5 Continued Fractions and Gauss Map

For $x \in (0,1) \setminus \mathbb{Q}$ a unique way of writing

$$x = [0; a_1 a_2 a_3 \ldots] = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

as continued fraction

$$\pi : \mathbb{N}^{* \mathbb{N}^*} \longrightarrow (0,1) \setminus \mathbb{Q}$$

$$\pi(a_1 a_2 \ldots) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

$a_1, a_2, a_3, \ldots \geq 1$. We write

$$\frac{p_n}{q_n} = [0; a_1 a_2 \ldots a_n]$$ for the nth convergent of the cont. fract.

p_n, q_n co-prime

Given x we get $a_1 = \lfloor \frac{1}{x} \rfloor$ and then

$$T(x) = [0; a_2 a_3 \ldots] = \frac{1}{x} - a_1 = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$$

or $$T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = \{ \frac{1}{x} \}$$ fractional part.

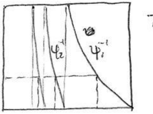
$T : \Omega = (0,1) \setminus \mathbb{Q} \circlearrowleft$ is called the Gauss map.

The Gauss map

$\Omega = (0,1)\setminus\mathbb{Q}$. We define

$T: \Omega \to \Omega$

$x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = \{ \frac{1}{x} \}$



$T: (\frac{1}{n+1}, \frac{1}{n}) \to (0,1)$ bijection diffeo.

Inverse branches:

$\psi_n : \Omega \to \Omega$

$y \mapsto \frac{1}{n+y}, \quad n \geq 1.$

$(0,1) \to (\frac{1}{n+1}, \frac{1}{n})$ diffeo.

Thm 5.1 [Gauss,]

The probability measure $d\mu(x) = \frac{1}{\ln 2} \frac{1}{1+x} dx$ is T-invariant.

proof: For $A \in L^\infty((0,1))$ write

$$\int_0^1 A\circ T(x) \cdot \frac{1}{1+x} dx = \sum_{n\geq 1} \int_{1/(n+1)}^{1/n} A\circ T(x) \cdot \frac{1}{1+x} dx =$$

$$\sum_{n\geq 1} \int_0^1 A(y) \cdot \frac{1}{1+\psi_n(y)} |\psi_n'(y)| dy =$$

$$\int_0^1 A(y) \sum_{n\geq 1} \frac{1}{1+\frac{1}{n+y}} \frac{1}{(n+y)^2} dy =$$

$$\int_0^1 A(y) \sum_{n\geq 1} \frac{1}{n+y} \cdot \frac{1}{n+1+y} dy =$$

$$\int_0^1 A(y) \sum_{n\geq 1} \left(\frac{1}{n+y} - \frac{1}{n+1+y}\right) dy =$$

$$\int_0^1 A(y) \frac{1}{1+y} dy$$

Thm (later) $d\mu = \frac{1}{\ln 2} \frac{1}{1+x} dx$ is mixing, whence ergodic for T.

proof: postponed to later ~~where a more~~ in a more general context.

Thm 5.2 For Leb a.e $x \in (0,1)\setminus\mathbb{Q}$ the number $a \geq 1$ appears in the cont fraction expansion for x with asymptotic frequency:

$$\rho_a = \frac{1}{\ln 2} \ln \frac{(a+1)^2}{a(a+2)}$$

proof: In $x = [0; a_1 a_2 a_3 \ldots]$ we have

$a_k = a$ iff $T^{k-1}(x) \in (\frac{1}{a+1}, \frac{1}{a})$

χ_a char fct $\mathbb{1}_{(\frac{1}{a+1}, \frac{1}{a})}$

The frequency of a among the n first numbers

$$\frac{1}{n} \#\{1 \leq k \leq n : a_k = a\} = \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{(\frac{1}{a+1}, \frac{1}{a})}(T^k x)$$

By Birkhoff + ergodicity

$$\xrightarrow{n} \int \mathbb{1}_{(\frac{1}{a+1}, \frac{1}{a})} d\mu(x) = \int_{1/(a+1)}^{1/a} \frac{1}{\ln 2} \frac{1}{1+x} dx = \frac{1}{\ln 2} \left(\ln(1+\tfrac{1}{a}) - \ln(1+\tfrac{1}{a+1})\right)$$

$\rho_1 = 0{,}415\ldots$

$\rho_2 = 0{,}170\ldots$

$\rho_3 = 0{,}093.$

$\rho_7 =$

$x\in(0,1)$, $a_1 = n(x) = \lfloor \frac{1}{x} \rfloor$

and $T(x) = -\lfloor \frac{1}{x} \rfloor + \frac{1}{x} = \{ \frac{1}{x} \} \in [0,1)$

