3 Extension, Factors, Product

Kolmogorov extension

$(X_i, \mathcal{B}_i)_{i\in\mathbb{N}}$ countable family

$X = \prod_{i\in\mathbb{N}} X_i$ product space.

$\forall F \subset \mathbb{N}$ finite: Rectangles over F:
$\mathcal{R}_F = \{ R = \prod_{i\in F} A_i \times \prod_{j\notin F} X_j : A_i \in \mathcal{B}_i \}$

$\mathcal{B}_F = \sigma(\mathcal{R}_F)$

$\mathcal{B} = \sigma(\bigcup_{F \text{ finite}} \mathcal{R}_F)$

Consistent family of probas

$(X, \mathcal{B}_F, \mu_F)_{F\subset\mathbb{N} \text{ finite}}$

$\forall F \subset G$ finite; $B \in \mathcal{B}_F \subset \mathcal{B}_G$
$\mu_F(B) = \mu_G(B)$

Compact clan: $\mathcal{C}^F \subset \mathcal{B}^F$ is a compact clan if it has the finite intersection property (think of compact subsets)

$\bigcap_{k\geq 1} C_k = \emptyset \Rightarrow \exists N: \bigcap_1^N C_k = \emptyset$

countable family here

(tight)

μ_F is inner regular wrt $\mathcal{C}_F$ if

$\forall B \in \mathcal{B}_F : \mu_F(B) = \sup_{\substack{C\in\mathcal{C}_F \\ C\subset B}} \mu_F(C)$

Examples: when X_i is a complete separable metric sp (or homeom. to such) [called a Polish space] with $\mathcal{C}_i$ being the compact subsets.

$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \{1, \dots, d\}$ etc.
(and finite products thereof)

Thm 3.1 [Kolmog.]

Let $X = \prod_{\mathbb{N}} X_i \quad \mathcal{B} = \sigma(\bigcup_{F \text{ finite}} \mathcal{R}_F)$

Assume $(X, \mathcal{B}_F, \mu_F)_{F\subset\mathbb{N} \text{ fin}}$ is a consistent ~~countable~~ family with each μ_F being inner regular wrt a compact clan (in $\mathcal{B}_i$) $i\in\mathbb{N}$. Then

1) There is a unique proba on $(X, \mathcal{B})$ which extends the family: $\forall F \subset \mathbb{N}$ finite $B \in \mathcal{B}_F$: $\mu(B) = \mu_F(B)$

2) [Approximation] $\forall B \in \mathcal{B}, \epsilon > 0$ there is a finite disjoint union of rectangles
$B' = \bigsqcup_{\text{finite}} R_\alpha \quad R_\alpha \in \bigcup_{F \text{ finite}} \mathcal{R}_F$
such that
$\mu(B \triangle B') < \epsilon$

Shift spaces.

$(X_0, \mathcal{B}_0, \mu_0)$ fixed proba space

$$X = \prod_{k\in\mathbb{Z}} X_0 \quad \text{or} \quad X_+ = \prod_{k\in\mathbb{N}_0} X_0$$

σ-algebra generated by rectangles. We assume that ~~each~~ μ_0 is regular wrt a compact class, so that Kolmogorov extension applies. Product mesure on rects

$$R = \prod_{j\in F} A_j \times \prod_{j\notin F} X_0 \qquad F \text{ finite},\ A_j \in \mathcal{B}_0$$

$$\mu(R) = \prod_{j\in F} \mu_0(A_j) \quad (\mu_+ = ..)$$

then extends to X (or X_+)

Shift transformation

invertible $\sigma : X \to X \quad (\sigma(x))_k = x_{k+1},\ k\in\mathbb{Z}$

non-inv. $\sigma_+ : X_+ \to X_+ \quad (\sigma(x))_k = x_{k+1},\ k\in\mathbb{N}$

$\mu(\sigma^{-1}R) = \mu(R)$ (and $\mu_+(\sigma_+^{-1}R) = \mu_+(R)$) are then shift inv. measures.

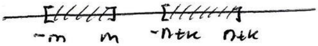
Thm 3.2 $(X, \mathcal{B}, \mu, \sigma)$ (and $(X_+, \mathcal{B}_+, \mu_+, \sigma_+)$) are mixing, hence ergodic.

