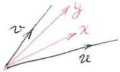
6 ~~4~~. Hilbert metric + Birkhoff inequality 2016

✱ 2.1

~~a~~ Cross-ratio (birapport)

Let E be a real ~~topol.~~ vec. sp of dim ≥ 2.



For ~~all~~ u, v independent vectors in E we write
$V = V_{u,v} = \text{Span}\{u, v\}$.

Def ~~4~~.1 Let $x, y \in V^* \;(V^* = V\setminus\{0\})$ and suppose that at least one of x or y is ~~madeg~~ not parallel to neither u nor v.
Then there is a unique decomp

$$(*) \quad \begin{aligned} x &= au + bv \\ y &= cu + dv \end{aligned}$$

for $a, b, c, d \in \mathbb{R}$ with at most one of the numbers being zero. We define the cross ratio of x, y with respect to u, v:

$$R = R[x, y; u, v] = \frac{a \cdot d}{b \cdot c}.$$

Prop ~~4~~.2 $R[x, y; u, v]$ is a projective invariant

Proof: $R[\lambda x, y; u, v] = \frac{(\lambda a) \cdot d}{(\lambda b) \cdot c} = R.$

$R[x, y; \lambda u, v] = \frac{(a/\lambda) \cdot d}{b \cdot (c/\lambda)} = R.$

so R depends only on the projective class of each of x, y, u, and v. //

Prop ~~4~~.3 $R[x, y; u, v]$ is a linear invariant.

Proof: If $A : E \to E$ is linear and Au, Av are <u>independent</u> then
$$\left.\begin{aligned} Ax &= a\,Au + b\,Av \\ Ay &= c\,Au + d\,Av \end{aligned}\right\} \text{unique}$$
so $R[Ax, Ay; Au, Av] = R[x, y; u, v]$

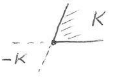
The Hilbert metric

Def. 4 E real topol. vector space. $K \subset E$ is a proper, closed convex cone (also called an $\mathbb{R}$-cone) iff closed and:

(a) $\mathbb{R}_+ K = K$

(b) $K + K = K$

(c) $K \cap (-K) = \{0\}$



Def. 5 For $x, y \in K^* = K \setminus \{0\}$ we set:

$$\beta_K(x,y) = \inf\{t > 0 : tx - y \in K\}$$

with values in $[0, +\infty]$ (by prop (c))

$$d_K(x,y) = \log(\beta_K(x,y) \cdot \beta_K(y,x))$$

$$d_K : K^* \times K^* \to \mathbb{R} \cup \{+\infty\}$$

Prop. 6 d_K defines a projective metric:

1) $d_K(x,y) \geq 0$, $= 0$ iff $x \parallel y$

2) $d_K(\lambda x, \lambda' y) = d_K(x,y)$, $\lambda, \lambda' > 0$

3) $d_K(x,y) = d_K(y,x)$

4) $d_K(x,z) \leq d_K(x,y) + d_K(y,z)$

proof: 1) $t_1 = \beta(x,y)$, $t_2 = \beta(y,x)$

$u = t_1 x - y \in K$, $v = t_2 y - x \in K$ $\Rightarrow$ $t_2 u + v = (t_1 t_2 - 1)x \in K$, $u + t_1 v = (t_1 t_2 - 1) y \in K$

which implies $t_1 t_2 - 1 \geq 0$

equality $\Rightarrow$ $t_2 u + v = 0 \Rightarrow u = v = 0$

2) $\beta(x,y) = \frac{1}{\lambda} \beta(\lambda x, y) = \lambda \beta(x, \lambda y)$

3) obvious

4) $t_1 x - y \in K$, $s_1 y - z \in K$ $\Rightarrow$

$t_1 s_1 x - z \in K$ sim: $t_2 s_2 z - x \in K$

$d(x,z) \leq \log(t_1 s_1)(t_2 s_2)$

Take inf over t_1, s_1, t_2, s_2

Def. 7 Given $u, v \in E$ the cone generated by u, v:

$$\mathbb{R}_+(u,v) = \{su + tv : s, t \geq 0\}$$

Relation with cross-ratio

Prop. 8 Let $u, v \in E$ be indep. Take $x, y \in K^* = \mathbb{R}_+(u,v)^*$. Suppose that either

1) $x \nparallel y$

2) $x \parallel y$ but $x \nparallel u, v$

Then

$$d_K(x,y) = |\log R(x,y;u,v)|$$

proof $t_1 = \beta(x,y)$, $t_2 = \beta(y,x)$

~~Case 1~~ suppose $t_1, t_2 < \infty$

Then

$t_1 x - y \parallel u$, $t_2 y - x \parallel v$ (or the converse)

so $u' = \lambda u = t_1 x - y \neq 0$

$v' = \lambda' v = t_2 y - x \neq 0$

whence:

$t_2 u' + v' = (t_1 t_2 - 1) x$

$u' + t_1 v' = (t_1 t_2 - 1) y$

$R(x,y;u,v) =$

$R(x,y;u',v') =$

$\frac{t_2 : t_1}{1 : 1}$ (or $\frac{1 : 1}{t_1 : t_2}$)