Then $x = \frac{1}{a+T(x)}$, $a = \lfloor \frac{1}{x} \rfloor$, $T(x) \in [0,1)$

Lemma Let $\frac{p_n(x)}{q_n(x)} = [0; a_1 a_2 .. a_n]$ be the n'th convergent for $x = [0; a_1 a_2 a_3 \dots]$. We have:

So in particular $\boxed{p_n(x) = q_{n-1}(Tx)}$

$$\begin{pmatrix} p_n(x) \\ q_n(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} p_{n-1}(Tx) \\ q_{n-1}(Tx) \end{pmatrix} \quad \text{with } a_1 = \lfloor \tfrac{1}{x} \rfloor$$

proof

$$\frac{p_n(x)}{q_n(x)} = [0; a_1 a_2 .. a_n] = \frac{1}{a_1 + [0; a_2 a_3 .. a_n]}$$

$$= \frac{1}{a_1 + \frac{p_{n-1}(Tx)}{q_{n-1}(Tx)}} = \frac{q_{n-1}(Tx)}{a_1 q_{n-1}(Tx) + p_{n-1}(Tx)}$$

when $p_{n-1}(Tx)$ and $q_{n-1}(Tx)$ are co-prime then so are $p_n(x) = q_{n-1}(Tx)$ and $q_n(x) = a_1(x) q_{n-1}(Tx) + p_{n-1}(Tx)$

Lemma: Set $p_0 \equiv 0$, $q_0 \equiv 1$. Then $p_n = p_n(x)$, $q_n = q_n(x)$ verifies

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

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proof:

$$\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

Corollary: $\det \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = p_{n-1} q_n - p_n q_{n-1} = (-1)^n$

so $$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}$$

Coroll $p_n \geq F_n$, $q_n \geq F_{n+1}$
where $(F_n)_{n\geq 0} = (0,1,1,2,3,5,\dots)$
is the Fibonacci seq. $F_{n+2} = F_{n+1} + F_n$

One has $F_n \geq 2^{\lfloor n/2 \rfloor - 1}$, $n \geq 1$

proof:

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \geq \overbrace{\begin{pmatrix}0&1\\1&1\end{pmatrix}\cdots\begin{pmatrix}0&1\\1&1\end{pmatrix}}^{n} = \begin{pmatrix}0&1\\1&1\end{pmatrix}^n \underset{\text{induction proof}}{=} \begin{pmatrix}F_{n-1}&F_n\\F_n&F_{n+1}\end{pmatrix}$$

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}^2 = \begin{pmatrix}1&1\\1&2\end{pmatrix} \geq \begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}^m = 2^m \cdot \begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$\begin{pmatrix}p_1&p_2\\q_1&q_2\end{pmatrix} \geq \begin{pmatrix}1&1\\1&1\end{pmatrix}$$

Lemma $x - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{(q_{n-1}T^n(x) + q_n)q_{n-1}}$

and

$M_n M_{n-1} \leq \left| x - \frac{p_{n-1}}{q_{n-1}} \right| \leq 2 M_n M_{n-1}$

where $M_n(x) = x \cdot T(x) \cdots T^{n-1}(x)$

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proof: For $x > 0$: $\frac{1}{x} = \underset{\in[0,1)}{T(x)} + \underset{\in\mathbb{N}^*}{a_1(x)}$

Thus,

$$\begin{pmatrix}x\\1\end{pmatrix} = \begin{pmatrix}1\\1/x\end{pmatrix} x = \begin{pmatrix}1\\T(x)+a_1\end{pmatrix} x = \begin{pmatrix}0&1\\1&a_1\end{pmatrix}\begin{pmatrix}T(x)\\1\end{pmatrix} x.$$

Iterating this:

$$\begin{pmatrix}x\\1\end{pmatrix} = \begin{pmatrix}0&1\\1&a_1\end{pmatrix}\cdots\begin{pmatrix}0&1\\1&a_n\end{pmatrix}\begin{pmatrix}T^n(x)\\1\end{pmatrix} x \cdot T(x) \cdots T^{n-1}(x)$$

$$= \begin{pmatrix}p_{n-1}&p_n\\q_{n-1}&q_n\end{pmatrix}\begin{pmatrix}T^n x\\1\end{pmatrix} M_n(x) \quad \text{or}$$

$$\begin{pmatrix}x\\1\end{pmatrix} = \begin{pmatrix}p_{n-1}T^n(x) + p_n\\ q_{n-1}T^n(x) + q_n\end{pmatrix} M_n(x)$$