Proof: For $A, B \in \mathcal{B}$ we need to show that $\mu(\sigma^{-k}A \cap B) \xrightarrow[k]{} \mu(A)\mu(B)$

When A, B are rectangles

$$A = \prod_{|i|\le n} A_i \times \prod_{|i|>n} X_0$$

$$B = \prod_{|j|\le m} B_j \times \prod_{|j|>m} X_0$$



For $k > n+m = N(A,B)$

$$\mu(\sigma^{-k}A \cap B) = \prod_{|i|\le n} \mu_0(A_i) \prod_{|j|\le m} \mu_0(B_j)$$
$$= \mu(A)\mu(B)$$

For finite disjoint union of rectangles

$$A = \bigsqcup_{\text{finite}} A_\alpha \qquad B = \bigcup_{\text{finite}} B_\beta$$

for $N > \max_{\alpha,\beta} N(A_\alpha, B_\beta)$, $k \ge N$

$$\mu(\sigma^{-k}A \cap B) = \sum_{\alpha,\beta} \dots = \mu(A)\mu(B)$$

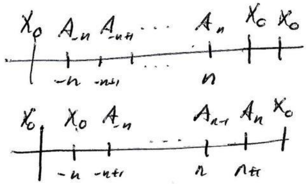
By thm 3.1 Given $A, B \in \mathcal{B}$, $\epsilon > 0$ there are A', B' finite disjoint union of rects s.t.

$$\mu(A \triangle A') < \varepsilon \quad \mu(B \triangle B') < \varepsilon$$

and then

$$\limsup_{k\to\infty} |\mu(\sigma^{-k}A \cap B) - \mu(A\cap B)|$$

$$\le 4\epsilon$$



Thm 3.3

Ex 3 Strong law of large numbers

$(\Omega, \mathcal{B}_\Omega, \mu_\Omega)$ proba space.

$Y_0, Y_1, \ldots$ sequence of i.i.d random vars.

$$\lambda = Y_{i*}\mu_\Omega \quad \text{common law (on } (\mathbb{R}, \mathcal{B}_\mathbb{R}))$$

We assume $\mathbb{E}|Y_i| < \infty$

$$= \int_\mathbb{R} |x|\, d\lambda(x) < \infty$$

Then: For μ_Ω a.e. $\omega \in \Omega$: $\frac{1}{n}(Y_0(\omega) + \ldots + Y_{n-1}(\omega)) \to \mathbb{E}Y_i$

Aim: Show strong law of large numbers using Birkhoff's thm.

Define: $X = \prod_{i \geq 0} \mathbb{R}$ and

$$\boxed{\begin{aligned} \phi &: \Omega \to X \\ &\omega \mapsto (Y_0(\omega), Y_1(\omega), \ldots) \end{aligned}}$$

Measurable rectangle: $(A_i \in \mathcal{B}_\mathbb{R})$

$$R = A_0 \times A_1 \times \ldots \times A_{k-1} \times \prod_{\geq k} \mathbb{R}$$

$\mathcal{R}$ collection of meas. rects.
$\mathcal{B}_X = \sigma(\mathcal{R})$ σ-alg generated.

$$\phi^{-1}(R) = Y_0^{-1}(A_0) \cap \ldots \cap Y_{k-1}^{-1}(A_{k-1}) \in \mathcal{B}_\Omega$$

so $\phi^{-1}(\mathcal{R}) \subset \mathcal{B}_\Omega \Rightarrow$

$$\phi^{-1}\mathcal{B}_X = \phi^{-1}(\sigma(\mathcal{R})) \subset \sigma(\mathcal{B}_\Omega) = \mathcal{B}_\Omega.$$

The map

$$\phi : (\Omega, \mathcal{B}_\Omega) \to (X, \mathcal{B}_X)$$

is thus measurable and we may define the push-forward of μ_Ω:

$$\mu_X := \phi_* \mu_\Omega \quad (\text{on } (X, \mathcal{B}_X))$$

For rectangles $\mu_X(R) = \mu_\Omega(\phi^{-1}R)$
$= \mu_\Omega(Y_0^{-1}(A_0) \cap \ldots \cap Y_{k-1}^{-1}(A_{k-1}))$
$= \mu_\Omega(Y_0^{-1}(A_0)) \times \ldots \times \mu_\Omega(Y_{k-1}^{-1}(A_{k-1}))$ (indep)
$= \lambda(A_0) \ldots \lambda(A_{k-1})$

product meas.

