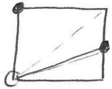
4. ~~Measures~~ Invariant Measures on compact spaces

Example $[0,1]$ compact metric space



$T \curvearrowright [0,1]$

discontinuous

$T(x) = \lambda x \quad , 0 < x \leq 1$
$\lambda < 1$

$T(0) = 1$

has no T-invariant Borel probability meas.

proof suppose μ T-invariant proba

$$\mu((0,a]) = \mu(T^{-1}(0,\lambda a]) = \mu((0,\lambda a]) = \cdots = \mu((0,\lambda^n a])$$

By Dominated convergence

$$\to \mu\left(\bigcap_n (0,\lambda^n a]\right) = \mu\emptyset = 0.$$

$\mu(\{0\}) = \mu(T^{-1}\{0\}) = \mu\emptyset = 0.$

$\mu(\{1\}) = \mu(T^{-1}\{1\}) = \mu\{0\} = 0$

Invariant measures for continuous transformations

In the following let (X,d) be a compact metric space and $T: X \to X$ a continuous transformation

$M_+^1(X)$ space of proba measures

By Riesz we identify $M_+^1(X)$ with

$F = \{\Lambda \in C(X)' : \Lambda \geq 0,\ \Lambda \mathbb{1} = 1\}$

Since F is a closed subset of the unit ball in $C(X)'$ and the unit ball is sequentially weak-* compact we obtain (Banach-Alaoglu)

Thm 4.1: $M_+^1(X)$ is sequentially weak-* compact, ie. given any sequence $(\nu_n)_{n\geq 1}$ in $M_+^1(X)$ we may extract a convergent subsequence: $\exists \nu \in M_+^1(X)$, subseq n_k:

$\forall f \in C(X)$: $\int f\, d\nu_{n_k} \longrightarrow \int f\, d\nu$

The (~~continuous~~ not needed here) map $T: X \to X$ induces a map on $M(X)$:

$T_*: M(X) \to M(X)$ $\quad (T_*\mu)(A) = \mu(T^{-1}A)$, $A \in \mathcal{B}$

Remarks

ex Dirac mass en $p \in X$: δ_p

$\delta_p(A) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{if not.} \end{cases}$

$T_*\delta_p = \delta_{T(p)}$ $\quad$ $\delta_p(T^{-1}A) = \delta_{Tp}(A)$

Given $f \in C(X)$:

$$\int f\, d(T_*\mu) = \int f\circ T\, d\mu.$$

$$\int f\, T_*^j d\mu = \int f\circ T^j\, d\mu$$

~~Rec~~ Recall μ is T-invariant iff $\int f\circ T\, d\mu = \int f\, d\mu \quad \forall f \in C(X)$.

Thm ~~Prop~~ ~~Thm~~ 4.2: $T: X \to X$ cont transf of a compact metric sp.

Let (ν_n) be a sequence in $M_+^1(X)$ and define

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} T_*^j \nu_n$$

Then any weak-* limit point μ of $(\mu_n)_{n\geq 1}$ is a T-invariant proba meas.

In particular,

$M_+^1(X,T)$ is non-empty.

(set of T-inv proba)

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proof: Let $\mu_{n_k} \longrightarrow \mu \in M_+^1(X)$ and let $f \in C(X)$.

Since T is continuous so is $f\circ T$ so we have

$\int (f\circ T - f)\, d\mu = \lim_k \int (f\circ T - f)\, d\mu_{n_k}$

Now:

$\int f\circ T\, d\mu_{n_k} - \int f\, d\mu_{n_k} =$

$\frac{1}{n_k} \int (f\circ T - f) \sum_{j=0}^{n_k-1} T_*^j d\nu_{n_k} =$

$\frac{1}{n_k} \int \sum_{j=0}^{n_k-1} (f\circ T^{j+1} - f\circ T^j)\, d\nu_{n_k} =$

$\frac{1}{n_k} \int (f\circ T^{n_k} - f)\, d\nu_{n_k}$

and

$\frac{1}{n_k} \left| \int f\circ T^{n_k} - f\, d\nu_{n_k} \right| \leq \frac{2}{n_k} |f|_\infty \xrightarrow[k\to\infty]{} 0.$

ex Pick $\nu_n = \delta_p$ arbitrary fixed p

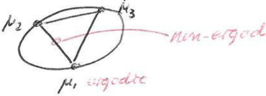
$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j p}$$

(Dirac masses along the orbit of p)

