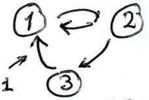
9 Gibbs measures

Def 1: A transition matrix $(d\times d)$ $A=(a_{ij})$ $\in\{0,1\}$ is said to be topological mixing if $\exists n\geq 1: (A^n)_{ij}\geq 1\ \forall i,j$

<u>ex</u>



$A_{31}=1$

$$A=\begin{pmatrix}0&1&0\\1&0&1\\1&0&0\end{pmatrix}$$

$$A^5=\begin{pmatrix}2&1&1\\2&2&1\\1&1&1\end{pmatrix}$$ so topol mixing $N=5$

Def 2: Given a $(d\times d)$ transition matrix A we associate to this a (one-sided) shift space of finite type:

$$\sigma \circlearrowright \Sigma_A^+=\{\xi=(\xi_n)\in\Sigma_d : A_{\xi_n\xi_{n+1}}=1\ \forall n\}$$

shift $\quad$ ($n\in\mathbb{Z}$ or) $\mathbb{N}$.

<u>ex</u> $(231212\ldots)\in\Sigma_A^+$

In the following we consider a topol mixing transition matrix A $(d\times d)$ and a function space ~~compact~~)

$$X=Lip(\Sigma_A^+, d_\theta)$$

with $0<\theta<1$ and

$$d_\theta(x,y)=\sup\{\theta^k : x_k\neq y_k\}$$

Thm 3 (Ruelle-Perron-Frobenius) Let $g\in X$ and define for $y\in\Sigma_A^+$, $\phi\in X$:

$$\mathcal{L}\phi(y)=\mathcal{L}_g\phi(y)=\sum_{x:\sigma x=y} e^{g(x)}\phi(x)$$

This operator has a spectral gap, i.e.: $\exists\lambda>0$, $h\in X_+$, $\mu\in M_+^1(\Sigma_+, d_\theta)$

$$\mathcal{L}h=\lambda h,\quad \mu\mathcal{L}=\lambda\mu,\quad \mu(h)=1$$

and $\forall\phi\in X$:

$$\|\lambda^{-n}\mathcal{L}^n\phi-h\int\phi\,d\mu\|_X\leq C\eta^{n-1}\|\phi\|_X$$

with $0<\eta<1$.

The measure $d\nu=h\,d\mu$ is σ-invariant and mixing (whence ergodic).

Proof: (sketch):

1) Use a modified log-Lipsch cone: For $a>0$, $b>0$ let

$$K_{a,b}=\{\varphi\in X :$$
$\forall x,y\in[i]$, $1\leq i\leq d$ (same 1-cylinder) $\varphi(x)\leq e^{a d_\theta(x,y)}\phi(y)$,
$\forall x,y: \varphi(x)\leq b\,\phi(y)\}$.

Show that there are $a>0$, $b>0$, $0<\sigma<1$ s.t.:

$$\mathcal{L}^N K_{a,0}\subset K_{\sigma a,b}$$ (N = mixing time)

with $\Delta=\mathrm{diam}_{K_{a,0}} K^*_{\sigma a,b}<\infty$.
Deduce the conclusion for the operator $\mathcal{L}^N$ using our cone contraction thm.

2) Extend the conclusion to $\mathcal{L}$ itself.

Def ~~Often~~ one writes

$$P(g)=\log\lambda_g\in\mathbb{R}$$

~~an~~ It is called the "pressure" of g (and the shift-space)

We define n-cylinders in Σ_A^+ as usual

$$Z_n = \{ [x_0 \dots x_{n-1}] \subset \Sigma_A^+ \}$$

One has the bound:

$$\boxed{\begin{array}{l}\text{Lemma: For } C \in Z_n \text{ non-empty} \\ \quad \operatorname{var}_C S_n g \le \operatorname{Lip} g \, \frac{\theta}{1-\theta} \\ \text{uniformly in } n \ge 1.\end{array}}$$

proof: Let $x, y \in C$.