$\sigma : X \to X \quad (x_0, x_1, \ldots) \to (x_1, x_2, \ldots)$
is then meas. pres. on rects:
$\mu_X(\sigma^{-1}R) = \mu_X(R)$
whence on $\mathcal{B}_X$. ($\cap$ rect $\subset$ rect, $\cup$ rect not)

As $\mathcal{B}_X = \sigma(\mathcal{R})$, $\mathcal{R}$ semi-alg, any $B \in \mathcal{B}_X$ may be approx. by a finite disjoint union of rects:

$\forall B \in \mathcal{B}_X, \epsilon > 0$
$\exists R_1, \ldots, R_N \in \mathcal{R}: B' = \dot{\bigcup}_1^N R_i$
$\mu(B \Delta B') < \epsilon$

Then $(X, \mathcal{B}_X, \mu_X, \sigma)$ is (Thm 3.2) mixing whence ergodic

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Strong law of large numbers

Def $f : X \to \mathbb{R}$

$$x = (x_0, x_1, \ldots) \mapsto x_0$$

$$\mathbb{E}|f| = \int |x_0|\, d\mu_X = \int |x_0|\, d\lambda(x_0) = \mathbb{E}|Y_0| < \infty.$$

Birkhoff $\Rightarrow$
There is a meas subset $\Lambda \subset X$ of full measure st for μ_X-a.e. $x \in \Lambda$:

$$\frac{1}{n} S_n f(x) = \frac{1}{n}(f(x) + \ldots + f \circ \sigma^{n-1} x) \to \mathbb{E}f = \mathbb{E}Y_i$$

$\Omega' = \phi^{-1}\Lambda \subset \Omega$ has full measure in Ω and for every $\omega \in \Omega'$; $(x = \phi(\omega))$

$$\frac{1}{n}(Y_0(\omega) + \ldots + Y_{n-1}(\omega)) = \frac{1}{n}(f(x) + \ldots + f \circ \sigma^{n-1}(x)) \to \mathbb{E}Y_i \quad //$$

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Def 3.4

$\mathcal{X} = (X, \mathcal{B}_X, \mu_X, T_X)$

$\mathcal{Y} = (Y, \mathcal{B}_Y, \mu_Y, T_Y)$

two meas pres ~~map~~ transformations (proba).

We say that $\mathcal{Y}$ is a <u>factor</u> of $\mathcal{X}$ or that $\mathcal{X}$ is an <u>extension</u> of $\mathcal{Y}$ provided there are sets of full measures

$X' \in \mathcal{B}_X$, $Y' \in \mathcal{B}_Y$ with

$T_X(X') \subset X'$ $T_Y(Y') \subset Y'$ (invariant)

and a measure pres map

$\pi : X' \longrightarrow Y'$; $\pi \circ T_X = T_Y \circ \pi$

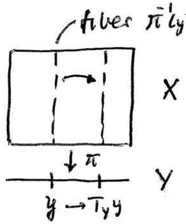
π: Also called a homomorphism (mod 0)

Commuting diagram

$$\begin{tikzcd} X' \arrow[r, "T_X"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & X' \arrow[d, "\pi"] \\ Y' \arrow[r, "T_Y"'] & Y' \end{tikzcd}$$

We say that $\mathcal{X}$ and $\mathcal{Y}$ are <u>isomorphic</u> if $\mathcal{X}$ is ~~also a factor of $\mathcal{Y}$~~ (mod 0) if π is a bimeasurable bijection

with ~~and~~ π is an invertible bijection, bimeasurable



exists more refined definitions

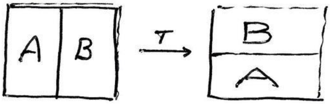
Factor maps

Examples 3.5

$I = [0,1)$

$I \times I = [0,1) \times [0,1)$

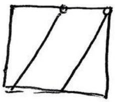
$$T(x,y) = \begin{cases} (2x, y/2) & \text{if } 0 \le x < \frac{1}{2} \\ (2x-1, \frac{y}{2}+\frac{1}{2}) & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$



invertible map
preserves Lebesgue (2dim)

