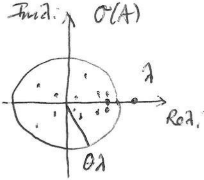
7B. Cone contraction and spectral gap.

example: Perron-Frobenius for matrices.

$A \in M_n(\mathbb{R})$ $\quad A = (a_{ij}),\ a_{ij} > 0.$



$\lambda_j \in D(0,\theta)$

Theorem

$\lambda = r_{sp}(A) > 0$ is a simple eigenvalue for A. Every other eval is of strictly smaller modulus.

The eigenvectors of A and A^T associated with λ may be chosen <u>strictly</u> positive

$\Downarrow$

$\lambda h = A h, \quad \lambda \ell = A^T \ell, \quad \ell^T h = 1$

$h = (h_i),\ h_i > 0, \qquad \ell = (\ell_j),\ \ell_j > 0$

$\exists\, 0<\theta<1,\ C<\infty$ such that $\forall x \in \mathbb{C}^n$

$|\lambda^{-n} A^n x - h(\ell^T x)| \leq C\theta^n |x|$

In the following
E is a Banach space.
We write for the norm either of:
$|\cdot|,\ \|\cdot\|,\ |\cdot|_E,\ \|\cdot\|_E$

Cones in a normed space (or a Banach space) E, over the reals.

<u>Regularity</u>

Def 5.1 K a proper $\mathbb{R}$-cone in E

We say that:

a) K is <u>outer regular</u> iff:

$\exists\, \ell \in E'$, $k>0$ (nontrivial) s.t.:

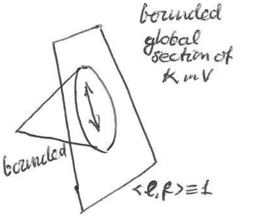
$\forall f \in K: \ \langle \ell, f\rangle \geq k\, \|f\|_E \cdot \|\ell\|_{E'}$

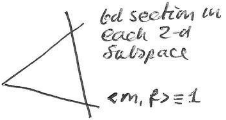
b) K is of <u>k-bounded sectional aperture</u> iff $(k>0)$

$\forall f_1, f_2 \in K^*$, $V_{12} = \mathrm{Span}_{\mathbb{R}}\{f_1, f_2\}$

$\exists\, m \in V_{12}'$ s.t. $\forall f \in K \cap V_{12}$:

$\langle m, f\rangle \geq k\, \|f\|_V \, \|m\|_{V'}$





<u>Remark</u>: a) $\Rightarrow$ b) just take $m=\ell$.

<u>ex</u> $\mathbb{R}^n_+$ is outer reg. in any norm

$x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$. Take:

$\langle \ell, x\rangle = \sum x_i$. Then

$k=1$ for $\|\cdot\|_1$ since $\ell^T x = \|x\|_1$ when $x \in \mathbb{R}^n_+$

$k = \frac{1}{\sqrt{n}}$ for $\|\cdot\|_2$

$k = \frac{1}{n}$ for $\|\cdot\|_\infty$

<u>ex</u> $E = L^p(\Omega, \mu)$. The cone $K = L^p_+(\Omega, \mu)$

$p=1$: K is outer regular $(k=1)$

$\langle \ell, f\rangle = \int f\, d\mu = \|f\|_1$

$p>1$: K is not outer regular but it is of bd sectional aperture. One may take $k = 1/2$.

$p > 1$

$K = L^p_+(\Omega, \mu)$ is ofibd sectional aperture:

Take $f_1, f_2 \in K^e$, linearly independent.

