4 Conformal repellers and dimension

(Ω, d) metric space (e.g. $\mathbb{R}^n$)
$J \subset \Omega$ ~~bounded~~ ~~compact~~ measurable subset.
$B(x,r)$ open ball rad $r>0$ centered at $x \in \Omega$.

Def 1. For $r>0$ let $N(r,J)$ be the minimal # of r-balls necessary to cover J. Then the upper box-dimension of J:

$$\overline{\dim}_B J = \limsup_{r \to 0^+} \frac{\log N(r,J)}{\log 1/r}$$

(lower box-dim $\underline{\dim}_B$ similarly defined, just box-dim $\dim_B$ when limit $\exists$)

ex: $J \subset \Omega$ $\mathrm{Card}\, J < \infty \Rightarrow \dim_B J = 0$

ex: $K \subset \mathbb{R}^n$ $\mathrm{int}\, K \neq \emptyset$ bounded
$\dim_B K = n$.

ex $1/3$-Cantor $= J$

$N(r_n = \frac{1}{3^n}, J) = 2^n$
$\dim_B J = \log 2 / \log 3$.

ex $\dim_B (\{0\} \cup \{\frac{1}{n} : n \geq 1\}) = \frac{1}{2}$
$\dim_B (\mathbb{Q} \cap [0,1]) = 1$.

Def 2 For $\delta > 0$ let δ-cover(J) denote the collection of countable covers of J with open sets of diam $< \delta$. We define for $s \geq 0$:

$$M_s(\delta, J) = \inf\{ \sum_{i \geq 1} |U_i|^s : (U_i)_{i \geq 1} \in \delta\text{-cover } J \}$$

and then

$$M_s(J) = \lim_{\delta \to 0} \uparrow M_s(\delta, J) \in [0, +\infty]$$

the s-dimensional Hausdorff measure.

$\dim_H(J) = \inf\{ s \geq 0 : M_s(J) < \infty \}$
is the Hausdorff dim of J.

Prop 3 $\exists!\, s^* \in [0, +\infty]$:
$\forall\, 0 \leq s < s^* : M_s(J) = +\infty$
$\forall\, s^* < s \leq +\infty : M_s(J) = 0$

$s^* = \dim_H(J)$.
One calls $M_{s^*}(J) \in [0, +\infty]$ the Hausdorff measure of J.

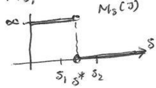
proof. When $0 \leq s_1 < s_2 < \infty$ and $\delta > 0$:

$$|U_i|^{s_2} = |U_i|^{s_2 - s_1} |U_i|^{s_1} \leq \delta^{s_2 - s_1} |U_i|^{s_1}$$

$$\Rightarrow M_{s_2}(\delta, J) \leq \delta^{s_2 - s_1} M_{s_1}(\delta, J)$$

which yield(s) the implication

$M_{s_2}(J) > 0 \Rightarrow M_{s_1}(J) = +\infty$
$M_{s_1}(J) < \infty \Rightarrow M_{s_2}(J) = 0$



exo $\dim_H(\mathbb{Q}) = 0$

Def 4 Given a ~~pp~~ positive measure μ on (Ω, d) we define its dimension:

$$\dim_H \mu = \inf\{ \dim_H A : A \subset \Omega, \mu(\Omega \setminus A) = 0 \}$$

Clearly

$$\dim_H \mu \leq \dim_H \overline{\mathrm{supp}\, \mu}$$

μ proba on Ω — support in J

Def 5 $\mu \in M^1_+(J)$. We say that μ is upper (respectively lower) Ahlfors s-regular ($s \geq 0$) if there is $\delta_0 > 0$, $C < \infty$, s.t. for all $x \in J$ and every $0 < r < \delta_0$

$$\mu(B(x,r)) \leq C r^s \quad \text{(upper)}$$

respectively ~~for all x~~

$$\frac{1}{C} r^s \leq \mu(B(x,r)) \quad \text{(lower)}$$

Remark: If true for a.e. $x \in J = \operatorname{supp} \mu$ then it is true for all $x \in J$.

Thm 6 If μ is upper

1) Ahlfors s-regular then $\dim_H J \geq s$.

2) If μ is lower A. s-reg then

$$\dim_H J \leq \dim_B \Lambda \leq s$$

proof: Let $(U_i)_{i \geq 1}$ be a

1) $\delta_0/2$-cover (J). For every U_i pick $x_i \in U_i$ an s-reg point. Since

$$U_i \subset B(x_i, |U_i|)$$

we get

$$1 \leq \sum \mu(U_i) \leq \sum \mu(B(x_i, |U_i|)) \leq \sum C |U_i|^s$$

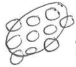
So that $M_\delta(\delta_0/2, J) \geq 1/C > 0$. Thus $\dim_H J \geq s$.