$I = [0,1)$

$f(x) = 2x \bmod 1$
preserves Lebesgue (1D)



Let π_1 = projection onto 1st factor

$$\begin{tikzcd} I \times I \arrow[r, "T"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowright"] & I \times I \arrow[d, "\pi_1"] \\ I \arrow[r, "f"'] & I \end{tikzcd}$$

$\pi_* Leb_2 = Leb_1$

$\mathcal{X} = (I \times I, \mathcal{B}_{I\times I}, \mu_{Leb_2}, T)$

$\mathcal{Y} = (I, \mathcal{B}_I, \mu_{Leb_1}, f)$

$\mathcal{Y}$ is a factor of $\mathcal{X}$.

$\mathcal{X}$ is an extension of $\mathcal{Y}$.

~~Full~~ Shift Spaces of Finite type

Def 3.6 A uni-directional full shift space on d symbols is defined as

$$\Sigma_d^+ = \prod_{n\geq 0} \{1,\dots,d\} =$$

$$\{ \underline{s} = (s_n)_{n\geq 0} : s_n \in \{1,\dots,d\}\}.$$

The shift operator is:

$$\sigma^+((s_n)_{n\geq 1})_k = s_{k+1}, \quad k \geq 0$$

A bi-directional shift space (d symbols) is:

$$\Sigma_d = \prod_{\mathbb{Z}} \{1,\dots,d\} =$$

$$\{ \underline{s} = (s_n)_{n\in\mathbb{Z}} : s_n \in \{1,\dots,d\}\}$$

Shift on the same, except $k\in\mathbb{Z}$.

Topology:
We define the topol on Σ_d^+ (Σ_d) to be the product topology of $\{1,\dots,d\}$ (discrete top.)

A base for that topo is given by cylinder sets:

$$[b_0,\dots,b_{n-1}] = \{b_0\}\times\dots\times\{b_{n-1}\}\times \prod_{k\geq n}\{1,\dots,d\}$$

$\mathcal{C}_n$ = collection of such n-cylinder

$$\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$$

(base for rectangles)

Prop 3.7 Σ_d^+ (Σ_d) ~~is~~ equipped with the prod topo is compact, metrizable, totally disconnected and perfect.

proof: ~~$\prod$~~ $\{1,\dots,d\}$ is compact
~~Also~~ Tychonoff $\Rightarrow \prod\{1,\dots,d\}$ is (sequentially) compact for the prod topo.

Given any $0<\theta<1$ one verifies (exo) that

$$d_\theta(x,y) = \sup\{\theta^k : x_k \neq y_k\}$$

gives a metric with the same topology.

(We skip the two last properties & eventually without interest here

3.8

Example $d \geq 2$

$X_i = \{0, 1, \dots, d-1\}$

$\mathcal{B}_i = 2^{X_i}$ (all subsets)

$\mu_i = \vec{p} = (p_0, \dots, p_{d-1})$ proba

$\sigma : \cancel{*}\ \Sigma_d = \prod_{\mathbb{Z}} X_i \circlearrowleft$ shift

$\sigma_+ : \Sigma_d^+ = \prod_{\mathbb{N}} X_i \circlearrowleft$ shift

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The cylinder sets provides the rectangles for the measure:

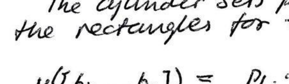
$$\mu_{\vec{p}}([b_i, \dots, b_j]) = p_{b_i} \cdots p_{b_j}$$

$\mathcal{B}$ (or $\mathcal{B}^+_{\phi}$) σ-algebra generated by cylinders.