Thm 4.3 [Krylov and Bogoliubov]
X compact metric sp. $T: X \to X$ cont. Then $M_+^1(X,T)$ is non-empty, convex and compact.

proof: Prop 4.2 $\Rightarrow$ non-empty
$M_+^1(X)$ is convex compact
$M_+^1(X,T)$ [space of T-invariant proba measures) is a closed convex subset of $M_+^1(X)$,

closed since T is <u>cont</u>:
$\mu_n \in M_+^1(X,T) \xrightarrow{w^*} \mu \in M_+^1(X)$, $f \in C(X)$
$\int f\circ T \, d\mu_n = \int f \, d\mu_n \Rightarrow \int f \circ T \, d\mu = \int f \, d\mu$.



An extremal point of a convex set is a point which may not be written as a proper convex combination of two distinct points in the set.

$\boxed{\text{Denote by } \mathcal{E}(X,T) \text{ the extremal points of } M_+^1(X,T)}$

only measurable here:

Thm 4.4 $(X,\mathcal{B})$ measurable space. $T: X \to X$ measurable. Then the extremal points $\mathcal{E}(X,T)$ in $M_+^1(X,T)$ (if non-empty) are precisely the ergodic measures in $M_+^1(X,T)$

proof: μ not ergodic $\Rightarrow$
$\exists B \in \mathcal{B}: B = T^{-1}B$ with $0 < \underbrace{\mu(B)}_{=t} < 1$.
Then $\mu = t\left(\frac{1}{\mu B}\mu_{|B}\right) + (1-t)\left(\frac{1}{\mu B^c}\mu_{|B^c}\right)$ is a proper convex comb of distinct T-inv proba meas.

Suppose μ ergodic but $\mu = t\mu_1 + (1-t)\mu_2$, $0 < t < 1$ with $\mu_1, \mu_2 \in M_+^1(X,T)$.
Then $0 \le \mu_1 \le \frac{1}{t}\mu$
so $\mu_1 \ll \mu$ (abs cont.)

Birkhoff:
Since μ is ergodic, for $A \in \mathcal{B}$:
$\frac{1}{n} S_n \mathbb{1}_A(x) \xrightarrow[n]{} \mathbb{1} \cdot \int \mathbb{1}_A \, d\mu = \mu(A)$
for μ-ae x $\Rightarrow$ for μ_1-a.e. x
By dom convergence

$$\mu_1(A) = \lim_n \frac{1}{n}\mu_1(S_n \mathbb{1}_A) = \mu_1(\mu(A)\cdot\mathbb{1}) = \mu(A)\,\mu_1(\mathbb{1}) = \mu(A)$$
so $\mu = \mu_1$

<u>similarly for μ_2</u>.

Two measures μ_1 and μ_2 are said to be mutually singular (written $\mu_1 \perp \mu_2$) iff $\exists A \in \mathcal{B}$:
$\mu_1(A) = \mu_2(A^c) = 0$
$\overset{\text{proba}}{\Leftrightarrow}$ $\mu_1(A^c) = \mu_2(A) = 1$

Prop 4.5 Let $\mu_1, \mu_2 \in \mathcal{E}(X,T)$.
Then $\mu_1 \neq \mu_2 \Rightarrow \mu_1 \perp \mu_2$

proof: $\mu_1 \neq \mu_2$ implies that there is $B \in \mathcal{B}$ with
$\mu_1(B) \neq \mu_2(B)$
Birkhoff + ergodicity:
$\exists A_1 \in \mathcal{B}, \mu_1(A_1) = 1$, s.t $\forall x \in A_1$
$\frac{1}{n} S_n \mathbb{1}_B(x) \to 1 \cdot \mu_1(B)$
$\exists A_2 \in \mathcal{B}; \mu_2(A_2) = 1$, s.t $\forall x \in A_2$
$\frac{1}{n} S_n \mathbb{1}_B(x) \to 1 \cdot \mu_2(B)$
Thus $A_1 \cap A_2 = \emptyset$ and $\mu_1 \perp \mu_2$

Def 4.6 $(X, \mathcal{B})$ a measurable space. $T: X \to X$ measurable is said to be uniquely ergodic iff

$M_+^1(X,T)$ consists of a single measure (ergodic since extremal)

Thm 4.7 X a compact metric space and $T: X \to X$ continuous. The following are equivalent:

1) T is uniquely ergodic

2) $\forall f \in C(X)$, $\forall x \in X$:

$$\frac{1}{n} S_n f(x) \xrightarrow[n]{} \Lambda(f) \qquad (*)$$

with $\Lambda(f)$ indep of x.