2) let $r < \delta_0/2$, $N = N_r$. Let $(x_i)_{i=1}^N$ be a maximally r-separated set in J. Then $\forall\, i \neq j$

$$B(x_i, r/2) \cap B(x_j, r/2) = \emptyset$$

and

$$\bigcup_i B(x_i, r) \supset J$$



Thus, $1 \geq \sum \mu(B(x_i, r/2)) \geq \frac{1}{C} N (r/2)^s$

so that

$$N_r \leq (2^s C) r^{-s}$$

Since N is an upper bound for the cardinality of an r-cover we have

$$\dim_B J \leq \lim_{r \to 0} \frac{\log N_r}{\log 1/r} \leq s. \quad //$$

Besicovitch cover.

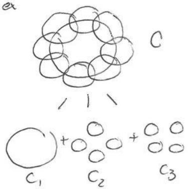
Def 7 (X,d) metric space, $A \subset X$.

I. A Besicovitch cover of A, $\mathcal{B}(A)$ is a collection of open balls centered at points in A, such that every $a \in A$ is the center of at least one ball in $\mathcal{B}(A)$.

$\mathcal{B} \subset \{ B(a,r) : a \in A, r > 0 \}$

$\forall a \in A \; \exists r > 0 : B(a,r) \in \mathcal{B}$

ex



II $\mathcal{B}(A)$ is said to be a fine cover if $\forall \delta > 0$

$\mathcal{B}_\delta(A) = \{ B(a,r) \in \mathcal{B} : a \in A, r < \delta \}$

is again a Besic. cover of A.

→ i.e. we have a Besic. cover of arbitrarily small diameter

Rem For a fine cover we have equivalently $\forall a \in A \; \exists r_n \to 0^+$ such that $B(a, r_n) \in \mathcal{B}(A)$.

Thm 8 [Besicovitch covering Lemma]

Case $X = \mathbb{R}^D$ $d =$ Euclidean.

$\exists \, b_D \in \mathbb{N}$ (dep on the dim D only) such that if $\mathcal{B}$ is a Besicovitch cover of A then $\exists$ a countable subcover $C \subset \mathcal{B}$ such that any $x \in \mathbb{R}^D$ is contained in at most b_D balls in C.

$\Leftrightarrow$ $\exists$ disjointed families $C_1, \dots, C_k \subset \mathcal{B}$, $k \leq b_D$ s.t. $C = C_1 \cup \dots \cup C_k$ covers A.

Rem: We say that a metric sp. (Ω, d) has the Besicovitch property if $\exists \, b = b_\Omega$ for which the conclusion in Thm 8 holds

Pointwise dimension

$\frac{\log \mu B(x,r)}{\log r} > s \Leftrightarrow \mu B(x,r) < r^s$ when $(0<r<1)$

μ Borel proba measure on Ω

Def 9

We define for $\delta<1$, $x\in\Omega$:

$$\underline{d}_\mu(x,\delta) = \inf_{0<r\le\delta} \frac{\log\mu(B(x,r))}{\log r}$$

~~Then~~ and set

$$\underline{d}_\mu(x) = \lim_{\delta\to 0^+}\uparrow \underline{d}_\mu(x,\delta)$$

We call $\underline{d}_\mu(x)$ the pointwise dimension at x.

We set $\underline{d}_\mu^* = \operatorname{ess\,sup}_{x\in\Omega} \underline{d}_\mu(x)$

One has for all $s<\underline{d}_\mu(x,\delta)$, $0<r\le\delta$: $\mu(B(x,r)) < r^s$ and $\underline{d}_\mu(x,\delta)$ is the smallest number for which this holds.

We set $\underline{d}_\mu(x)=+\infty$ if $\mu(B(x,r))=0$ for some $r>0$. (but such points have meas zero and do not appear in ess sup)

When Ω has the Besicovitch property:

Thm 10 $\dim_H\mu = \underline{d}_\mu^*$

proof Let $s<\underline{d}^*_\mu$. Then the set $\{s<\underline{d}_\mu(x)\}$ has (strictly) pos meas.

Now $\{s<\underline{d}_\mu(x)\} = \bigcup_{n\ge1}\{s<\underline{d}_\mu(x,\tfrac1n)\}$ so there must be $\delta_0>0$ for which

$$D = \{s<\underline{d}_\mu(x,\delta_0)\}$$

has pos. meas. Set $\epsilon=\mu(D)>0$.

Let $A\subset\Omega$ be a meas subset of full measure. and let $(U_i)_{i\ge1}$ be a δ_0-cover(A). ~~Then~~ It also covers $A\cap D$ which has measure $\epsilon>0$.

For each $i\in I$ s.t. $U_i\cap D\cap A\neq\emptyset$ pick $x_i\in U_i\cap D\cap A$. Then

$$0<\epsilon=\mu(D)=\mu(A\cap D)\le \sum_{i\ge1}\mu(U_i\cap A\cap D) \le \sum_{i\in I}\mu(U_i)\le\sum\mu(B(x_i,|U_i|))$$

$\le \sum|U_i|^s$ so $M_s(D\cap A)\ge\epsilon>0$ and $\dim_H(A)\ge s$.

