo)

so if $C \in \bigvee_{i=0}^{n-1} T^{-i}\xi$ then

C is an intersection of $T^{-i}C_j$ for which each $\partial T^{-i}C_j$ has zero meas

$A = \mathring{A} \cup \partial A$ (disjoint)

$T^{-1}A = \underbrace{T^{-1}\mathring{A}}_{\text{open}} \cup T^{-1}\partial A$ (not nec disj.)

$\partial(T^{-1}A) \subset \overline{T^{-1}A} \setminus T^{-1}\mathring{A}$

$T^{-1}\mathring{A} \subset T^{-1}A \subset T^{-1}\bar{A}$ so

$T^{-1}\mathring{A} \subset (T^{-1}A)^\circ,\ \overline{(T^{-1}A)} \subset T^{-1}\bar{A}$

$\partial(T^{-1}A) \subset T^{-1}\bar{A} \setminus T^{-1}\mathring{A} = T^{-1}\partial A$

3.19

Lemma: Let E_n be a (possibly maximal) (n,ε)-sep set, $\nu_n = \frac{1}{|E_n|} \sum_{x \in E_n} \delta_x$. ~~let~~ and ~~Let be an accum pt.~~

Let $\mu_n = \frac{1}{n} \sum_{0}^{n-1} T^{-i}\nu_n$.

~~Let μ be an accum pt taed~~

Then (μ_n) has an accum pt μ ~~which~~ T-invar and: $\overline{\lim_n} \frac{1}{n} \log |E_n| \le h_\mu(T)$

Pick n_k s.t. ~~$\frac{1}{n_k}\log|E_{n_k}| \to$~~

$\frac{1}{n_k} \log |E_{n_k}| \to \overline{\lim} \frac{1}{n} \log |E_n|$

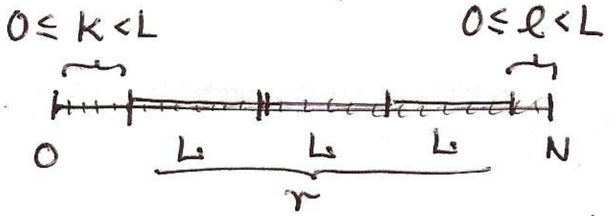
and let μ be an accum pt of μ_{n_k}.

Consider a partition ξ with diam $< \varepsilon$ and $\mu(\partial\xi) = 0$

Each element in $\xi_n = \bigvee_0^{n-1} T^{-i}\xi$ contains at most one element of E_n with ν_n-meas $\frac{1}{|E_n|}$

so $\log |E_n| = H_{\nu_n}(\xi_n)$

2) If ~~$\mu(\partial C)$~~ $\mu(\partial C) = 0$ then $\lim_n \mu_n(C) = \mu(C)$



$$\xi_n = \bigvee_{i=0}^{n-1} T^{-i}\xi$$

$$N = k + (\underbrace{L+L+\dots+L}_{r}) + \ell = k + r.L + \ell.$$

$$\xi_N = \bigvee_0^{N-1} T^{-i}\xi = \underbrace{\xi_k}_{(k<L)} \vee T^{-k}(\xi_L \vee T^{-L}\xi_L \vee \dots \vee T^{-(r-1)L}\xi_L) \vee \underbrace{T^{-(N-\ell)}\xi_\ell}_{(<L)}$$

$$H(\xi_N) \le 2L\, H(\xi) + \sum_{j=0}^{r-1} H(T^{-k-jL}\xi_L)$$ [Summing over $0 \le k < L$]

$$\sum_{k=0}^{L-1}: \quad L\,H(\xi_N) \le 2L^2 H(\xi) + \sum_{i=0}^{N-1} H(T^{-i}\xi_L)$$

$$\left(\nu_n = \frac{1}{|E_n|} \sum_{x \in E_n} \delta_x \right) \qquad \mu_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i_* \nu_n$$

Now, for any measure m and any finite partition ξ one has $H_m(T^{-1}\xi) = H_{T_* m}(\xi)$
so

$$L\,H_{\nu_N}(\xi_N) \le 2L^2 H(\xi) + \sum_{i=0}^{N-1} H_{T^i_*}(\xi_L)$$
$$\le 2L^2 H(\xi) + N H_{\mu_N}(\xi_L)$$

facile:
$p H_{\mu_1} + (1-p) H_{\mu_2} \le H_{p\mu_1 + (1-p)\mu_2}$
concavity of ϕ

or
$$\boxed{\frac{1}{N} H_{\nu_N}(\xi_N) \le \frac{2L\,H(\xi)}{N} + \frac{1}{L} H_{\mu_N}(\xi_L)}$$