Then find $u_1, u_2 \in K^a$, $|u_1|_p = |u_2|_p = 1$ and such that ~~$\forall q$~~

$V_{12} = \mathrm{Span}\{f_1, f_2\} = \mathrm{Span}\{u_1, u_2\}$

$$V_{12} \cap K = \mathbb{R}_+[u_1, u_2]$$
$$= \{ t_1 u_1 + t_2 u_2 : t_1, t_2 \geq 0 \}$$

Define now the auxiliary function

$$g = u_1^{p-1} + u_2^{p-1}$$
$$= u_1^{p/q} + u_2^{p/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ or $\frac{p}{q} = p - 1$

Define for $f \in V = L^p(\Omega, \mu) \cap V_{12}$

$$\langle m, f \rangle = \int f g \, d\mu, \quad m \in V_{12}'$$

Then

$$\|m\|_{V_{12}'} = \|g\|_{L^q}$$
$$\leq \|u_1^{p/q}\|_q + \|u_2^{p/q}\|_q$$
$$= \left(\int u_1^p \, d\mu\right)^{1/q} + \left(\int u_2^p \, d\mu\right)^{1/q}$$
$$= 1 + 1 = 2.$$

For $u = s_1 u_1 + s_2 u_2 \in V_{12} \cap K$

$$\langle m, u \rangle = \int (u_1^{p/q} + u_2^{p/q})(s_1 u_1 + s_2 u_2) \, d\mu$$
$$\geq \int u_1^p s_1 + u_2^p s_2 \, d\mu = s_1 + s_2$$
$$\geq |u|_{L^p}.$$ So we have

$$\langle m, u \rangle \geq \frac{1}{2} |m|_q \cdot |u|_p$$

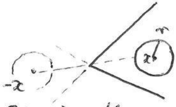
Def 5.2 Let K be a proper $\mathbb{R}$-cone in a normed vector space E. We say that

a) K is <u>inner regular</u> iff $\operatorname{int} K \neq \emptyset$

b) K is <u>regenerating</u> iff $\exists g < \infty$ such that $\forall f \in E$:

$\exists f_1, f_2 \in K : f = f_+ - f_-$ and
$$|f_+|_E + |f_-|_E \leq g \cdot |f|_E$$

Prop 5.3 a) $\Rightarrow$ b)



Suppose $B(x,r) \subset K$

$B(-x,r) \subset -K$

Given $f \in E$, $f \neq 0$:
$$\tilde{f}_+ = x + r\frac{f}{|f|} \in B(x,r) \subset K$$
$$\tilde{f}_- = x - r\frac{f}{|f|} \in B(x,r) \subset K$$

we have
$$f = \frac{1}{2r}|f|(\tilde{f}_+ - \tilde{f}_-) = f_+ - f_-$$

where
$$|f_+| + |f_-| \leq \frac{1}{2r}|f| \cdot 2(|x| + r) = \underbrace{\left(\frac{1}{r}|x| + 1\right)}_{g} \cdot |f|$$

$$g = \frac{|x|}{r} + 1$$

Lemma 8.4

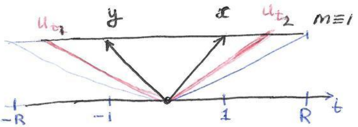
Let K be an $\mathbb{R}$-cone in E of k-bounded sect aperture. Then $\forall x, y \in K^*$:

$$\left| \frac{x}{|x|_E} - \frac{y}{|y|_E} \right|_E \leq \frac{2}{k} d_K(x,y)$$

<u>proof</u>: Let $V = \text{Span}\{x,y\}$ and pick $m \in V'$ so that $\|m\|=1$ and:

$$\langle m, u \rangle \geq k \cdot \|u\| \ , \quad u \in V \cap K.$$

First we normalize x and y so that $\langle m,x\rangle = \langle m,y\rangle = 1$



Define for $t \in \mathbb{R}$:

$$u_t = \frac{1+t}{2} x + \frac{1-t}{2} y$$

for which

$$\langle m, u_t \rangle = \frac{1+t}{2} + \frac{1-t}{2} = 1$$

<u>Suppose $u_t \in K$ $(\cap V)$</u> then:

$\langle m, u_t\rangle = 1 \geq k \cdot |u_t|_E$ so $|u_t|_E \leq 1/k$.