Let $s>\underline{d}^*_\mu$. Then $D=\{\underline{d}_\mu(x)\ge s\}$ has zero measure. Then for a.e $x\in\Omega$:

$\underline{d}_\mu(x)<s$ the set of such points [illegible]

Let $A\subset\Omega$ be of full measure. Then for every $x\in A$ there must be a sequence $r_n\to0^+$ st

$$\mu(B(x,r_n))>r_n^s$$

The cover $\mathcal{E}=\{B(x,r_n(x)): x\in A, n\ge1\}$ is thus a fine Besicovitch covering: Given $\delta_0>0$ $\exists C\subset\mathcal{E}_{\delta_0}$ a countable δ_0-cover(A) for which $\forall x\in A: \#\{B\in C: x\in B\}\le b_\Omega<\infty$

But then

$$\sum_{B\in C}|B(x,r_i)|^s\le\sum(2r_i)^s$$

$$\le 2^s\sum\mu(B(x,r_i))$$

$$\le 2^s b_\Omega\,\mu(A)<+\infty \Rightarrow$$

$$M_s(A)\le 2^s b_\Omega\,\mu(A)<\infty \Rightarrow$$

$$\dim_H A\le s$$

(*)

Normal families and Montel's thm

Carleson - Gamelin
Complex Dynamics

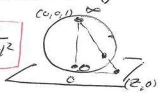
$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $d_{\overline{\mathbb{C}}}$ spherical metric

$\overline{\mathbb{C}} \setminus \{0\} = \overline{\mathbb{C}}^* \rightarrow \mathbb{C} = \overline{\mathbb{C}} \setminus \{\infty\}$

$z \mapsto 1/z$ chart-map.

holomorphic ~~distan~~ isometry on $\overline{\mathbb{C}}$

$$\boxed{ds = \frac{|dz|}{1+|z|^2}}$$



$(\overline{\mathbb{C}}, d_{\overline{\mathbb{C}}})$ is a compact Riemann ~~ian~~ surface.

A meromorphic map on open ~~conn~~ connected $D \subset \overline{\mathbb{C}} \xrightarrow{f} \overline{\mathbb{C}}$ is a

~~sets prof f~~ which cont map ~~which~~ is holomorphic at every point ~~of~~ in D
(in some charts)

Let $\mathcal{F} \subset$ Meromorphic maps on D. We say that $\mathcal{F}$ is a normal family if every seq $\{f_n\}$ in $\mathcal{F}$ contains a subseq that converges uniformly on compact subsets in D (wrt to the spherical metric)

By Arzela-Ascoli this is equiv to say that $\mathcal{F}$ is equicontinuous on compact subsets of D

e.g: $f(z) \equiv \infty$ is meromorphic

C-G
Thm 1.9 [C-G] $M < \infty$
The family ~~of~~ $\mathcal{F}$ of unif M-bd analytic fcts in D is normal.

C-G
Thm 7.2 [Montel's Thm]:
Let $\mathcal{F}$ be a family of merom fcts on D. If there are 3 points in $\overline{\mathbb{C}}$ that are omitted by every $f \in \mathcal{F}$ then $\mathcal{F}$ is normal (a hence equicont on compact subsets of D)

Coroll
In particular if normal for every $f_n \in \mathcal{F}$, $z \in D$

$$D_{\overline{\mathbb{C}}} f_n(z) = |f_n'(z)| \frac{1+|z|^2}{1+|f_n(z)|^2}$$

(the deriv wrt sph metric

~~is bounded~~ is ~~bd~~ unif bd.

Complex dynamics

(*)

$f : \mathbb{C} \to \mathbb{C}$ polyn. of $d^\circ \geq 2$.

Def 11 $K(f) = \{z \in \mathbb{C} : f^n(z) \not\to \infty\}$ is called the filled Julia set.
$J(f) = \partial K(f)$ is the Julia set of the map f.
(exo K and J are compact sets)

ex: $f(z) = z^2$
$K(f) = \{|z| \leq 1\} = \overline{\mathbb{D}}$
$J(f) = \{|z| = 1\} = S^1$

f holomorphic, $f'(z) \neq 0$
$\Rightarrow$ f' is conformal, i.e preserves angles of tangent vectors at z.
$\Rightarrow$ f' maps balls to balls.

