1. Topological Dynamical Systems TD M1
Minimality

Let (X, τ) be a compact topol sp. and let $T: X \to X$ be cont.

Def 1.1 $T: X \to X$ is said to be minimal iff $\forall x \in X$:
$O_+(x) = \{ f^k x : k \geq 0 \}$ is dense in X.

A closed subset $E \subset X$ is said to be minimal for T iff $T(E) \subset E$ and $T_{|E}$ is minimal.

Thm 1.2. The following are equivalent:

(i) $T: \cancel{X} \to \cancel{X}$ is minimal

(ii) If $E \subset X$ is a closed subset, with $T(E) \subset E$ then $E = \emptyset$ or X

(iii) If $U \subset X$ is a non-empty open subset then $\bigcup_{k \geq 0} T^{-k} U = X$

proof: (i) $\Rightarrow$ (ii). Let $E \subset X$ be closed and forward invariant. If $x \in E$ then $X = Cl\, O_+(x) \subset E$ so $E = X$.

(ii) $\Rightarrow$ (iii) Let U be open ~~o~~non-empty. Then $E = X \setminus \bigcup_{k \geq 0} T^{-k} U$ is closed and forward invariant. But $E \neq \cancel{\emptyset} X$ so $E = \cancel{X} \emptyset$ and $\bigcup_k T^{-k} U = X$

(iii) $\Rightarrow$ (i) Let $x \in X$ and pick any open set $U \subset X$. Then $\bigcup_k T^{-k} U = X$ so there must be k with $T^k x \in U$.

Thm 1.3: Any topol dyn sys on a compact space has a minimal subset.

proof: Let $\mathcal{E}$ denote the collection of non-empty, closed (whence compact) forward T-invar subsets of X.

Then $\mathcal{E}$ is partially ordered under inclusion, non-empty (since $X \in \mathcal{E}$) and every linearly ordered collection E_α, $\alpha \in A$ has a least element $\bigcap_\alpha E_\alpha$ (which is compact, non-empty and forward T-invariant)

By Zorn $\mathcal{E}$ admits a ~~least~~ (minimal) element E^*.

~~By thm~~ If $E \subset E^*$ is a strictly smaller closed forward T-invar subset, minimality of E^* implies $E = \emptyset$ so by Thm (ii) $T_{|E^*}$ is minimal.

Rem: easy counter-exs on non-compact spaces.
$]0,1[\circlearrowleft x \mapsto x/2$

Thm 1.4 Let (X,τ) be compact and $T: X \to X$ continuous. Suppose that $(X, \mathcal{B}_\tau, T)$ is uniquely ergodic with measure μ.
Then T is minimal on $E = \operatorname{supp} \mu$.

<u>proof</u>:
$V = X \setminus E = \bigcup \{U : U \text{ open}, \mu(U) = 0\}$
$= $ largest open subset of zero measure.
$\mu(T^{-1}V) = \mu(V) = 0$ so $T^{-1}V \subset V$
and $T^{-1}(X \setminus E) = X \setminus T^{-1}E \subset X \setminus E$
shows that $E \subset T^{-1}E$ so
E is forward T-invariant.

~~Suppose T is not minimal on E. Then $\exists F$~~

Suppose F is a closed subset of E which is forward-T-invar, non-empty.
By the Krylov - Bogoliubov Thm there exists a T-invariant proba measure on F. By unique ergodicity it must be μ and $E = \operatorname{supp} \mu \subseteq F \subseteq E$
shows that $F = E$ so
$T_{|E}$ is indeed minimal

Topological Mixing

Def 1.5 Let (X, τ, T) be a topol dyn sys. It is said to be

1) Topological transitive iff
$\forall U, V \subset X$ open, non-empty:
$\exists m \geq 0: \ U \cap T^{-m} V \neq \emptyset$

2) Topological Mixing iff
$\forall U, V \subset X$ open, non-empty:
$\exists M = M(U, V) \ \forall m \geq M: \ U \cap T^{-m} V \neq \emptyset$.

Lemma 1.6 Let $(x_n)_{n \geq 0}$ be a dense sequence in a metric sp (X, d). Then every $p \in X$ is

* either an isolated point
* or an accumulation pt of the sequence.
(i.e. $\exists n_k : 0 < d(p, x_{n_k}) \xrightarrow[k]{} 0$)

proof: For $n \geq 0$ let

$r_n = \min \{ d(p, x_j) > 0 : 0 \leq j \leq n \} > 0$.
with $\min \{\emptyset\} = +\infty$.

Then r_n is decreasing whence has a limit $r^* = \lim_n r_n \geq 0$.

Either $r^* > 0$ so $d(p, x_n) \geq r^* \ \forall n$ whence $d(p, y) \geq r^* \ \forall y \in X$ so p is isolated

Or $r^* = 0$: $\forall k \geq 1 \ \exists n_k$ with $0 < d(x_{n_k}, p) < 1/2^k$ so p is accum pt of (x_n).

Thm 1.7 Let (X, d) be a metric space which is separable and locally compact and without isolated points.

A cont map $T: X \to X$ is topologically transitive iff it has a dense forward orbit, i.e. $\exists x \in X$: $O_+(x) = \{ f^k x : k \geq 0 \}$, $\overline{O_+(x)} = X$.

proof Given U, V open non-emp. and a dense orbit $O_+(x) = \{x_0, x_1, x_2, \dots\}$ the sequence (x_n) enters ∞ many times in both U and V

(since any $p \in U$ is accum pt of the seq by Lemma 1.6)

$\exists n_1 < n_2$ with $T^{n_1}_x \in U$, $T^{n_2}_x \in V$
so $T^{n_1}_x \in U \cap T^{-(n_2 - n_1)} V \neq \emptyset$.

