2.2 Symbolic Dynamics ~~and~~ (Topological Dynamical Systems)

Defs 2.1: Let $d \geq 2$ and $A \in M_d(\{0,1\})$ a $d \times d$ matrix taking values 0 or 1. We call A a transition matrix or adjacency matrix (matrice d'adjacence ou transition).

We associate to A a shift-space one-sided / two-sided:

$$\Sigma_A^+ = \{(b_i)_{i \geq 0} : 1 \leq b_i \leq d,\ A_{b_i b_{i+1}} = 1,\ \forall i\}$$

$$\Sigma_A = \{(b_i)_{i \in \mathbb{Z}} : \quad " \quad \}$$

and the shift map:

$$\sigma : \Sigma_A^+ \circlearrowleft \quad (\sigma(b))_i = b_{i+1}$$

(Σ_A^+, σ) and (Σ_A, σ) are called SUBSHIFT of finite type.

ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Sigma_A^+ = \{(111\ldots), (222\ldots)\} = \{\overline{1}, \overline{2}\}$$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Sigma_A^+ = \{(121212\ldots), (212121\ldots)\} = \{\overline{12}, \overline{21}\}$$

periodic points of period 2.

Cylindersets:

$$Z_n^+ = \{[b_0 \ldots b_{n-1}] \neq \emptyset\}$$

$$[b_0 \ldots b_{n-1}] = \{\xi \in \Sigma_A^+ : \xi_0 = b_0, \ldots, \xi_{n-1} = b_{n-1}\}$$

$$Z_{m,n} = \{[b_m \ldots b_n] \neq \emptyset\} \quad m \leq n \in \mathbb{Z}$$

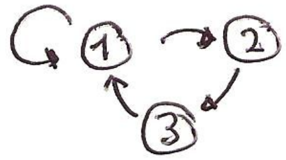
(pareil)

$$\mathrm{Card}\, Z_n^+ = \mathrm{Card}\{b_0 \ldots b_{n-1} : A_{b_0 b_1} \ldots A_{b_{n-2} b_{n-1}} = 1\} = \sum_{b_0 \ldots b_{n-1}} A_{b_0 b_1} \ldots A_{b_{n-2} b_{n-1}} = \sum_{b_0, b_{n-1}} (A^n)_{b_0 b_{n-1}} = \sum_i \sum_j (A^n)_{ij}$$

Transition Graphe:

$$A = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \quad i \to j$$

$\{1,2,3\} \to \{1,2,3\}$



11231223… ok

13 forbidden transition
21
32 ; 33

Topology: The topology on Σ_A^+ or (Σ_A) is the one generated from the cylinder sets as a base.

n-paths or
Admissible path of length n

$b_0 b_1 \ldots b_{n-1}$ is admissible if $A_{b_0 b_1} = 1,\ A_{b_1 b_2} = 1, \ldots, A_{b_{n-2} b_{n-1}} = 1$

11231223 an admissible path len = 8

Prop 2.2 $\forall n \geq 1$, $i, j \in \{1, \dots, d\}$

$A^n_{ij} = \mathrm{Card}\,\{$ admissible paths of length n from i to j in the graph$\}$. and $\mathrm{Card}\, Z^+_n = \sum_{i,j} A^n_{ij}$

Proof: $i_0 i_1 \dots i_{n-1}$ is a path in from $i_0 = i$ to $i_{n-1} = j$ iff

$A_{i_0 i_1} = A_{i_1 i_2} = \dots = A_{i_{n-2} i_{n-1}} = 1$ or

$\cancel{(A^n)_{ij} = A}$

$A_{i_0 i_1} A_{i_1 i_2} \cdots A_{i_{n-2} i_{n-1}} = 1$

So number of ~~pe~~ n-paths from i to j =

$$\sum_{i_1 \dots i_{n-1}} A_{i i_1} A_{i_1 i_2} \cdots A_{i_{n-2} j} = (A^n)_{ij}.$$

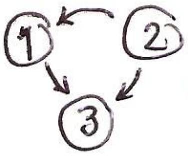
$\mathrm{Card}\, Z^+_n = \sum$

ex
$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $A^3 = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$(A^3)_{12} = 3$: 1112
1122
1222

Prop 2.3 Σ^+_A (and Σ_A) are ~~non~~ empty iff A is nilpotent. (We suppose that this is not the case from now on)

Σ^+_A (and Σ_A) are compact subsets of Σ^+_d (and Σ_d), metrisable and σ is contin.