We have:

$$|t(x-y)|_E = |2u_t - (x+y)|_E \leq 2|u_t| + |x| + |y| \leq \frac{4}{k}$$

so $|t| \leq R :\underset{\text{def}}{=} \frac{4}{k} \cdot \frac{1}{|x-y|}$

$K \cap V = \mathbb{R}_+ [u_{t_1}, u_{t_2}]$ where

$$-R \leq t_1 < -1 < 1 < t_2 \leq R$$

The ~~[illegible]~~ Hilbert distance between x, y in K:

$$d = d_K(x,y) = d_{u_{t_1}, u_{t_2}}(x,y)$$

$$= \log [-1, 1; t_1, t_2]$$

$$= \log \cancel{[illegible]}$$

$$\frac{1-t_1}{-1-t_1} \cdot \frac{t_2+1}{t_2-1}$$

$$= \log \frac{-t_1 \mp 1}{-t_1+1} \ \frac{t_2+1}{t_2-1}$$

lower bound $\geq \log \frac{-R \mp 1}{-R+1} \cdot \frac{R+1}{R-1}$ or

$$d \geq \log \left(\frac{R+1}{R-1}\right)^2 \iff$$

$$e^{d/2} \geq \frac{R+1}{R-1} = \frac{1+1/R}{1-1/R} \iff$$

$$\cancel{R}\ \frac{1}{R} \leq \tanh \frac{d}{4} \leq \frac{d}{4}$$

Thus

$$|x-y| \leq \frac{4}{k} \cdot \frac{1}{R} \leq \frac{1}{k} \cdot d(x,y)$$

Normalizing to unit norm

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| = \left| \frac{x|y| - y|y| + y|y| - y|x|}{|x| \cdot |y|} \right|$$

$$\leq \frac{|x-y|}{|x|} + \frac{1}{|x|} \big| |y| - |x| \big|$$

$$\leq 2|x-y| \quad \text{as } |x| \geq \langle m, x\rangle = 1$$

$$\leq \frac{2}{k} d_K(x,y)$$

Theorem 8.5
Suppose E a Banach space. Let $K \subset E$ be a proper closed $\mathbb{R}$-cone. Suppose $T \in L(E)$ is a linear contraction of the cone
$$TK^* \subset K^*, \quad \Delta = \operatorname{diam}_K TK^* < +\infty$$
and that K is of k-bd sectional aperture. Then

a) K contains one unique T-invariant (half) line $\mathbb{R}_+ h$

b) Writing $Th = \lambda h$ with $\lambda > 0$ there are consts $M, C < \infty$ ~~so that~~ and a function
$$\rho : K \longrightarrow \mathbb{R}_+ \quad \text{s.t.:}$$
$\forall x \in K,\ n \geq 1$:
$$|\lambda^{-n} T^n x - h \cdot \rho(x)|_E \leq C \cdot \eta^{n-1} |x|_E$$
$$|h \cdot \rho(x)|_E \leq M \cdot |x|_E$$
where $\eta = \tanh \Delta/4$.

Proof. Note we assume here E a Banach space and K closed for the Banach space topology.

Pick $e_0 \in K$ of norm one and define recursively:
$$e_{n+1} = \frac{Te_n}{|Te_n|_E}, \quad n \geq 0$$
Then $e_n \in T^n K^*$, $e_{n+m} \in T^{n+m} K^* \subset T^n K^*$ for $n, m \geq 1$ so we have
$$d_K(e_{n+m}, e_n) \leq \eta\, d_K(e_{n+m-1}, e_n)$$
$$\leq \dots \leq \eta^{n-1} d_K(e_{m+1}, e_1)$$
$$\leq \eta^{n-1} \cdot \Delta. \quad \text{By Lemma 8.4}$$
$$(*) \quad |e_{n+m} - e_n| \leq \frac{2}{k} \cdot \eta^{n-1} \cdot \Delta \xrightarrow[n \to \infty]{} 0$$
so (e_n) is Cauchy in E (Ban. sp)
Let $h = \lim_n e_n$, $|h|_E = 1$