$S_n g(x) = g(x) + g \circ T(x) + \dots + g \circ T^{n-1}(x)$

$|g \circ T^k(x) - g \circ T^k(y)| \le \operatorname{Lip} g \cdot \theta^{n-k}$

same $n-k$-cylinder

so $|S_n g_x - S_n g_y| \le \operatorname{Lip} g \sum_{j \ge 1} \theta^j$ //

$$\boxed{\text{Lemma : For } D \in Z_1 : \nu(D) = e^{O(1)}}$$

proof: $A_{ij}^N \ge 1 \ \forall i,j$. Then $\exists k < \infty$:

$\frac{1}{k} 1\!\!1 \le \mathcal{L}^N 1\!\!1_D h \le k \cdot 1\!\!1$

so that $\mu(\mathcal{L}^N 1\!\!1_D h) = e^{O(1)}$

but LHS $=$

$\mu(h \mathcal{L}^N 1\!\!1_D) = \lambda^N \mu(h 1\!\!1_D) e^{O(1)}$

$= e^{O(1)}_N \nu(D)$

Since $h = e^{O(1)}$ (as in $K_{a,b}$)

$$\boxed{\begin{array}{l}\text{Thm (Gibbs measure).} \\ d\nu = h\, d\mu \text{ is the unique} \\ \sigma\text{-inv proba measure s.t.} \\ \forall n \ge 1,\ C \in Z_n,\ x_0 \in C: \\ \quad \nu(C) = e^{-nP + S_n g(x_0) + O(1)} \\ \text{for some } P \in \mathbb{R},\ P = P(g) \quad \text{indep of } n, C, x_0 \\ \text{i.e.:} \\ \quad \frac{1}{K} \le \frac{\nu(C)}{e^{-nP + S_n g(x_0)}} \le K\end{array}}$$

proof: Let $x, x_0 \in C \in Z_n$,
$y = \sigma^{n-1}(x) \in D = \sigma^{n-1}(C) \in Z_1$

$\mathcal{L}^n(h 1\!\!1_C)(\sigma y) = e^{S_n g(x)} h(x)$

$= e^{S_n g(x_0)} h(x_0) \cdot e^{O(1)}$

$\nu(C) = \lambda^{-n} \mu(\mathcal{L}^n h 1\!\!1_C)$

$= e^{S_n g(x_0)} h(x_0) e^{O(1)}$ #

Uniqueness: Suppose η is another such Gibbs meas.
Then $\eta \ll \nu$, $d\eta = f d\nu$.
and $1/k \le f \le k$, $k < \infty$

σ-invariance of η and $\nu \Rightarrow$
$f = f \circ \sigma$ and ν ergodic $\Rightarrow$
$f = $ const a.s $\Rightarrow f = 1$ a.s. //

Variational principle for Gibbs measure

Let $g \in Lip(\Sigma_A^+, d_\theta)$ and let ν_g be the associated Gibbs m.

$$\boxed{\text{Thm } P(g) = h_{\nu_g}(\sigma) + \int g \, d\nu_g}$$

proof: In order to calculate the entropy:
$\mathcal{A} = \langle Z_1 \rangle$ generates $\mathcal{B}$:
$\mathcal{A}_n = \langle Z_n \rangle$, $\mathcal{B} = \sigma(\bigcup \mathcal{A}_n)$.
Thus Kolmo-Sinai:
$$h_\nu(\sigma) = h_\nu(\sigma, \mathcal{A})$$

For $C \in Z_n$, $x_C \in C$ we have by the Gibbs property:
$$\nu(C) = e^{-nP + S_n g(x_C) + O(1)}$$

$$\frac{1}{n} H(\mathcal{A}_n) = -\frac{1}{n} \sum_{C \in Z_n} \mu(C) \log \mu(C)$$
$$= \sum_{C \in Z_n} (P(g) - \tfrac{1}{n} S_n g(x_C)) + O(\tfrac{1}{n})$$
$$= \sum_C \int_C (P(g) - \tfrac{1}{n} S_n g) \, d\nu + O(\tfrac{1}{n})$$
$$= \int (P(g) - \tfrac{1}{n} S_n g) \, d\nu + O(\tfrac{1}{n})$$
$$= P(g) - \int g \, d\nu + O(\tfrac{1}{n})$$
Take limits //

Def: For $g \in C^0(\Sigma_A^+)$ one defines more generally
$$\Omega_n = \sum_{C \in Z_n} \sup_C e^{S_n g}$$
$$P(g) = \overline{\lim} \, \frac{1}{n} \log \Omega_n$$

Rem: 1) $P(g) = \lim \frac{1}{n} \log \Omega_n$ since $\Omega_{n+m} \le \Omega_n \Omega_m$ $n, m \ge 1$ (submultipl.)