$T_z\mathbb{C}$ $\quad\quad$ $T_{f(z)}\mathbb{C}$

(main property which relates $\dim_H$ and complex dynamics)

$U_\infty = \{z \in \mathbb{C} : f^n(z) \to \infty\}$
$= \mathbb{C} \setminus K(f)$
is an open nghd of ∞.
$fU_\infty = U_\infty = f^{-1}U_\infty$

~~Topological mixing:~~
~~Lemma: $\forall z \in J(f), r > 0$~~
~~$\mathbb{C} \setminus \bigcup_{n \geq 0} f^n B(z,r)$ contains at most two points~~

$fK(f) = K(f) = f^{-1}K(f)$
$fJ(f) = J(f) = f^{-1}J(f)$

Def 12 $J(f)$ is said to be (unif) hyperbolic if
$\exists \beta > 1$, $C > 0$ s.t
$\forall z \in J_f, n \geq 1 : |(f^n)'(z)| \geq C\beta^n$

Thm 13 Let $J(f)$ be hyperb.
Then for every $z \in J(f)$
$\bigcup_{n \geq 0} f^n B(z,r)$ omits at most ~~two~~ one points in $\mathbb{C}$
In particular $f : J \to J$ is topologically mixing

proof: $\{f^n|_{B(z,r)}\}_{n \geq 1}$ is not equi contin.
Since $f^n(z)$ stays unif bounded. and

$$Df^n(z) = \underbrace{|(f^n)'(z)|}_{\geq C\beta^n} \underbrace{\frac{1+|z|^2}{1+|f^n(z)|^2}}_{0 < bd < \infty}$$

By Montel's thm $\bigcup_{n \geq 0} f^n B(z,r)$ omits at most 2 pts in $\mathbb{C}$ but one of them is ∞.

Rem The thm holds for any Julia set but the proof is more tricky. In fact one may define

$J(f) = \{z : \forall r > 0$
$(f^n|_{B(z,r)})_{n \geq 1}$ is not a normal family$\}$

See C-G.

Special Case study

$f_c(z) = z^2 + c \quad |c| < \frac{1}{10}$

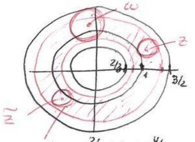
Then

$$\begin{cases} |z| < \frac{3}{4} \Rightarrow |f_c(z)| < \frac{2}{3} \\ |z| > \frac{4}{3} \Rightarrow |f_c(z)| > \frac{3}{2} \end{cases}$$

Define for $0 < \rho < 1$

$A_\rho = \{ \rho < |z| < \frac{1}{\rho} \}$

Then $f^{-1} \overline{A_{2/3}} \subset A_{3/4}$



$3/4 < |z| < 4/3$

Any $w \in \overline{A}_{2/3}$ has two preimages $f^{-1}(w) = \{z_1, z_2\}$ both $z_1, z_2 \in A_{3/4}$

Let $\delta_0 = \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3}\right) = \frac{1}{24} > 0$

$\beta = 3/2 > 1$

Then given $w \in A_{3/4}$ and ~~the~~ a preimage $z \in A_{3/4}$ there is a unique inverse map to f

$$\psi : B(w, 2\delta_0) \longrightarrow B\left(z, \frac{2\delta_0}{\beta}\right)$$

which is a ~~strict~~ $1/\beta$-Lipschitz contraction

<u>proof</u>: given $u \in B(w, 2\delta_0) \subset A_{2/3}$ and $v = \psi(u) \in A_{3/4}$

$$|\psi'(u)| = \frac{1}{|f'(v)|} = \frac{1}{|2v|} \le \frac{1}{6/4} = \frac{2}{3} = \frac{1}{\beta}$$

$B(w, 2\delta_0)$ is convex. MVT.

Lifting a path:

Given $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \mathrm{Crit}\, f$

$w_0 = f(z_0) \in \mathbb{C} \setminus f\,\mathrm{Crit}\, f$

and a path $\gamma : [0,1] \xrightarrow{C^0} \mathbb{C} \setminus f(\mathrm{Crit}\, f)$ with $\gamma(0) = z_0$. There is a unique lift

$\tilde{\gamma} : [0,1] \to \mathbb{C} \setminus \mathrm{Crit}\, f$

s.t. $f(\tilde{\gamma}(t)) = \gamma(t), \ 0 \le t \le 1$

(implicit fct thm)

The local inverse ψ is defined through such a lift.

Given an orbit

$z \in J(f)$, $z_0 = z$, $z_n = f^n z_0$

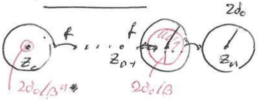
we have a sequence of inverse univalent maps: $\forall n \geq 0$

$$\boxed{\psi_{z_n} : B(z_{n+1}, 2\delta_0) \to B(z_n, 2\delta_0)}$$

which are $\frac{1}{\beta}$-lipschitz, so

$$\boxed{\psi^{(n)}_{z_0} = \psi_{z_0} \circ \dots \circ \psi_{z_{n-1}} : B(z_n, 2\delta_0) \to B(z_0, 2\delta_0)}$$

is $(\frac{1}{\beta^n})$-lipschitz contr.



