$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

$\mathrm{Aut}^{\omega}(\mathbb{D}) = \{ \phi : \mathbb{D} \to \mathbb{D} \text{ bihol.} \}$

every $\mathrm{Aut}^{\omega}(\mathbb{D})$ has the form of a Möbius transformation

$$M(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}, \quad \theta \in \mathbb{R}, a \in \mathbb{D}$$

One checks

$$\left|\frac{dM}{dz}\right| = \left| e^{i\theta} \frac{\cancel{z}+|a|^2}{(1-\bar{a}z)^2} \right| = \frac{1-|M(z)|^2}{1-|z|^2}$$

Thus the following differential identity

$$\frac{|dz|}{1-|z|^2} = \frac{|dM|}{1-|M(z)|^2}$$

Thm 1 [Poincaré] Let $(\mathbb{D}, d_{\mathbb{D}})$ be the unit disk equipped with the Riemannian metric

$$ds = \frac{2|dz|}{(1-|z|^2)} \qquad ds^2 = \frac{4(dx^2+dy^2)}{(1-(x^2+y^2))^2}$$

Then $\mathrm{Isom}(\mathbb{D}, d_{\mathbb{D}}) = \mathrm{Aut}^{\omega}(\mathbb{D})$

Proof $\mathrm{Aut}^{\omega}\mathbb{D} \subset \mathrm{Isom}\,\mathbb{D}$ by the above calculation.
For the converse we need

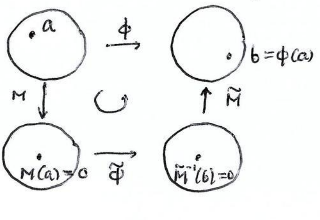
Lemma [Schwarz] Let $\phi : \mathbb{D} \xrightarrow{C^{\omega}} \mathbb{D}$ with $\varphi(0) = 0$. Then $|\varphi'(0)| \le 1$ with equality iff $\varphi(z) = e^{i\theta} z$.

Define the conformal derivative

$$\frac{D\phi}{Dz} = |\phi'(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2}$$

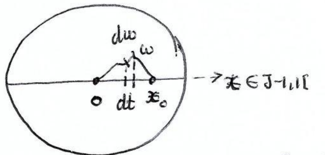
Thm 2 For $\phi \in C^{\omega}(\mathbb{D};\mathbb{D})$, Either $\forall z \in \mathbb{D}$ $\left|\frac{D\phi}{Dz}(z)\right| < 1$ or $\forall z \in \mathbb{D}$ $\left|\frac{D\phi}{Dz}\right| = 1$

Proof: pre- and post compose to map 0 to 0. Use Schwarz on $\tilde{\Phi}$



proof (end of) of Thm 1.
An isometry of $(\mathbb{D}, d_{\mathbb{D}})$ fixing 0 and say $\phi'(0) = 1$ must be the identity so $\in \mathrm{Aut}^{\omega}(\mathbb{D})$.
Then use Thm 2.

Geodesics in $(\mathbb{D}, d_{\mathbb{D}})$



Through orthog proj onto $]-1,1[$ (eucl. metric in $\mathbb{C}$) we see that

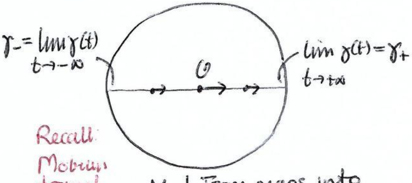
$$\frac{|dx|}{1-x^2} \le \frac{|dw|}{1-|w|^2}$$

$\Rightarrow$ geodesic between 0 and x_0 is the straight line segment $[0; x_0]$

$$\mathrm{len}([0;x_0]) = \left| \int_0^{x_0} \frac{dq}{1-q^2} \right| = \ln \frac{1+|x_0|}{1-|x_0|}$$

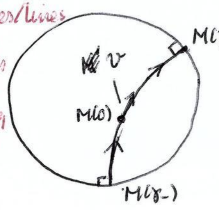
$\Rightarrow$ Geodesic flow through 0:

$$\gamma(t) = \tanh \frac{t}{2}, \quad t \in \mathbb{R}$$



Recall: Mobius transf maps circles/lines into circles/lines preserving angles

M ↓ Isom maps into circles $\perp$ $\partial\mathbb{D}$



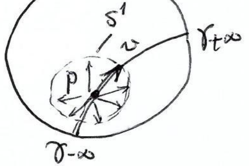
v tangent vector

$ds(v) = 1$

Geodesic flow

$$\varphi : \mathbb{R} \times \mathcal{U}^1(\mathbb{D}) \longrightarrow \mathcal{U}^1(\mathbb{D})$$