2) For $g \in Lip$ this def coincides with the $P(g)$ coming from Gibbs properties

Variational principle

Thm $P(g) = \sup\limits_{\mu \in M_+^1(\sigma)} (h_\mu(\sigma) + \int g \, d\mu)$ for any $g \in C^0(\Sigma_A^+)$

Lemma: $a_1, \dots, a_k \in \mathbb{R}$, $p_1, \dots, p_k \ge 0$ $\sum p_i = 1$:
$$-\sum p_i \log p_i + \sum a_i p_i \le \log \sum e^{a_i}$$
with equality iff
$$p_i = e^{a_i} / \sum e^{a_j}$$

proof: Set $\Omega = \sum e^{a_j}$. Then with $\varphi(t) = -t \log t$ concave:
$$0 = \varphi(1) = \varphi(\sum p_i) = \varphi\left(\sum \frac{e^{a_i}}{\Omega} \frac{p_i \Omega}{e^{a_i}}\right)$$
$$\underset{\text{Jensen}}{\ge} \sum \frac{e^{a_i}}{\Omega} \varphi\left(\frac{p_i \Omega}{e^{a_i}}\right)$$
$$= -\sum p_i \log \frac{p_i \Omega}{e^{a_i}}$$
$$= -\log \Omega - \sum p_i \log p_i + \sum p_i a_i \; //$$

proof of thm: (assume $g \in Lip$)
$$H_\mu(\mathcal{A}_n) + n \int g \, d\mu \overset{\sigma\text{-inv.}}{=}$$
$$\sum_{C \in Z_n} -\mu(C) \log \mu(C) + \int_C S_n g \, d\mu =$$
$$\sum_C \mu(C) \left(-\log \mu(C) + \int_C S_n g \frac{d\mu}{\mu(C)}\right) \le$$
$$\log \sum_C \exp \int_C S_n g \frac{d\mu}{\mu(C)} =$$
$$\log \sum_C \exp(S_n g(x_C) + O(1)) =$$
$$\log \sum_C (\nu(C) e^{nP(g) + O(1)}) =$$
$$nP(g) + O(1) \rightarrow$$
$$\underline{\text{K-S}} \quad h_\mu(\sigma) = \lim \frac{1}{n} H_\mu(\mathcal{A}_n) \le P(g) - \int g \, d\mu$$

Rem: The sup is realized (max) for the Gibbs measure ν_g when $g \in Lip$ and ν_g is then uniquely determined by this property (not shown here)

Analytic Perturbation Theory

[Kato: Pert. theory for linear operators Chap VII §1]

Thm Let $\mathcal{L}_t$ be an analytic family of bd lin operators $t \in \mathcal{U} \mapsto \mathcal{L}_t \in L(X)$, X Banach

$\mathcal{U}$ nghd of 0:

Suppose λ_0 is a simple isolated eval of $\mathcal{L}_{t=0}$ with 1-dim projection $h_0 \otimes \ell_0$, $h_0 \in X$, $\ell_0 \in X'$ and $\langle \ell_0, h_0 \rangle = 1$.

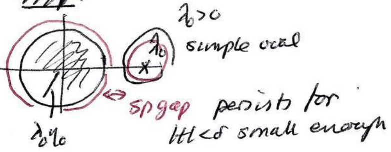
1) Then there is $\delta > 0$ and analytic functions
$$t \in B_{\mathbb{C}}(0,\delta) \to \lambda_t \in \mathbb{C},\ h_t \in X,\ \ell_t \in X'$$
such that λ_t is an isolated simple eval of $\mathcal{L}_t$ with 1-dim projection $h_t \otimes \ell_t$

2) Furthermore, the connected component of ∞ of the resolvent set $\hat{\mathbb{C}} \setminus \sigma(\mathcal{L}_t)$ depends continuously upon t.

Coroll

Interpretation for operators with spectral gap

~~Thm~~ ~~Prop~~



Suppose $\exists\, C < \infty,\ \eta_0 < 1$:
$$\| \lambda_0^{-n} \mathcal{L}_0^n \phi - h_0 \langle \ell_0, \phi \rangle \|_X \le C \eta_0^{n-1} \|\phi\|_X$$
Then for $\delta > 0$ small enough we have $\exists\, \eta_\delta < 1,\ C_\delta < \infty$:

$\forall\ |t| < \delta$:
$$\| \lambda_t^{-n} \mathcal{L}_t^n \phi - h_t \langle \ell_t, \phi \rangle \| \le C_\delta \eta_\delta^{n-1} \|\phi\|_X$$

Also $P(t) = \log \lambda_t$, $t \in \mathbb{R}$ depends analytically upon t, $|t| < \delta$ (small enough for unique analytic cont.)