Conversely as X is separable there is a countable base for the topol $U_1, U_2, U_3, \dots$

Pick an open set V_0 (non-empty).
There is N_1 with $V_0 \cap T^{-N_1} U_1 \neq \emptyset$
By local compactness there is V_1 non-empty open so that
$V_1 \subset \overline{V_1} \subset V_0 \cap T^{-N_1} U_1$
Construct recursively V_k open non-empty with
$V_k \subset \overline{V_k} \subset V_{k-1} \cap T^{-N_k} U_k$
Then $\bigcap_{n \geq 0} \overline{V_k}$ is compact non-empty and if $x \in \bigcap \overline{V_k}$ then $T^{N_k} x \in U_k \ \forall k \geq 1$

Topol Dyn Syst

(X,d) compact metric space
$T: X \to X$ a continuous map.

Def 1.8 For $x \in X$
$\omega(x) = \{\text{limit pts of } (T^n x)_{n\geq 0}\}$
$= \{y \in X : \exists n_k \to \infty,\ T^{n_k} x \to y\}$
When T is a homeomorphism
$\alpha(x) = \{\text{limit pts of } (T^{-k} x)_{k \geq 0}\}$.

Remarks
1) $T^{-1}\omega(x)$ may be strictly larger than $\omega(x)$ (clear).

2) If E is a minimal subset for T then $\omega(x) = E$, $\forall x \in E$

Thm 1.9 $\omega(x)$ is closed, non-empty and T invariant
$T\omega(x) = \omega(x) = \omega(Tx)$
One has:
(*) $\omega(x) = \bigcap_{n\geq 0} Cl\{T^k x : k \geq n\}$.

proof: Let $K = \bigcap_n Cl\{T^k x : k \geq n\}$
Suppose $y \in \omega(x)$, i.e. $\exists n_k \to \infty$ with $T^{n_k}(x) \to y$. For any $n \geq 0$ we have $n_k \geq n$ for k large enough so $y \in Cl\{T^k x : k \geq n\}$ $\forall n$.

Conversely if $y \in K$, $\forall N\ \exists k_N \geq N$ with $d(y, T^{k_N} x) < \frac{1}{N}$ so $y \in \omega(x)$.

K is the intersection of a decr. seq of compact non-empty sets whence compact, non-empty.

Clearly $T\omega(x) \subset \omega(x) = \omega(Tx)$

Let $y \in \omega(x)$, $T^{n_k} x \to y$.
The sequence $T^{n_k - 1} x$ admits a conv. sub-seq
$T^{n_{k_j} - 1} x \xrightarrow[j]{} z$ and so $z \in \omega(x)$
and $y = \lim_j T\, T^{n_{k_j} - 1} x = Tz$
so $y \in T\omega(x)$ //

Def 1.10 $T: X \to X$ cont. A point $x \in X$ is called <u>wandering</u> for T if $\exists \mathcal{U}$ a nghd of x such that $\forall n > m \geq 0$
$T^{-n}\mathcal{U} \cap T^{-m}\mathcal{U} = \emptyset$

"errant" français

The <u>non-wandering set</u> for T, $\Omega(T)$, consists of all points that are not wandering. One has

$$\Omega(T) = \{x \in X : \forall \mathcal{U} \text{ nghd of } x : \exists n \geq 1 \text{ with } T^{-n}\mathcal{U} \cap \mathcal{U} \neq \emptyset\}$$

Remark: For $n > m \geq 0$:
1) $T^{-n}\mathcal{U} \cap T^{-m}\mathcal{U} = T^{-m}(T^{-(n-m)}\mathcal{U} \cap \mathcal{U})$
so it suffices to consider the case $m = 0$.
2) T homeo $\Rightarrow$ $\Omega(T) = \Omega(T^{-1})$

Thm 1.11
(i) $\Omega(T)$ is closed
(ii) $\bigcup_{x \in X} \omega(x) \subset \Omega(T)$ (so $\neq \emptyset$)
(iii) All periodic pts $\in \Omega(T)$
(iv) $T\Omega(T) \subset \Omega(T)$ and if T is a homeo: $T\Omega(T) = \Omega(T)$

Proof: (i) $X \setminus \Omega(T)$ is clearly open (every point in the set $\mathcal{U}$ are wandering).

(ii) Let $x \in X$, $y \in \omega(x)$, and let V be a nghd of y. Since $\exists n_k \to \infty$: $T^{n_k}x \to y$ there are $n_\ell < n_m$ with $T^{n_\ell}x, T^{n_m}x \in V$ and $T^{n_\ell}x \in V \cap T^{-(n_m - n_\ell)}V$.

(iii) $x = T^p x \Rightarrow \mathcal{U} \cap T^{-p}\mathcal{U} \neq \emptyset$ $\forall \mathcal{U} \in \text{nghd}(x)$

(iv) Let $x \in \Omega(T)$, $V \in$ nghd of $T(x)$. Then $T^{-1}V$ is a nghd of x so $\exists n \geq 1$ $T^{-(n-1)}V \cap T^{-1}V \neq \emptyset$ and $T^{-n}V \cap V \neq \emptyset$.