We deduce:

(1) $x = \frac{x}{1} = \frac{p_{n-1}T^n(x) + p_n}{q_{n-1}T^n(x) + q_n}$

$$x - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{(q_{n-1}T^n(x) + q_n) q_{n-1}} = \frac{(-1)^{n-1}}{(q_{n-1}T^n(x) + q_n) q_{n-1}}$$

as well as:

$$\begin{cases} x = (p_{n-1}T^n(x) + p_n) M_n(x) \\ 1 = (q_{n-1}T^n(x) + q_n) M_n(x) \end{cases}$$

Since $q_{n-1} < q_n$ and $0 < T^n(x) < 1$:

$$q_n M_n(x) \leq 1 \leq 2 q_n M_n(x)$$

~~Therefore~~

$\cancel{\left| x - \frac{p_{n-1}}{q_{n-1}} \right| \leq \frac{1}{q_n q_{n-1}} \leq 2M_n M_{n-1}}$

~~and~~

$\cancel{\left| x - \frac{p_{n-1}}{q_{n-1}} \right| \geq}$

$$\left| x - \frac{p_{n-1}}{q_{n-1}} \right| = \frac{1}{(q_{n-1}T^n x + q_n) q_{n-1}} = \frac{M_n}{q_{n-1}} \leq 2 M_n M_{n-1}$$

and

$$\left| x - \frac{p_{n-1}}{q_{n-1}} \right| = \frac{M_n}{q_{n-1}} \geq M_n M_{n-1}$$

Thm For Leb-a.e $x \in (0,1) \setminus \mathbb{Q}$:

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = -\frac{\pi^2}{6\log 2}$$

(thus: $|x - \frac{p_n}{q_n}| \sim e^{-n\frac{\pi^2}{6\log 2}} \sim e^{-2{,}37n}$)

proof: We have

$$M_{n+1}M_n \leq \left|x - \frac{p_n}{q_n}\right| \leq 2M_{n+1}M_n$$

where $M_n = M_n(x) = \prod_{j=0}^{n-1} T^j(x)$

As T is ergodic w.r.t. $d\mu = \frac{1}{\log 2}\frac{1}{1+x}dx$:
For μ a.e x = Leb a.e. x:

$$\frac{1}{n}\log M_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \log T^j(x) \xrightarrow[n]{}$$

(by Birkhoff)

$$I = \int_0^1 \log(x)\, d\mu(x) = \int_0^1 \frac{\log x}{\log 2} \cdot \frac{1}{1+x}\, dx$$

Using $\int_0^1 x^p \log x \, dx \overset{I.B.P.}{=} -\frac{1}{(p+1)^2}$

$$I = \frac{1}{\log 2}\left[\int_0^1 \log x\,(1 - x + x^2 - \dots)\,dx\right]$$
$$= \frac{1}{\log 2}\left[\sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{(p+1)^2}\right]$$
$$= -\frac{1}{\log 2} \cdot \frac{\pi^2}{12}$$

[admitting $\sum_{n\geq 1} \frac{1}{n^2} = \pi^2/6$ one has

$$\sum_{n\geq 1} \frac{(-1)^n}{n^2} = 2\sum_{n\geq 1}\frac{1}{(2n)^2} - \sum_{n\geq 1}\frac{1}{n^2}$$
$$= -\frac{1}{2}\sum \frac{1}{n^2} = -\pi^2/12]$$

Now for Leb a.e x:

$$\lim_n \frac{1}{n}\log\left|x - \frac{p_n}{q_n}\right| = \lim_n \frac{1}{n}\log M_{n+1}M_n$$
$$= 2\cdot I = -\frac{\pi^2}{6\log 2}$$

Prop One has the expansion $\forall x \in (0,1) \setminus \mathbb{Q}$:

$$x = \sum_{k\geq 0} \frac{(-1)^k}{q_k q_{k+1}}$$

proof:

$$\frac{p_n}{q_n} = \frac{p_n}{q_n} - \frac{p_0}{q_0} =$$
$$\sum_{k\geq 0}^{n-1}\left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) =$$
$$\sum_{k=0}^{n-1} \frac{(-1)^k}{q_k q_{k+1}}$$

and $p_n/q_n \xrightarrow[n]{} x$