We want to relate the size of the n'th preimage

$$\boxed{B_n(z, \delta_0) = \bigcap_{k=0}^{n} f^{-k} B(z_k, \delta_0)}$$

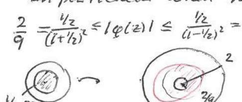
(n'th-Bowen ball, rad δ_0) to the conformal derivative of f^n at z:

$$\boxed{\begin{aligned}\lambda_n(z) &= |(f^{(n)})'(z)| \\ &= \prod_{k=0}^{n-1} f'(z_k)\end{aligned}}$$

We let $M = \sup |f'(z)|$

(Distortion) Lemma (CG Thm 1.6)
Let $\varphi : B(0,1) \to \mathbb{C}$ be univalent with $\varphi(0) = 0$, $\varphi'(0) = 1$, then
$\forall z \in \mathbb{D}: \frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \leq \frac{|z|}{(1-|z|)^2}$
In particular when $|z| = 1/2$

$$\frac{2}{9} = \frac{1/2}{(1+1/2)^2} \leq |\varphi(z)| \leq \frac{1/2}{(1-1/2)^2} = 2$$



By scaling (exo)

$$\boxed{\begin{gathered}\text{Coroll 14: As } \psi^{(n)} : B(z_n, 2\delta_0) \to B(z_0, 2\delta_0) \text{ is univalent:} \\ B(z, \tfrac{4\delta_0}{9\lambda_n}) \subset B_n(z, \delta_0) \subset B(z, \tfrac{4\delta_0}{\lambda_n})\end{gathered}}$$

with $\frac{1}{\lambda_n} = |(\psi^{(n)})'(z_n)|$

~~**Lemma** Let $z \in J(f)$, $r > 0$ and let $n = n(z,r)$ be the minimal n for which~~

~~$$B_n(z, \delta_0) \subset B(z, r)$$~~

~~Then~~

~~$$\frac{4\delta_0}{9\lambda_n} \leq r \leq \frac{4\delta_0 M}{\lambda_n}$$~~

~~proof: The first ineq is clear. Since $B_{n-1}(z,\delta_0) \not\subset B(z,r)$ but $B_{n-1}(z,\delta_0) \subset B(z, \frac{4\delta_0}{\lambda_{n-1}})$ we have~~

~~$$r < \frac{4\delta_0}{\lambda_{n-1}} \leq \frac{4\delta_0 M}{\lambda_n(z)}$$~~

Lemma 15 Let $\delta_1 = \delta_0 / 9M$.
For $z \in J$, $r > 0$ let $n = n(z, r)$ be the maximal n st

$$B(z, r) \subset B_n(z, \delta_0)$$

Then

$$\delta_1 \leq \tfrac{1}{4} \lambda_n(z) \cdot r \leq \delta_0$$

and

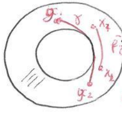
$$B_n(z, \delta_1) \subset B(z, r)$$

proof: We have $r \leq \frac{4\delta_0}{\lambda_n}$
Also $B(z,r) \not\subset B_{n+1}(z, \delta_0) \Rightarrow$

$$r > \frac{4\delta_0}{9\lambda_{n+1}} \geq \frac{4\delta_0}{9M\lambda_n} = \frac{4\delta_1}{\lambda_n}$$

and as $\delta_1 < \delta_0$:

$$B_n(z, \delta_1) \subset B(z, \tfrac{4\delta_1}{\lambda_n}) \subset B(z, r) \quad //$$



$\tilde{\gamma}$ (cone component)

$d(x_1,x_2) \le \frac{1}{\beta} d(y_1,y_2)$

(for suitable choices of preimages)

Let d_{A_ρ} be the metric in A_ρ [Riem length of curve in A_ρ]

$J = J(f_c) \subset A_\rho$ equipped with the metric d_{A_ρ}.

$X = Lip(J, d_{A_\rho})$

Standard log-lipschitz cone family $a>0$

$$K_a = \{\varphi \ge 0 : \varphi(x) \le e^{a d(x,y)} \varphi(y) \quad \forall x,y \in J\} \subset C^0_{\mathbb{R}^+}(J)$$

Let $g \in X$ and consider the Ruelle transfer op

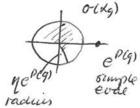
$$\mathcal{L}_g \phi(w) = \sum_{z : f_c z = w} e^{g(z)} \varphi(z), \quad w \in J.$$

Then $\mathcal{L}_g$ is a unif contraction of K_a for a large enough. K_a is inner and outer reg in X so the sp gap thm applies

$\exists! P(g) \in \mathbb{R}$, $\mu_g \in M^1_+(J)$, $h_g \in K_a^*$ $\mu_g(h_g) = 1$ s.t: $\forall n \ge 1$:

$$\forall \phi \in X: \| e^{-nP(g)} \mathcal{L}_g^n - h_g \textstyle\int \phi \, d\mu_g \|_X \le C \eta^{n-1} \|\phi\|_X$$

with $0<\eta<1$.



$d\nu_g = h_g \, d\mu_g$ is then an f_c invariant mixing (hence ergodic) proba measure on $J(f_c)$.

Lemma 16 The support of ν_g is all of $J(f_c)$. More precisely $\forall \delta > 0 \ \exists k_\delta > 0$:

$$\forall z \in J : \nu_g(B(z,\delta)) \ge k_\delta$$

proof: By the Coroll to Montel's thm: $\forall z \in J, d>0$ $\exists N = N(z,d)$: $f^N B(z, d/2) \supset J$.