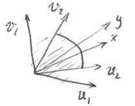
and $d_K(x,y) = \log t_1 t_2$

Prop 10 Let

$u_1, v_1 \in E^*$, $K_1 = \mathbb{R}_+(u_1, v_1)$

$u_2, v_2 \in K_1^*$, $K_2 = \mathbb{R}_+(u_2, v_2)$

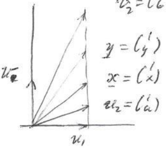
$x, y \in K_2^*$



There are 3 Hilbert-distances $d_{K_1}(x,y)$, $d_{K_2}(x,y)$ and $\Delta = d_{K_1}(u_2, v_2)$. One has

$$d_{K_1}(x,y) \le \tanh\frac{\Delta}{4}\, d_{K_1}(x,y)$$

Proof: using cross-ratios. We may assume $x \neq y$ (or else $d_{K_1}(x,y) = 0$) and also $d_{K_1}(x,y) < +\infty$.



$$u_2 = u_1 + a v_1, \qquad 0 \le a, b \le +\infty$$
$$v_2 = u_1 + b v_1$$

$$\Delta = \left|\log\left(\frac{1 \cdot b}{1 \cdot a}\right)\right| \le +\infty$$

$$\underline{x} = u_1 + x v_1$$
$$\underline{y} = u_1 + y v_1$$

$$d_{K_1}(\underline{x}, \underline{y}) = \left|\log\frac{y}{x}\right| = \boxed{\left|\int_x^y \frac{1}{t}\,dt\right|}$$

Writing $\underline{x}, \underline{y}$ as lin comb. of u_2, v_2:

$$\underline{x} = \frac{b-x}{b-a} u_2 + \frac{x-a}{b-a} v_2$$
$$\underline{y} = \frac{b-y}{b-a} u_2 + \frac{y-a}{b-a} v_2$$

So $d_{K_2}(x,y) =$

$$\left|\log\frac{(b-x)(y-a)}{(x-a)(b-y)}\right| =$$

$$\left|\log\frac{y-a}{b-y} - \log\frac{x-a}{b-x}\right| =$$

$$\boxed{\left|\int_x^y \frac{b-a}{(b-t)(t-a)}\,dt\right|}$$

Let us bound the ratio (note $a \le x, y \le b$)

$$\eta = \sup_{a<t<b} \frac{(b-t)(t-a)}{(b-a)} \cdot \frac{1}{t}$$

max attained for $t = \sqrt{ab}$:

$$\eta = \frac{(b-\sqrt{ab})(\sqrt{ab}-a)}{(b-a)\sqrt{ab}}$$

$$= \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right.$$

$$= \frac{\sqrt{b/a}-1}{\sqrt{b/a}+1} = \frac{e^{\Delta/2}-1}{e^{\Delta/2}+1}$$

$$= \frac{e^{\Delta/4} - e^{-\Delta/4}}{e^{\Delta/4} + e^{-\Delta/4}} = \tanh\frac{\Delta}{4}$$

Then

$$d_{K_1}(\underline{x}, \underline{y}) = \left|\int_x^y \frac{1}{t}\,dt\right|$$

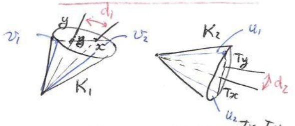
$$\le \left|\int_x^y \eta \frac{b-a}{(b-t)(t-a)}\,dt\right|$$

$$= \eta\, d_{K_2}(x,y)$$

Theorem 4.11 [Garrett Birkhoff 57] (son of George David B.)

Let $T: V_1 \to V_2$ be a linear map between $\mathbb{R}$-vectorspaces.
Let $K_1 \subset V_1$, $K_2 \subset V_2$ be proper $\mathbb{R}$-cones, s.t. $TK_1^* \subset K_2^*$.
Set $\Delta = \operatorname{diam}_{K_2} TK_1^* \in [0, +\infty]$.
Then for any $x, y \in K_1^*$:

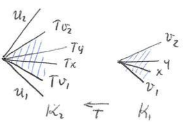
$$d_{K_2}(Tx, Ty) \le \tanh \tfrac{\Delta}{4}\, d_{K_1}(x, y)$$



proof: We may assume Tx, Ty lin independent, (or else $d_2 = 0$) whence x and y, too. | Also assume $d_{K_1}(x,y) < +\infty$

$$K_1(x,y) = V_1(x,y) \cap K_1 = \mathbb{R}_+[v_1, v_2] \ni x, y$$

$$K_2(Tx,Ty) = V_2(Tx,Ty) \cap K_2 = \mathbb{R}_+[u_1, u_2] \ni Tx, Ty$$



$$d_{K_2}(Tx,Ty) = d_{u_1,u_2}(Tx,Ty)$$

Prop 4.8 $\longrightarrow$ $\le \eta \cdot d_{Tv_1,Tv_2}(Tx,Ty)$

Prop 4.3 $\longrightarrow$ $= \eta \cdot d_{v_1,v_2}(x,y)$

$= \eta \cdot d_{K_1}(x,y)$

where $\eta = \tanh \frac{1}{4} d_{u_1,u_2}(Tv_1, Tv_2) = \tanh \frac{1}{4} d_{K_2}(Tv_1, Tv_2)$

$\le \tanh \frac{1}{4} \Delta$. //