$\left.\begin{array}{l}(\Sigma_d, \mathcal{B}, \mu_p, \sigma) \\ (\Sigma_d^+, \mathcal{B}^+_{\phi}, \mu_p^+, \sigma_+^{\phi})\end{array}\right\}$ both mixing $\Rightarrow$ ergodic (Thm 3.2)

$\pi : \Sigma_d \to \Sigma_d^+$

$(x_i)_{i \in \mathbb{Z}} \mapsto (x_i)_{i \in \mathbb{N}}$



$$\begin{tikzcd} \Sigma_d \arrow[r, "\sigma"] \arrow[d, "\pi"] & \Sigma_d \arrow[d, "\pi"] \\ \Sigma_d^+ \arrow[r, "\sigma_+"] & \Sigma_d^+ \end{tikzcd}$$

invertible extension

factor

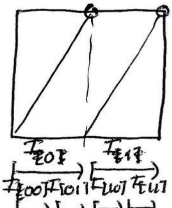
Example

$\mathcal{X} = ([0,1), \mathcal{B}_{[0,1)}, \lambda_{Leb}, T_2)$

$T_2(x) = 2x \bmod 1$

$\mathcal{Y} = (\Sigma_2^+, \mathcal{B}, \mu_{\frac{1}{2}, \frac{1}{2}}, \sigma_+)$

are isomorphic as measurable dynamical systems.



$$\mathcal{C}_n = \left\{ I = \left[\tfrac{k}{2^n}, \tfrac{k+1}{2^n}\right) : 0 \leq k < 2^n \right\}$$

binary expansion

$.b_0 b_1 \dots b_{n-1}$

$$T_2 I_{[b_0 \dots b_{n-1}]} = I_{[b_1 \dots b_{n-1}]}$$

$$b(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \quad I_{[0]} \\ 1 & \text{if } 1/2 \leq x < 1 \quad I_{[1]} \end{cases}$$

$\phi : [0,1) \to \Sigma_2^+$ defined by

$$\phi(x) = (b(x), b(Tx), b(T^2x), \dots)$$

not continuous but bi-measurable

$$\phi^{-1}([b_0 \dots b_{n-1}]) = I_{[b_0 \dots b_{n-1}]}$$

Let $N \in \Sigma_2^+$ consist of the countable set of pts of the form:

$$N = \{ (b_0 b_1 b_2 \dots b_{n-1} 1 1 1 \dots) \}_{\text{all } n, b_i}$$

Then $\phi([0,1)) = \Sigma_2^+ \setminus N$ is a meas pres bijection

Clearly by construction

$$\sigma \circ \phi = \phi \circ T$$

inverse $\phi^{-1}(\underbrace{(b_0 b_1 b_2 \dots)}_{b}) = \frac{b_0}{2} + \frac{b_1}{2^2} + \frac{b_2}{2^3} + \dots$

$b \notin N$

The natural invertible extension

(X_0, d) complete, separable metric space (or homeo to such)

$(X_0, \mathcal{B}_0, \mu_0)$ proba sp.
$T: X_0 \to X_0$ μ_0-invar. (not invertible)

extension invertible:

Thm[3.9] $\exists$ an invertible extension of $(X_0, \mathcal{B}_0, \mu_0, T)$:
$(X', \mathcal{B}', \mu', T')$ i.e. T' invertible
there is $\pi_0 : X' \to X_0$ meas s.t.

$$\pi_0 \circ T' = T \circ \pi_0$$
$$\pi_{0*}\mu' = \mu_0$$

Proof: Let $\hat{X} = \prod_{i \in \mathbb{Z}} X_i$
where $(X_i, \mathcal{B}_i, \mu_i)$ are identical copies of X_0

For $F \subset \mathbb{Z}$ finite define rectangles $\mathcal{R}_F$:

$$R = \prod_{i \in F} A_i \times \prod_{j \notin F} X_j, \quad A_i \in \mathcal{B}$$

$$\hat{\mathcal{B}} = \sigma\Big(\bigcup_{F \text{ finite}} \mathcal{R}_F\Big) \quad \sigma\text{-alg.}$$

$$\mu_F(R) = \mu_0\Big(\bigcap_{i \in F} T^{b-i} A_i\Big)$$

extend to meas on $\mathcal{B}_F = \sigma(\mathcal{R}_F)$

$F = \{2, 3, 4\}$
$b = \min F$

$A_2 \quad A_3 \quad A_4$

$$\mu_F(R) = \mu_0(A_2 \cap T^{-1} A_3 \cap T^{-2} A_4) =$$
$$\mu_G(R) = \mu_0(T^{-2}(A_2 \cap T^{-1} A_3 \cap T^{-2} A_4))$$

$$G = \{0, 1, \dots, 5\} \supset F$$

Kolmogorov consistency on rects
$F \subset G$, $R \in \mathcal{B}_F$: $\mu_G(R) = \mu_F(R)$