$\mathcal{U}^1(\mathbb{D})$ = unit tangent bundle
$\simeq \mathbb{D} \times \mathbb{S}^1$



Given $\xi = (p, v) \in \mathbb{D} \times S^1$
construct the geodesic
parametrized by $t \in \mathbb{R}$
γ_t such that $\gamma_0 = p$,

$$ds(\dot{\gamma}_t) \equiv 1 \quad \text{(unit speed)}$$

$$\frac{\dot{\gamma}_t}{|\dot{\gamma}_t|} = v \in \mathbb{S}^1$$

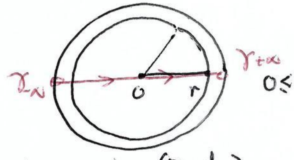
Then $\varphi_t(\xi) = \left(\gamma_t, \frac{\dot{\gamma}_t}{|\dot{\gamma}_t|}\right)$

Note that $ds(\dot{\gamma}_t) = 1$
means

$$\frac{2|\dot{\gamma}_t|}{1 - |\gamma_t|^2} = 1$$

so $|\dot{\gamma}_t| \longrightarrow 0$ as $|t| \to \infty$.

Perimeter of geodesic ball



$0 \leq r < 1$

Length in $(\mathbb{D}, d_{\mathbb{D}})$

$$s = \int_0^r \frac{2\,dt}{1-t^2} = \ln \frac{1+r}{1-r} \quad \text{or}$$

$$r = \tanh \frac{s}{2}$$

So the geodesic through 0
in the direction $1 \in \mathcal{U}^1$ is

$$t \in \mathbb{R} \to \gamma_t = \tanh \frac{t}{2}$$

$$\varphi_t(0) = \left(\tanh \frac{t}{2}, \underset{\text{const. direction}}{1}\right)$$

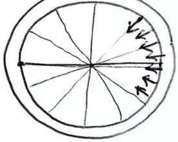
Perimeter of
$\partial B_{\mathbb{D}}(0, s)$ with $s > 0$; $r = \tanh \frac{s}{2}$

metric $\frac{2|dz|}{1-|z|^2} = \frac{2r\,d\theta}{1-r^2}$

$$= 2\,\mathrm{th}\frac{s}{2} \cdot \mathrm{ch}^2 \frac{s}{2}\, d\theta$$

$$= 2\,\mathrm{sh}\frac{s}{2}\,\mathrm{ch}\frac{s}{2}\, d\theta = \mathrm{sh}(s) \cdot d\theta$$

$$= \frac{1}{2}(e^{s} - e^{-s})\, d\theta$$



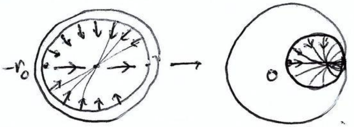
Consider the inward flow from a circle of radius

$$r_0 = \tanh \frac{s_0}{2}$$

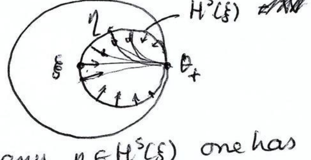
when $0 \le t \ll s_0$ the perimeter is of order $\pi e^{s_0 - t}$ so shrinks at exponential speed., Thus nearby points behave as if on the same stable manifold (up to time $\ll s_0$).

In order to get a true stable manifold we use a trick mapping $-r_0 \mapsto 0$, $0 \mapsto r_0$

$$M(z) = \frac{z + r_0}{1 - r_0 z}$$



taking $r_0 \to 1$ we obtain the stable horosphere, a circle tangent at $\theta_+ = \lim_{t \to \infty} \varphi_t(\xi)$

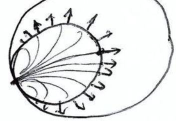


For any $\eta \in H^s(\xi)$ one has

$$d(\varphi_t \xi, \varphi_t \eta) \le Ce^{-t}, \ \forall t \ge 0$$

The stable manifold of ξ is precisely $H^s(\xi)$ with inward directed tangent vectors

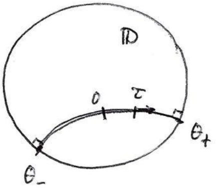
Similarly one construct the unstable horosphere $H^u(\xi)$ as a circle ~~at~~ tangent at $\theta_- = \lim_{t \to -\infty} \varphi_t(\xi)$



one has for any $\eta \in H^u(\xi)$

$$d(\varphi_t \xi, \varphi_t \eta) \le Ce^{t}, \ \forall t \le 0.$$

so indeed the unstable manifold of ξ.



diagonale

$(\theta_-, \theta_+) \in (S^1 \times S^1) \setminus \Delta$

$\gamma = \gamma_{\theta_-, \theta_+}$ gedodesic $\theta_- \to \theta_+$

eg. $\gamma(0)$ midpoint on arc.