Taking limits in $(*)$ we also obtain
$$|h - e_n|_E \leq \frac{2}{k} \cdot \Delta \eta^{n-1}, \quad n \geq 1$$

Denote
$$\lambda = |Th| > 0 \quad (\text{as } h \in K^*)$$
We have
$$0 = |Te_n|_E \cdot e_{n+1} - Te_n$$
$$\xrightarrow[n]{} |Th|_E \cdot h - Th$$
$$= \lambda \cdot h - Th$$
so $Th = \lambda \cdot h$ is invariant (exc unique)

b) For $x \in K^*$ we set $x_0 = x$,
$$x_n = \lambda^{-n} T^n x, \quad n \geq 1$$
Then:
$$(**) \quad \left|h - \frac{x_n}{|x_n|}\right| \leq \frac{2}{k} \eta^{n-1} \Delta$$
Also
$$|x_{n+1} - x_n|_E = |(\lambda^{-1} T - 1) x_n|$$
$$= |(\lambda^{-1} T - 1)(x_n - |x_n| h)|$$
$$\leq (\lambda^{-1} \|T\| + 1) \frac{2}{k} \eta^{n-1} \Delta \cdot |x_n|_E$$
$$\underset{\text{def}}{=:} R \cdot \eta^{n-1} \cdot |x_n|$$
Thus: $|x_{n+1}| \leq (1 + R\eta^{n-1}) |x_n|$
so
$$|x_n| \leq M |x_0| = \prod_{n \geq 1} (1 + R\eta^{n-1}) |x_0|$$
and then
$$|x_{n+m} - x_n| \leq \left(\sum_{k \geq 0} R \eta^{n+k-1}\right) M |x_0|$$
$$(***) \quad = \frac{R}{1-\eta} \eta^{n-1} M \cdot |x|$$
so (x_n) is Cauchy in E.
Write
$$\rho(x) := \lim_n |x_n|$$
Then
$$|x_n - h \cdot \rho(x)|_E \leq$$
$$|x_n - h|x_n||_E + ||x_n|_E - \rho(x)| \leq$$
$$\frac{2}{k} \eta^{n-1} \Delta \cdot M|x|_E + \frac{R}{1-\eta} \eta^{n-1} M |x|_E$$
$$= C \cdot \eta^{n-1} |x|_E$$

Thm 8.6 [Spectral gap]
E Banach space.
Let $K \subset E$ be a proper $\mathbb{R}$-cone which is of k-bd sect aperture and regenerating in E

Let $T \in L(E)$ be a strict contraction of K^*
$TK^* \subset K^*$, $\Delta = \operatorname{diam}_K TK^* < \infty$.

Then $\exists!\ \lambda > 0$, $h \in K^*$, $\ell \in E'$ with $|h| = 1$, $\langle \ell, h\rangle = 1$ such that

$$\forall x \in E,\ n \geq 1 :$$
$$|\lambda^{-n} T^n x - h\langle \ell, h\rangle|_E \leq C\eta^{n-1}|x|_E$$
where $C < \infty$, $\eta = \tanh\frac{\Delta}{4} < 1$.

Spectral interpretation:
λ is a simple eval of T.
with spectral projection $h \otimes \ell$

The operator $T - \lambda h \otimes \ell$ has spectral radius $\leq \eta \cdot \lambda < \lambda$.

Proof: Let $x \in E$. By the regenerating property we may find $x_+, x_- \in K$ so that $x = x_+ - x_-$ and
$$|x_+| + |x_-| \leq g \cdot |x|$$
g fixed const.