Kolomogoro ext. applies (compact class exist... (exercise)) nice

$\hat{\mu}$ extended measure

Admitted (not trivial)

detail ↓ next page

$$\boxed{\operatorname{supp} \hat{\mu} \subset X' = \{(x_k)_{k \in \mathbb{Z}} : T x_k = x_{k+1}\}}$$

Then $T' : ((x_k)_k) \to ((x_{k+1})_k)$
shift map (invertible) preserves $\hat{\mu}$

$\pi_0 : X' \to X_0$ projection
$(x_k)_k \to x_0$

$$T \pi_0(\underline{x}) = \pi_0(T' \underline{x})$$

$x_{-1} \quad x_0 \quad x_1 = Tx_0 \quad x_2$

Tx_0

$$\mu'(\pi_0^{-1}(A_0)) =$$
$$\mu'\Big(A_0 \times \prod_{i \neq 0} X_i\Big) =$$
$$\mu_0(A_0)$$

$$\Rightarrow \quad \pi_{0*}\mu' = \mu_0$$

Lemma: $(X, \mathcal{B}_X, \mu)$ proba.
(Y, d) metric space, complete separable (or homeo to such)

$T: X \to Y$ measurable.

ν ~~$\hat{\mu}$~~ $(A \times B) = \mu(A \cap T^{-1}B)$ extends to a measure on $\mathcal{B}_X \times \mathcal{B}_Y$ and one has

$G(T) = \{(x, Tx) : x \in X\}$

is measurable and of full $\hat{\mu}$ measure.

Proof: ~~Let~~ $\hat{d}(x,y) = d(Tx, y)$ is measurable so

$\hat{d}^{-1}(0) = G(T)$ is meas.

Let $\epsilon > 0$ and let $(y_i)_{i \geq 1}$ be a countable dense set in Y. Set $U_i = B(y_i, \epsilon/2)$.

Then $E_1 = T^{-1}U_1$, $E_2 = T^{-1}U_2 \setminus E_1, \ldots$

$E_k = T^{-1}U_k \setminus (E_1 \cup \ldots \cup E_{k-1}), \ldots$

gives a countable meas partition of X. Then

$E_i \subset T^{-1}U_i$ so

ν ~~$\hat{\mu}$~~ $(E_i \times U_i^c) = \mu(E_i \cap T^{-1}U_i^c) = 0.$

Thus $\bigcup_{i \geq 1} E_i \times U_i^c$ has null meas. and the complement $\bigcup_{i \geq 1} E_i \times U_i$ is thus of full measure.

When $(x,y) \in E_i \times U_i$ then

$d(y, y_i) < \epsilon/2$, $d(Tx, y_i) < \epsilon/2$

so also $d(y, Tx) < \epsilon$

So every such (x,y) is included in

$C_\epsilon = \{(x,y) : d(y, Tx) < \epsilon\}$

Which is thus of full measure

Then so is

$G(T) = \bigcap_{n \geq 1} C_{1/n}$

Given $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ consider

$G_m = \{(x)_{k \in \mathbb{Z}} : d(x_{m+1}, Tx_m) = 0\}$

The measure $\mu_{\{m, m+1\}} = \nu$ ($\mu = \mu_m$) verifies the condition of the Lemma so G_m is meas

$\hat{\mu}(G_m) = 1$

Then also

$X' = \bigcap_{m \in \mathbb{Z}} G_m =$

$\{(x_k)_{k \in \mathbb{Z}} : \forall k : d(x_{k+1}, Tx_k) = 0\}$

~~[illegible]~~ is meas and of full measure.