Application: (Σ_A^+, d)

$X = \mathrm{Lip}(\Sigma_A^+, d)$ is a Banach algebra; i.e.: $\forall\, \varphi, \psi \in X$:
$\varphi \cdot \psi \in X$, $\|\varphi \cdot \psi\|_X \le \|\varphi\|_X \|\psi\|_X$

$\sigma^+ : \Sigma_A^+ \to \Sigma_A^+$ shift

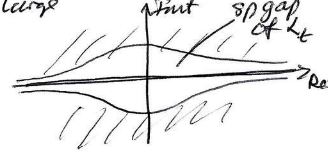
$g \in X$, $\phi \in X$, $y \in \Sigma_A^+$:
$$\mathcal{L}_g \phi(y) = \sum_{x : \sigma_+ x = y} e^{g(x)} \phi(x)$$

~~depends analytically upon g. Let $t \in \mathbb{C}$ and consider~~

For $t \in \mathbb{C}$, $a \in X$ consider
$$\mathcal{L}_{g+ta} \phi = \mathcal{L}_g (e^{ta} \phi)$$
This is analytic in t:
$$\mathcal{L}_{g+ta}(\cdot) = \mathcal{L}_g \Big(\sum \frac{t^n}{n!} a^n \phi \Big) = \sum \frac{t^n}{n!} \big(\mathcal{L}_g (a^n \cdot) \big) \in L(X)$$
$$\sum_{n \ge 0} \frac{t^n}{n!} \| \mathcal{L}_g a^n \| < \infty \quad \forall t \in \mathbb{C}$$

Thm $\exists\, \delta > 0$ As an analytic family of operators on X:
$$\lambda_t^{-n} \mathcal{L}_t^n = h_t \otimes \ell_t + R_t^n$$
with $\|R_t^n\| \le C \eta^{n-1}$, $n \ge 1$ uniformly in $|t| < \delta$.

For $t \in \mathbb{R}$ always a spectral gap (and a Gibbs measure μ_t). Spectral gap may disappear for ~~large~~ Im t large



$d\mu_t = h_t\, dm_t$ proba meas for $t \in \mathbb{R}$

$\sigma^{\pm}$-invar, mixing, ergodic

$(\Sigma_A, \mathcal{B}, \mu_0, \sigma)$ $\quad X = Lip(\Sigma_A, d_\theta)$

Correlation functions

$a, b \in X$

$$C_n(a,b) = \begin{cases} Cov(a\circ\sigma^n, b) & n \geq 0 \\ Cov(a, b\circ\sigma^{-n}) & n<0 \end{cases}$$

$$Cov(a\circ\sigma^n, b) = \mathbb{E}\left((a-\bar a)\circ\sigma^n \cdot (b-\bar b)\right)$$

Lemma $C_n(a,b) = O(q^n)$

proof: assume for simplicity $\lambda_0 = 1$, $\bar a = \bar b = 0$

$n \geq 0$

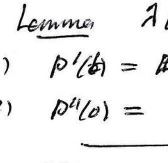
$$\langle l_0, a\circ\sigma^n \cdot b\, h_0\rangle = \langle l_0, \mathcal{L}_0^n(a\circ\sigma^n b h_0)\rangle$$
$$= \langle l_0, a\mathcal{L}_0^n(bh_0)\rangle$$
$$= \underbrace{\langle l_0, a h\rangle}_{=0}\langle l_0, bh_0\rangle + \underbrace{\langle l_0, a R_0^n(bh_0)\rangle}_{O(q^n)} \quad //$$

Lemma $\sigma^2_{a,b} := \sum_{n\in\mathbb{Z}} C_n(\underset{a-\bar a}{a}, \underset{b-\bar b}{b}) = \lim \frac{1}{n}\mathbb{E}(S_n a \cdot S_n b)$

Case $\bar a = \bar b = 0$ $\quad 0 \leq j,k < n$:

$$\mathbb{E}(a\circ\sigma^j \cdot b\circ\sigma^k) = \begin{cases} \mathbb{E}(a\circ\sigma^{j-k}\cdot b), & j\geq k \\ \mathbb{E}(a\cdot b\circ\sigma^{k-j}), & j<k \end{cases}$$

$$\frac{1}{n}\mathbb{E}(S_n a\, S_n b) = \frac{n}{n}C_0 + \left(C_1(a,b) + C_{-1}(a,b)\right)\frac{n-1}{n} + \cdots$$



$$\xrightarrow[n]{} C_0 + C_1 + C_{-1} + \cdots = \sum_{n\in\mathbb{Z}} C_n(a,b)$$