By thm 8.5, $\forall n \geq 1$
$$|\lambda^{-n}T^n x_+ - h\rho(x_+)| \leq C_1\eta^{n-1}|x_+|$$
$$|\lambda^{-n}T^n x_- - h\rho(x_-)| \leq C_1\eta^{n-1}|x_-|$$

Therefore:
$$|\lambda^{-n}T^n x - h(\rho(x_+) - \rho(x_-))|_E \leq$$
$$C_1\eta^{n-1}(|x_+| + |x_-|) \leq$$
$$C_1\eta^{n-1} g \cdot |x|$$

and
$$\lim_n \lambda^{-n}T^n x = h \cdot \underbrace{(\rho(x_+) - \rho(x_-))}_{\in \mathbb{R}}$$
By uniqueness of the limit
$$\rho(x) := \rho(x_+) - \rho(x_-)$$
is independent of the decomp and $|\rho(x)| \leq M|x_+| + M|x_-| \leq Mg|x|$

Thus:
$$|\lambda^{-n}T^n x - h\rho(x)| \leq C_1\eta^{n-1} g|x|$$
$$=: C\eta^{n-1}|x|$$

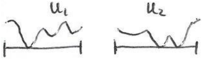
Now for $x, y \in E$, $a, b \in \mathbb{R}$
$$\lambda^{-n}T^n(ax + by) = a\,\lambda^{-n}T^n x + b\,\lambda^{-n}T^n y$$
$$\downarrow \qquad\qquad \downarrow$$
$$h \cdot \rho(ax + by) = h(a\rho(x) + b\rho(y))$$
shows that ρ is linear and since bounded we may write
$$\rho(x) = \langle \ell, x\rangle$$
with $\ell \in E'$ and
$$\|\ell\| \leq M \cdot g. \qquad //$$

Remark:
Suffices that
$K + (-K)$ is dense in E and for x in a dense subset of E we have
$$x = x_+ - x_-, \quad x_\pm \in K$$
$$|x_+| + |x_-| \leq g|x|$$
since for that dense subset we have
$$|\lambda^{-n}T^n x - h \cdot \rho(x)| \leq C\eta^{n-1}|x|$$
$$|\rho(x)| \leq Mg|x|$$
which is preserved when taking limits.

Cône L^p_+

Regularité ext de

$$K = L^p_+(\Omega;\mu)$$



$f_1, f_2 \in K^*$. $\quad t_1 = \beta(f_1, f_2) > 0$
lin indép $\quad t_2 = \beta(f_2, f_1) > 0$

$t_1 t_2 > 1$.

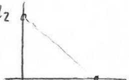
$u_1 = f_1 - \frac{1}{t_1} f_2 \in \partial K \qquad |u_1|_p \le |f_1|$

$u_2 = f_2 - \frac{1}{t_2} f_2 \in \partial K \qquad |u_2|_p \le |f_2|$

$K[f_1, f_2] = \mathbb{R}_+(u_1, u_2) = \{a u_1 + b u_2 : a, b \ge 0\}$.

$u_1 \mapsto u_1 / |u_1|_p$ (unit normalized)
$u_2 \mapsto u_2 / |u_2|_p$

A particular case is $(\mathbb{R}^n, |\cdot|_p)$ with the norm $|x|_p = (\sum_i |x_i|^p)^{1/p}$ corresponding to the counting measure μ on $\{1, \dots, n\}$.
$x : \{1, \dots, n\} \to \mathbb{R}$



$\frac{1}{p} + \frac{1}{q} = 1$

$pq = p + q$

$(p-1) q = p$

$g = u_1^{p-1} + u_2^{p-1}$

$\|g\|_q \le \|u_1^{p-1}\|_q + \|u_2^{p-1}\|_q$
$= (\int u_1^{(p-1)q})^{1/q} + \dots$
$= (\int u_1^p)^{1/q} + \dots$
$= 2$

Pour $f = a u_1 + b u_2$:

$\|f\|_p \le a + b$

$\int g f \le \|g\|_q \|f\|_p \le 2 \|f\|_p$

$\int g f \ge a \int u_1^p + b \int u_2^p = a + b$

d'ou

$$\|f\|_p \le \int g f \le 2 |f|_p$$

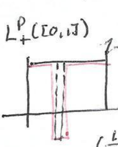
L^p_+ is a regenerating cone since $L^p = L^p_+ + (-L^p_+)$
$= L^p_+ - L^p_-$

$|f_+| + |f_-| = |f|$

so const $g = 1$.