For fixed L let $N_k \to \infty$ be such that

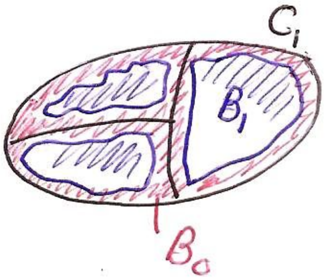
$$s^* = \overline{\lim_k} \frac{1}{N}\log|E_N| = \lim_k \frac{1}{N_k}\log|E_{N_k}|$$
$$\le 0 + \lim_k \frac{1}{L} H_{\mu_{N_k}}(\xi_L) \overset{\text{Lemma 3.18}}{=} \frac{1}{L}(H_\mu(\xi_L))$$

Letting $L \to \infty$: $s^* \le \lim \frac{1}{L} H_\mu(\xi_L) = h_\mu(T,\xi) \le h_\mu(T)$

Thm 3.20 Varational Principle
$f: X \to X$ cont map of a cmpt metric space (X,d). Then
$$h_{top}(f) = \sup_{\mu \in M_f^1(X,T)} h_\mu(f)$$

Let $\mu \in M_f^1(X,T)$

Proof: Let $C_1,\dots,C_k$ be a meas. partition of X. Since μ is Borel $\exists$ compact subsets $B_i \subset C_i$, $1 \le i \le k$, such that
$$\beta = \{B_0, B_1, \dots, B_k\}$$
with $B_0 = X \setminus (B_1 \cup \dots \cup B_k)$ is close to $\xi = \{C_1,\dots,C_k\}$,
$$H(\xi|\beta) < 1$$



then $h_\mu(f,\xi) \le h_\mu(f,\beta) + H(\xi|\beta)$
$\le h_\mu(f,\beta) + 1$

Now $\gamma = \{B_0 \cup B_1, \dots, B_0 \cup B_k\}$ is an open cover of X.

We have:
$$H_\mu\left(\bigvee_0^{n-1}\beta\right) \le \log \text{Card} \bigvee_0^{n-1}\beta \le \log 2^n \cdot \text{Card} \bigvee_0^{n-1}\gamma$$

{since for each $(B_{i_0} \cap B_{i_0}) \cap T^{-1}(B_1 \cap B_{i_1}) \cap \dots$ we may choose at each place B_0 or B_{i_j}} whence 2^n possibilities. So
$$H_\mu(\beta_n) \le n\log 2 + H_{top}(\gamma_n)$$
and
$$h_\mu(T,\beta) \le \log 2 + h_{top}(T,\gamma) \le \log 2 + h_{top}(T)$$

So $h_\mu(f,\xi) \le h_{top}(T) + \log 2 + 1$
valid for any cont map

replacing T by T^n we get
$n h_\mu(T) \overset{(ex\,0)}{=} h_\mu(T^n) \le h_{top}(T^n) + \log 2 + 1 \overset{(ex\,0)}{=} n\, h_{top}(T) + \log 2 + 1$ so $(1/n\,\times)$
$h_\mu(T) \le h_{top}(T)$. $(\forall n \ge 1)$

Applying Lemma 3.19 to E_n, a maximal (n,ε)-separated sets in X of cardinality $s_n(\varepsilon,X)$ we get with μ a corresponding accum pt in $M_f^1(X,T)$:
$$\lim_n \frac{1}{n} s_n(\varepsilon,X) \le h_\mu(T)$$
and so $s(\varepsilon,X,T) \le h_\mu(T) \le \sup_\nu h_\nu(T)$
and letting $\varepsilon \to 0$
$$h_{top}(T) \le \sup_{\nu \in M_f^1(X,T)} h_\nu(T)$$

Remark With μ being the above acc. pt we see that the sup is attained
$$h_{top}(T) = h_\mu(T)$$
provided there is $0 < \delta_0$ s.t. $s(\varepsilon,X,T) = s(\delta_0,X,T)$ $\forall\, 0 < \varepsilon < \delta_0$

This happens e.g. if T is expansive:
$$\sup_x \operatorname{diam} B_n(x,\delta_0) \xrightarrow[n\to\infty]{} 0$$