J compact $\Rightarrow J \subset \bigcup_{\text{finite}} B(z_i, d/2)$

Let $N = N_\delta = \max N(z_i, d/2)$. Then $\forall z \in J$ $z \in$ some $B(z_i, d/2)$ and $B(z,\delta) \supset B(z_i, d/2)$, so $f^N B(z,\delta) \supset J$, $N = N_\delta$.

Now

$\nu_g(B(z,\delta)) =$

$\mu_g(1_{B(z,\delta)} h) =$

$e^{-NP(g)} \mu_g(\mathcal{L}_g^N(1_{B(z,\delta)} h)) \ge$

$e^{-NP(g)} \min h \cdot \min e^{S_N g} \times \underbrace{\mu_g(1)}_{=1} = k_\delta > 0.$

Key – Lemma 17: For $z \in J$, $r > 0$ let $n = n(z,r)$ be the maximal value s.t.:

$$B(z,r) \subset B_n(z,\delta_0)$$

Then

$$(1) \quad \nu_g(B(z,r)) = e^{-nP(g) + S_n g(z) + O(1)}$$

and

$$(2) \quad r = \frac{1}{\lambda_n(z)} \cdot e^{O(1)}$$

proof: We have

$$\delta_1 \le \tfrac{1}{4} \lambda_n(z) \cdot r \le \delta_0$$

which yields (2). Also

$$B_n(z,\delta_1) \subset B(z,r) \subset B_n(z,\delta_0)$$

and $f^n: B_n(z,\delta_0) \to B(f^n z, \delta_0)$ is __univalent__, hence f^n is also univalent on $B(z,r)$ and

$$f^n B(z,r) \supset B(f^n z, \delta_1)$$

By Lemma

$$1 \ge \nu_g(f^n B(z,r)) \ge \nu_g(B(f^n z,\delta_1)) \ge k_{\delta_1} = e^{O(1)}.$$

so

$$\nu_g(B(z,r)) = e^{-nP(g) + S_n g(z) + O(1)}$$

//

~~$\nu_g(B(z,r))$~~

Furthermore, $\forall u \in B(z,r)$

$$|S_n g(u) - S_n g(z)| \le L_{pg} \cdot \left(\tfrac{1}{\beta} + \tfrac{1}{\beta^2} + \dots\right) \le \frac{L_{pg}}{\beta - 1} = O(1)$$

Then

$$\nu_g(B(z,r)) = \mu_g(1\!\!1_{B(z,r)} h) =$$

$$e^{-nP(g)} \mu_g (\mathcal{L}_g^n (1\!\!1_{B(z,r)} h))$$

$$\mathcal{L}_g^n 1\!\!1_{B(z,r)} h(\omega) = \begin{cases} 0 & \text{if } \omega \notin f^n B(z,r) \\ e^{S_n g(u)} h(u) & \text{if } \omega = f^n u,\ u \in B(z,r) \end{cases}$$

$$\sum_{u: f^n u = \omega} 1\!\!1_{B(z,r)}(u) h(u) \qquad e^{O(1)}$$

$$\mathcal{L}_g^n(1\!\!1_{B(z,r)} h) = e^{S_n g(z) + O(1)} (1\!\!1_{f^n B(z,r)} h)$$

$$\nu_g(B(z,r)) = e^{-nP(g) + S_n g(z) + O(1)} \times \nu_g(\underbrace{f^n B(z,r)}_{\supset B_n(f^n z, \delta_1)})$$

Local dimension

For $z \in J$ and $r > 0$ we have

$$(*) \quad \frac{\log \nu_g(B(z,r))}{\log r} = \frac{-nP(g) + S_n g(z) + O(1)}{-\log \lambda_n(z) + O(1)}$$

$$= \frac{P(g) - \frac{1}{n} S_n g(z) + O(\frac{1}{n})}{\frac{1}{n}\log \lambda_n(z) + O(\frac{1}{n})}$$

$(*_1)$ $\frac{1}{n} S_n g(z) = \frac{1}{n}(g(z) + g(fz) + \dots + g(f^{n-1}z))$ is a Birkhoff average.

$(*_2)$ $\frac{1}{n}\log \lambda_n(z) = \frac{1}{n}\log |f'(z)\cdot f'(f(z)) \cdots f'(f^{n-1}z)|$
$= \frac{1}{n} S_n(\log|f'|)$
is also a Birkhoff average

There exists $A_g^1 \subset J$ of full ν_g meas s.t $(*_1)$ converges and $A^2 \subset J$ st $(*_2)$ conv.