L^p_+ is not inner regular for $1 \le p < +\infty$ (in general)

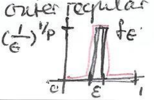
$1 \in L^p_+([0,1])$ but $\forall r > 0$ $1 - \frac{1}{r} f_\epsilon \notin L^p_+$ for ε small enough.



$(\frac{1}{\epsilon})^{1/p} > \frac{1}{r}$

$\epsilon < r^p$

$L^p_+(\Omega;\mu)$ is in general <u>not</u> outer regular for $1 < p \le +\infty$.



$\langle \ell, f \rangle = \int_0^1 f(x)\,dx$
$= \varepsilon \cdot \frac{1}{\epsilon^{1/p}} = \varepsilon^{1 - 1/p} \xrightarrow[\epsilon \to 0]{} 0$

$0 \le \sigma < 1$ Example 5.6

$K_\sigma = \{x \in \mathbb{R}^n_+ : x_i \ge \sigma x_j\}$

$K_\sigma \subset K_0$ inclusion stricte.

Pour $x, y \in K_\sigma^*$ on cherche $d_{K_0}(x,y)$.

$tx - y \in K_0 \Leftrightarrow$

$tx_i - y_i \ge 0 \quad \forall i \Leftrightarrow$

$t \ge \max_i y_i/x_i$. D'où

$t_1 = \beta(x,y) = \max_i y_i/x_i$

$t_2 = \beta(y,x) = \max_j x_j/y_j$

et

~~$d(x,y) = \log t_1 t_2 = \max$~~

$t_1 t_2 = \max_{i,j} \frac{y_i}{x_i} \frac{x_j}{y_j} \le \frac{1}{\sigma^2}$

d'où $\sqrt{t_1 t_2} = 1/\sigma$ et

$\Delta = \sup d \le 2 \log \frac{1}{\sigma}$

$\tanh \frac{d}{4} = \frac{e^{d/2} - 1}{e^{d/2} + 1} = \frac{\sqrt{t_1 t_2} - 1}{\sqrt{t_1 t_2} + 1}$

$\le \frac{1/\sigma - 1}{1/\sigma + 1} = \frac{1 - \sigma}{1 + \sigma}$

ℓ^p norm

$|x|_p = (\sum |x_i|^p)^{1/p}$

$|x|_\infty = \max\{|x_i|\}$

$\ell(x) = x_1 + \dots + x_n = |x|_1$

$= \sum_i 1 \cdot x_i$

$|\ell(x)| \le |(1)|_q |x|_p$

$= (\sum_i 1)^{1/q} |x|_p = (n)^{1/q} |x|_p$

$\|\ell\|_{\ell^{p'}} = (n)^{1/q} = n^{1 - 1/p}$

Also $\boxed{(\sum x_i^p)^{1/p} \le \sum x_i}$, all $x_i \ge 0$

$|x|_p \le \langle \ell, x \rangle \le n^{1-1/p} |x|_p$

so outer reg with const

$k = \frac{1}{n^{1-1/p}}$

Reg Int $K_0 + (-K_0) = \mathbb{R}^n$

$x = x_+ - x_-$ $\quad |x_+| + |x_-| \le 2|x|$ (ou mieux)

Reg Ext $x, y \in K_0$, $\ell(x) = \sum x_i$

Theorem [Perron-Frobenius]
Soit A une matrice reel d'élément strictement positive.
Alors A a un trou spectrale

proof: on peut supposer qu'il existe $\sigma \in (0,1)$, $M < \infty$ s.t.

$A = (a_{ij})$ $\quad 0 < \sigma M \le a_{ij} \le M < +\infty$

Then A maps $\mathbb{R}^n_+$ into $K_\sigma^{(n)}$ since for $x \in \mathbb{R}^n_+$, any k:

$\sigma M \sum x_j \le (Ax)_k \le M \sum x_j$