$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$A^3 = 0$

proof A nilpotent $\overset{\text{clear}}{\Rightarrow} \Sigma^+_A = \emptyset$

A not nilpotent $\Rightarrow$ $\forall n$: $\exists\, b_0 \dots b_{n-1}$ admissible

$K_n = \{ \xi = (\xi_i)_{i \geq 0} \in \Sigma^+_d : \xi_0 = b_0 \dots \xi_{n-1} = b_{n-1}$ $b_0 \dots b_{n-1}$ adm.$\}$

is a nested seq of compact ~~sets~~ non-empty subsets of Σ^+_d.

$\bigcap K_n = \Sigma^+_A$ is then $\neq \emptyset$.

(and compact, metrisable)

$\sigma : \Sigma^+_d \circlearrowleft$ is cont whence also its restriction to Σ^+_A.

$0 < \theta < 1$: $d(\xi, \eta) = \sup\{\theta^k : \xi_k \neq \eta_k\}$

Prop 2.4 (Σ_A^+, σ) is topologically
1) transitive iff $\forall i,j \ \exists m: \ (A^m)_{ij} \geq 1$.

2) (Σ_A^+, σ) is topologically mixing iff
$\exists m \ \forall i,j: (A^m)_{ij} \geq 1$

proof: Let U, V be open subsets of Σ_A^+.
Then $U \supset [b_0 \ldots b_{k-1}] \neq \emptyset$
$V \supset [c_0 \ldots c_{e-1}] \neq \emptyset$

1) Choose m so that $(A^m)_{b_{k-1} c_0} \geq 1$

Then
$\sigma^{m+k} U \cap V \supset [c_0 \ldots c_{e-1}] \neq \emptyset$.
so topol. transitive

2) Let m be such that $A^m \geq 1$
then $\forall n \geq k+m$
$\sigma^n U \cap V \supset [c_0 \ldots c_{e-1}] \neq \emptyset$
so topol mixing //

First do this calculation
Remark: For $1 \leq b \leq d$:
$\sigma([b]) = \bigcup_{c: A_{bc}=1} [c]$ $\quad \sigma^k([b]) = \bigcup_{c: A^k_{bc} \geq 1} [c]$

When $[b_0 \ldots b_{k-1}] \neq \emptyset$ then
$\sigma^j([b_0 \ldots b_{k-1}]) = [b_j \ldots b_{k-1}]$, $j < k$
$\sigma^k([b_0 \ldots b_{k-1}]) = \bigcup_{c: A_{b_{k-1} c}=1} [c]$

Periodic points and orbits

Def 2.5 (X, τ, T) a topol. dyn. sys.
$\xi \in X$ is <u>periodic</u> iff $\exists m \geq 1$: $\xi = T^m \xi$
The smallest such m is the <u>prime period</u>. We write

$$\cancel{Per_m}\ \mathrm{Fix}(T^m) = \{x : T^m x = x\}$$

when ~~x ∈ Fix T^m~~ x is periodic its orbit is

$$\mathcal{O}(x) = \{x, Tx, \ldots, T^{p-1}x\}$$

with p being the prime period.
~~We write~~

~~Per Per_m~~

Subshift case: (Σ_A^+, σ)

$\xi = (\xi_i)_{i \in \mathbb{N}} = \sigma^m(\xi)$ iff

$\xi_i = \xi_{i+m}$ $\forall i$ i.e.

$\xi = (\overline{\xi_0 \ldots \xi_{m-1}})$ ← bar = infinite repetition

<u>Prop</u> 2.6 $\# \mathrm{Fix}\, \sigma^m = \mathrm{tr}\, A^m$

proof: $\# \mathrm{Fix}\, \sigma^m = \# \{\xi_0 \ldots \xi_{m-1} : A_{\xi_0 \xi_1} \cdots A_{\xi_{m-1}\xi_0} = 1\} =$

$$\sum A_{\xi_0 \xi_1} \cdots A_{\xi_{m-1}\xi_0} = \sum_{\xi_0} (A^m)_{\xi_0 \xi_0} = \mathrm{tr}\, A^m$$

<u>Coroll</u> 2.7 Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A repeated acc. to multipl. Then

$$\# \mathrm{Fix}\, \sigma^m = \sum \lambda_i^m$$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$ 

$$\frac{1}{m} \log \# \mathrm{Fix}\, \sigma^m \to \log \frac{1+\sqrt{5}}{2}$$