For $z \in A = A^1 \cap A^2 \subset J$ (of full ν_g-measure)

$$\frac{1}{n} S_n g(z) \xrightarrow[n]{} \int g \, d\nu_g$$

$$\frac{1}{n} S_n(\log|f'|) \to \int \log|f'| \, d\nu_g$$

So for $z \in A$: $(*)$ converges when $r \to 0$ (so $n \to \infty$)

$$d_{\nu_g}(z) = \frac{P(g) - \int g \, d\nu_g}{\int \log|f'| \, d\nu_g}, \quad z \in A$$
$$= \lim \frac{\log \nu_g}{\log r}$$

and $\nu_g(A) = 1$.

By thm 10 ($\mathbb{C} \simeq \mathbb{R}^2$ has the Besic property)

$$\dim_H \nu_g = \frac{P(g) - \int g \, d\nu_g}{\int \log|f'| \, d\nu_g}$$

By the variational principle for Gibbs measures:

$$P(g) = h_{\nu_g} + \int g \, d\nu_g$$

so we get

Thm 18 For the Gibbs measure ν_g associated with $g \in Lip(J(f_c), d)$ we have

$$\dim_H \nu_g = \frac{P(g) - \int g \, d\nu_g}{\Lambda(\nu_g)} = \frac{h_{\nu_g}}{\Lambda(\nu_g)}$$

where

$$\Lambda(\nu_g) = \int_{J(f_c)} \log|f_c'| \, d\nu_g$$

is called the Lyapunov exponent of f_c wrt ν_g.

Bowen's formula

Special case (important!)

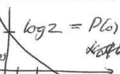
Let $g_s(z) = -s \log |f'(z)|$ and ~~set~~ write $e^{g} = \frac{1}{|f'|^s}$

~~Disv~~ ~~$P(g_s)$~~
$P(s) := P(-s \log |f'|)$

$\boxed{\begin{array}{l}\text{Thm 19 [Bowen]. } \cancel{\text{There is a unique value } s^* \in [0,2] \text{ s.t. } P(s)} \\ s^* = \dim_H(J(f_c)) \in [0,2] \\ \text{is the unique value such that} \\ \qquad P(s^*) = 0\end{array}}$

---

proof:



$\mathcal{L}_0 \mathbb{1} = 2 \cdot \mathbb{1}$ ~~$\mathcal{L}_0 \phi(w)$~~

$\mathcal{L}_s \phi(w) = \sum \frac{1}{|f'(z)|^s} \phi(z)$

For $s_1 > s_2$, $\phi \geq 0$:

$$\mathcal{L}_{s_1}\phi(w) = \sum \frac{1}{|f'(z)|^{s_1}} \phi(z)$$
$$\geq \frac{1}{\beta^{s_1 - s_2}} \sum \frac{1}{|f'(z)|^{s_2}} \phi(z)$$
$$= \frac{1}{\beta^{s_1 - s_2}} \mathcal{L}_{s_2}\phi(w)$$

$\Rightarrow P(s_2) - P(s_1) \geq (s_1 - s_2) \log \beta$

~~$\beta > 1$~~

$\Rightarrow$ $P(s)$ strictly decreasing and has a unique zero $s^* > 0$

When $P(s^*) = 0$ we get from the Key-Lemma

$$\cancel{\log}\, \nu_g(B(z,r)) = e^{S_n g(z) + O(1)} = \frac{1}{|f^{n\prime}(z)|^{s^*}} \cdot e^{O(1)}$$

and

$$r = \frac{1}{\lambda_n(z)} e^{O(1)} = \frac{1}{|(f^n)'(z)|} \cdot e^{O(1)}$$

so

$$\boxed{\nu_{g_{s^*}}(B(z,r)) = r^{s^*} \cdot e^{O(1)}}$$

for every $z \in J(f_c)$.

$\nu_{g_{s^*}}$ is thus s^*-Ahlfors regular and

$$\dim_H J(f_c) = s^*$$

Rmk
* {the s^*-dim Haussdorf measure is equivalent to ν_{s^*}}

** Bowen's formula holds for any hyperb. Julia set (also for a rational map)

*** For any ~~neo~~ rational map $J(f_c)$ may be defined as the set of points $z \in \overline{\mathbb{C}}$ s.t. $\forall r > 0$

$\{(f^n)_{|B(z,r)}\}_{n \geq 0}$ is not a normal family.

($J(f) \subset \overline{\mathbb{C}}$ compact subset)

Calculation of $P(s)$ or $P(g)$

Spectral radius in X and in $C(J)$ is the same. So

$$P(g) = \lim \sup (\|\mathcal{L}_g^n \mathbb{1}\|_\infty)^{1/n}$$

∂ $f_1, \dots, f_4$
$1/3$ contraction

Now for any $y \in J(f_c)$

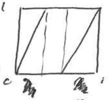
$$P(g) = \lim (\mathcal{L}_g^n \mathbb{1}(y))^{1/n}$$

and even for any $y \in A_p$

---

exs Bowen's formula is valid in many other cases:

d-branches



$f: I_1 \to I = [0,1]$
$I_2 \to I = [0,1]$

affine surj. maps

$r_1 \quad r_2$
$I_1 \quad I_2$

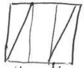
$f'_{|I_i} = 1/r_i$

$$\mathcal{L}_s \mathbb{1} = \sum r_i^s \mathbb{1} = (\sum r_i^s)\,\mathbb{1} = e^{P(s)}\,\mathbb{1}$$