3) The convergence in $(*)$ is uniform in x.

4) $(*)$ holds for a dense set in $C(X)$

Proof:

$1 \Rightarrow 2$: The measures $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x}$ must converge weakly towards the unique element $\mu \in M_+^1(X,T)$, or else there would be other limit points. So $\forall f \in C(X)$: $\frac{1}{n} S_n f(x) \to \int f \, d\mu$

$2 \Rightarrow 1$: Λ defines by Riesz an element $\mu_\Lambda \in M_+^1(X,T)$ (check the details)

Let $\mu \in M_+^1(X,T)$. Then by dominated convergence

$$\int f \, d\mu = \int \frac{1}{n} S_n f \, d\mu \to \int \Lambda(f) \, d\mu = \Lambda(f)$$

so $\mu = \mu_\Lambda$.

$1 \Rightarrow 3$ Suppose the contrary. Then $\exists f \in C(X)$ so that the conv in $(*)$ is non-uniform, i.e.:

$\underline{\exists \epsilon > 0}$ s.t. $\forall n$:

$$\sup_{k \geq n} \sup_{x \in X} \left| \frac{1}{k} S_k f(x) - \Lambda(f) \right| > \epsilon$$

Thus $\forall n$ $\exists k_n \geq n$, $x_n \in X$ with

$$\left| \frac{1}{k_n} S_{k_n} f(x_n) - \Lambda f \right| > \epsilon$$

~~While~~ or equivalently

$$\left| \int f \, d\mu_n - \Lambda f \right| > \epsilon$$

where

$$\mu_n = \frac{1}{k_n} \sum_{j=0}^{k_n - 1} T_*^j \delta_{x_n}$$

By Thm 4.2 any weak-* limit yields a T-invar. proba measure:

$$\mu_{n_\ell} \xrightarrow[\ell]{\text{weak-*}} \nu \in M_+^1(X,T)$$

Then

$$\left| \int f \, d\nu - \Lambda f \right| = \lim \left| \int f \, d\mu_{n_\ell} - \Lambda f \right| > \epsilon$$

which implies $\nu \neq \mu_\Lambda$ contrary to the unique ergod.

$3 \Rightarrow 4$ clear

$4 \Rightarrow 1$ If $\mu, \nu \in M_+^1(X,T)$ then

$$\int f \, d\mu = \Lambda f = \int f \, d\nu$$

for a dense set $\Rightarrow$ for all $f \in C(X)$ $\Rightarrow \mu = \nu = \mu_\Lambda$

or use $\left| \frac{1}{n} S_n (f - g)(x) \right| \leq \| f - g \|_\infty$ so Λ extends by continuity to all of $C(X)$

Equi-distribution

Def 4.8 (X,d) compact metr. sp. μ Borel proba measure. A sequence $(x_n)_{n\geq 0}$ is said to be equi-distributed with respect to μ if $\forall f \in C(X)$:

$$\frac{1}{n}\sum_{k=0}^{n-1} f(x_k) \xrightarrow[n]{} \int f\,d\mu$$

or, equivalently:

$$\frac{1}{n}\sum \delta_{x_n} \xrightarrow[n]{*} \mu$$

in the weak-* topology.

'Typical' Example 4.9:

A sequence $(x_n)_{n\geq 0}$ in $[0,1]$ is said to be equi-distributed (wrt Lebesgue) or uniformly distributed iff

1) $\forall f \in C(X)$; $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} f(x_k) = \int_0^1 f\,dx$

$\Updownarrow$

2) $\forall p \in \mathbb{Z}^*$: $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i p x_k} = 0$

$\Updownarrow$

3) $\forall\, 0\leq a<b\leq 1$: $\frac{1}{n}\#\{k\leq n : a\leq x_k\leq b\} \xrightarrow[n]{} b-a$

$\underbrace{\phantom{\#\{k\leq n : a\leq x_k\leq b\}}}_{\text{relative density in } [a,b]}$

$1\Leftrightarrow 2$: use trigo polyn dense in $C([0,1])$

$1\Rightarrow 3)$ $\forall \varepsilon>0$ find $0\leq f_- \leq \mathbb{1}_{[a,b]} \leq f_+$ with $f_-, f_+ \in C([0,1])$, $\int f_+ - f_- < \varepsilon$

$3\Rightarrow 1)$ Approximate $f\in C([0,1])$ by simple functions (lin combi of indicator fcts on intervals)

Def 4.10 Given $(X,\mathcal{B},\mu,T)$ on a compact metric space (X,d) ($\mathcal{B}$ Borel) we say that $x\in X$ is <u>generic</u> for (μ,T) iff the orbit $(T^n x)_{n\geq 0}$ is equi-distributed w.r.t. μ.