Lemma Given Leading eval $\lambda(\mathcal{L}_{g+ta}) = e^{P(t)}$ (t real)

1) $P'(t) = \mathbb{E}_{\mu_t}(a)$

2) $P''(0) = \sigma^2_{a,a} = \sigma_a^2 \ (\geq 0)$

1) $e^{P(t)} = \langle l_t, \mathcal{L}_t h_t\rangle$

$$P'(t)e^{P(t)} = \langle l_t, \mathcal{L}_t a h_t\rangle + \langle l_t', \mathcal{L}_t h_t\rangle + \langle l_t, \mathcal{L}_t h_t'\rangle$$
$$= e^{P(t)}\langle l_t, a h_t\rangle + e^{P(t)}\underbrace{(\langle l_t', h_t\rangle + \langle l_t, h_t'\rangle)}_{=0}$$
$$= e^{P(t)}\mathbb{E}_{\mu_t}(a) \quad (t \text{ real})$$

assume $\mathbb{E}_{\mu_0}(a) = \bar a = 0$

2) $e^{nP(t)} = \langle l_t, \mathcal{L}_t^n h_t\rangle = \langle l_t, \mathcal{L}_0^n e^{tS_n a} h_t\rangle$

$$nP'(t) = \langle l_t, \mathcal{L}_0^n e^{tS_n a} S_n a\, h_t\rangle$$

$\frac{\partial}{\partial t}|_{t=0}$: $\quad nP''(0) = \langle l_0, (S_n a)^2 h_0\rangle + \langle l_0', \mathcal{L}_0^n S_n a\, h_0\rangle + \langle l_0, \mathcal{L}_0^n S_n a\, h_0'\rangle$

$0 \leq k < n$

$$\langle l_0', \mathcal{L}_0^n a\circ\sigma^k h_0\rangle = \langle l_0', \mathcal{L}_0^{n-k}(a\mathcal{L}_0^k h_0)\rangle = \langle l_0', \mathcal{L}_0^{n-k}(a h_0)\rangle = O(q^{n-k})$$

so $\langle l_0', \mathcal{L}_0^n S_n a\, h_0\rangle = O(1)$

$$\langle l_0, \mathcal{L}_0^n a\circ\sigma^k h_0'\rangle = \langle l_0, \mathcal{L}_0^k a\circ\sigma^k h_0'\rangle = \langle l_0, a\mathcal{L}_0^k h_0'\rangle = O(q^k)$$
$$\langle l_0, \mathcal{L}_0^n S_n a\, h_0'\rangle = O(1)$$

$$P''(0) = \lim_n \frac{1}{n}\left(\mathbb{E}_0((S_n a)^2) + O(1)\right) = \sigma_{a,a} \ (\geq 0)$$

Thm $\forall t \in \mathbb{R}$: $\mathbb{E}_0(e^{i\frac{t}{\sqrt n}S_n a}) \xrightarrow[n\to\infty]{} e^{-t^2\sigma_a^2/2}$

$\bar a = 0$ so (by Levy) $\frac{1}{\sqrt n}S_n a \xrightarrow[law]{} N(0, \sigma_a^2)$

proof: LHS = (assume $P(0) = 0$)

$\cancel{\langle l_0, \mathcal{L}^n_{0+\frac{it}{\sqrt n}a} h_0\rangle}$

$$\langle l_0, e^{i\frac{t}{\sqrt n}S_n a} h_0\rangle = \langle l_0, \mathcal{L}^n_{\frac{it}{\sqrt n}} h_0\rangle =$$
$$e^{nP(\frac{it}{\sqrt n})}\underbrace{\left(\langle l_0, h_{\frac{it}{\sqrt n}}\rangle\langle l_{\frac{it}{\sqrt n}}, h_0\rangle\right)}_{=1+O(\frac{1}{\sqrt n})} + O(q^n)$$

$$e^{n\left(P(0) + \frac{it}{\sqrt n}P'(0) - \frac{t^2}{2n}P''(0) + O(\frac{t^3}{\sqrt{n}^3})\right)}$$
$$= e^{-t^2/2\, P''(0) + nO((\frac{t}{\sqrt n})^3)}$$
$$\xrightarrow[n\to\infty]{} e^{-t^2/2\, P''(0)} = e^{-t^2/2\, \sigma_a^2}$$

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