so $P(s) = 0 \Leftrightarrow$

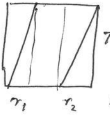
$$\sum_{i=1}^{d} r_i^s = 1$$

$1/3$ - Cantor



$2 \cdot (\frac{1}{3})^s = 1$

or $s = \frac{\log 2}{\log 3} = \dim_H \mathcal{C}_{1/3}$



T

$r_1 \quad r_2$ sizes

$e^g = p_1 \quad p_2$ weights $\sum p_i = 1$

Λ

$e^{P(g)} = 1$, $P(g) = 0$

$$\mathcal{L}_g \mathbb{1} = \sum p_i \mathbb{1} = \mathbb{1} = h_g$$

$$\mathcal{L}_g \phi(y) = \sum_{Tx_i = y} p_i \phi(x_i)$$

$$\nu_g(I_i) = \mu_g(I_i) = p_i$$

$$\nu_g(\log |T'|) = \sum p_i \log \frac{1}{r_i}$$

$$\dim_H \nu_g = \frac{P(g) - \int g\, d\nu_g}{\int \log T'} = \frac{-\sum p_i \log p_i}{\sum p_i \log 1/r_i} = \frac{\sum p_i \log p_i}{\sum p_i \log r_i}$$

exo

max value for $p_i \sim r_i^s$ with $\sum p_i = \sum r_i^s = 1$

i.e. $s = \dim_H \Lambda$

Dimension of harmonic measure for $f_c(z) = z^2 + c$.

Suppose that $(f_c^n(0))_{n\geq 1}$ stays bounded then

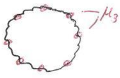
$\boxed{\text{Thm}^{20}\,[\text{Manning}]\ \dim_H \mu_{harm}(f_c) = 1}$

For $\omega \in J(f_c)$ let

$$(f_c^n)^{-1}(\omega) = \{z_1^{(n)}, \dots, z_{2^n}^{(n)}\}$$

be the collection of 2^n n'th preimages of ω. We define the equi-distribution measure

$$\mu_n^{(\omega)} = \frac{1}{2^n} \sum_{k=1}^{2^n} \delta_{z_k^{(n)}}$$



Then $\mu_n \xrightarrow{\omega} \mu_{harm}$ converges weakly as $n \to \infty$. The limit is called the harmonic measure of f_c

$$\mathcal{L}_s \phi(\omega) = \sum_{z: f_c(z) = \omega} \frac{1}{|f'(z)|^s} \phi(z)$$

$$\mathcal{L}_s^n \phi(\omega) = \sum_{f_c^n z = \omega} \frac{1}{\lambda_n(z)^s} \phi(z)$$

$s = 0$: $\mathcal{L}_0^n 1 = 2 \cdot 1$

$$\begin{array}{ccc} 2^{-n} \mathcal{L}_0^n \phi(\omega) & = & \int \phi \, \mu_n^{(\omega)} \\ \downarrow n & & \downarrow n \\ 1 \cdot \nu_0(\phi) & & \int \phi \, \mu_{harm} \end{array}$$

So $\mu_{harm} = \nu_0$ (Gibbs meas with $s=0$)

$$\dim \nu_0 = \frac{P(0) - 0 \cdot \int \log \lambda_1 \, d\nu_0}{\int \log \lambda_1 \, d\nu_0}$$

$$f_c^n(z) - \omega = z^{2^n} + (\text{terms } z \to z^{2^n - 1}) + f_c^n(0) - \omega = \prod_{k=1}^{2^n} (z - z_k) = z^{2^n} + (\dots) + \prod_{k=1}^{2^n} z_k$$

We have the identity:

$$\boxed{f_c^n(0) - \omega = \prod_{k=1}^{2^n} z_k}$$

&

$$\int \log |f_c'(z)| \, d\mu_n^{(\omega)} = \int \log |2z| \, d\mu_n^{(\omega)} = \frac{1}{2^n} \sum_1^{2^n} \log |2 z_k| = \log 2 + \frac{1}{2^n} \log \left|\prod_{k=1}^{2^n} z_k\right| = \log 2 + \frac{1}{2^n} \log |f_c^n(0) - \omega|$$

When $f_c^n(0)$ stays bd then $|f_c^n(0) - \omega| = e^{o(n)}$ so

$$\int \log |f'| \, d\nu_0 = \lim_n \int \log |f'| \, d\mu_n^{(\omega)} = \log 2$$

Since $P(0) = \log 2$ as well

$$\dim \nu_0 = \frac{\log 2}{\log 2} = 1$$

Remark: If $f_c^n(0) \to \infty$ then $\frac{1}{2^n} \log |f_c^n(0) - \omega| \to \alpha > 0$ and $\dim \nu_0 = \frac{\log 2}{\log 2 + \alpha} < 1$