Thm 4.11 X compact metric. $T: X\to X$ measurable. μ a T-invariant ergodic Borel proba. Then μ-a.e $x\in X$ is generic ~~wrt~~ for (μ,T).

proof:

$C(X)$ is separable (admitted)

Let $(f_j)_{j\geq 0}$ be dense in $C(X)$.

For each j there is $B_j \in \mathcal{B}$ so that (Birkhoff + ergodic)

$\forall x\in B_j$: $\frac{1}{n} S_n f_j(x) \xrightarrow[n]{} \int f_j\,d\mu$

$A = \bigcap_{j\geq 0} B_j$ is then of full meas. and we have Birkhoff conv for every f_j and $x\in A$

Given $f\in C(X)$, $\varepsilon>0$ find j so that $\|f-f_j\|_\infty < \varepsilon/2$.

Then $\forall x\in A$

$\limsup_n \left|\frac{1}{n}S_n f(x) - \int f d\mu\right| < \varepsilon$,

$\varepsilon>0$ was arbitrary //

Remark: Although μ-a.e. point is generic it may be impossible to give any explicit such point.

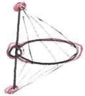
Theorem 4.12 [Krein-Milman]
[Rudin "Funct An." I, 3.23]

Ω locally convex topo. vect. sp.
$K \subset \Omega$ non-empty compact convex.
Then K is the closed convex hull of its extreme points $\mathcal{E}(K)$.

$$K = \overline{co(\mathcal{E}(K))}$$

Recall: $\mathcal{E}(K) \subset K$, and $x \in \mathcal{E}(K)$
if $\forall y, z \in K$, $0<t<1$
$(1-t)y + tz = x \Rightarrow y = z = x$.

$\mathcal{E}(K)$ need not be closed



Take in $\mathbb{R}^3$ a circle C and a segment I intersecting in $p \in C \cap I$

Then $\mathcal{E}(co(C \cup I))$ = ~~[illegible]~~
$= (C \setminus \{p\}) \cup \partial I$

Thm 4.13 [Choquet]

Ω loc. convex. topo. vect. sp.
$K \subset \Omega$ non-empty compact convex and separable, metrizable.

Then for every $\mu \in K$ there is a proba measure ν supported on $\mathcal{E}(K)$ (the extreme points) representing μ, i.e.:

for any cont lin functional f on Ω we have

$$\langle \mu, f \rangle = \int_{\mathcal{E}(K)} \langle \lambda, f \rangle \, d\nu(\lambda)$$

Application to ergodic th:
~~Example~~

X compact metric space
$T: X \to X$ continuous

$K = M_+^1(X,T)$ is compact, convex, separable, metrizable, so

Given $\mu \in M_+^1$ there is a Borel proba supported on $\mathcal{E}(X,T)$ (ergodic measures) ν s.t. $\forall f \in C(X)$:

$$\int_X f \, d\mu = \int_{\mathcal{E}(X,T)} \left(\int_X f \, d\lambda \right) d\nu(\lambda)$$

We write $\mu = \int_{\mathcal{E}(X,T)} \lambda \, d\nu(\lambda)$ and call it the ergodic decomposition of μ.

Example: $X = \overline{\mathbb{D}} \subset \mathbb{R}^2$

$T_\alpha : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ rotation by angle α with $\frac{\alpha}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$

Ergodic measures:
$\lambda_0 = \delta_0$ and λ_r = uniform distr on S_r circle of radius $r \in (0,1]$

Any $\mu \in M_+^1(X,T)$ may be decomposed as $\mu = \int_0^1 \lambda_r \, d\nu(r)$

$$\int_{\overline{\mathbb{D}}} f \, d\mu = \int_0^1 \left[\int_0^{2\pi} f(r,\theta) \frac{d\theta}{2\pi} \right] d\nu(r)$$

Example: normalized Lebesgue meas:

$$\frac{1}{\pi} \int_{\overline{\mathbb{D}}} f \, dx dy = \int_0^1 \left[\int_0^{2\pi} f(r,\theta) \frac{d\theta}{2\pi} \right] 2r \, dr$$