$\gamma(\tau)$ geodesic at time $\tau \in \mathbb{R}$

Liouville measure on

$(\theta_-, \theta_+, \tau) \in (S^1 \times S^1 \setminus \Delta) \times \mathbb{R}$

$$\boxed{d\mu = \rho(\theta_-, \theta_+)\, d\theta_- d\theta_+ d\tau}$$

ρ only depends upon

$d(\theta_-, \theta_+) = |e^{i\theta_-} - e^{i\theta_+}|$

and blows up as $d(\theta_-, \theta_+) \to 0$

$d\mu$ invariant under geod flow

The change of coordinates

$\psi : U^1(\mathbb{D}) \simeq \mathbb{D} \times U^1 \longrightarrow$

$(S^1 \times S^1 \setminus \Delta) \times \mathbb{R}$

$(x, y, \theta) \longmapsto (\theta_-, \theta_+, \tau)$

is a C^∞-diffeomorphism (admitted)

In ideal coordinates the stable manifold of $\xi \sim (\theta_-, \theta_+, \tau)$:

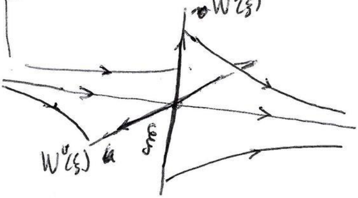
$W^s(\xi) = \{(u, \theta_+, \tau) : u \in \mathbb{R}\}$

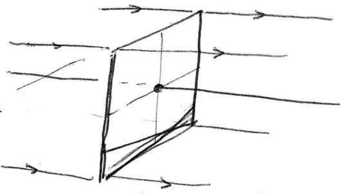
And the unstable manifold

$W^u(\xi) = \{(\theta_-, v, \tau) : v \in \mathbb{R}\}$

$$\boxed{\frac{4\, dx\, dy}{(1 - |x^2 + y^2|)^2} \cdot d\theta}$$

flow in (p, θ) coordinates:





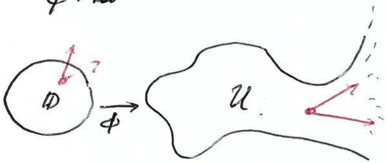
~~flows~~ ~~[illegible]~~ in $(\theta_-, \theta_+, \tau)$ coordinates

Riemann surfaces

Thm 3 [The Riemann mapping thm]

Let $\mathcal{U} \subset \mathbb{C}$ be open, non-empty, simply connected. Then there exists a biholomorphic mapping

$$\phi : \mathbb{D} \longrightarrow \mathcal{U}$$



The Poincaré metric induces a "hyperbolic" metric on $\mathcal{U}$: $d_{\mathcal{U}}$

$$ds_{\mathcal{U}} := ds_{\mathbb{D}} \quad \text{(on image of tangent vectors)}$$

$(\mathcal{U}, d_{\mathcal{U}})$ also verifies the uniform contraction principle

---

Covering maps (in $\mathbb{C}$) $\mathcal{U} \subset \mathbb{C}$ open.

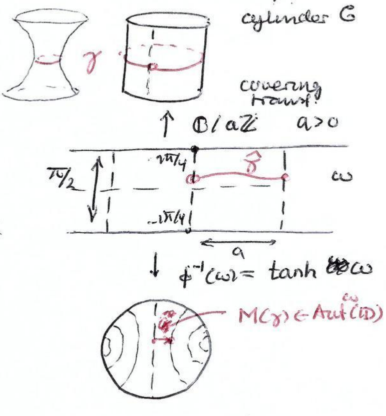
Def 4 $\phi : \mathbb{D} \xrightarrow{C^\omega} \mathcal{U}$ is said to be a covering map if it is surjective and to every $\omega \in \mathcal{U}$, $\exists$ open nghd V of ω such that $\phi^{-1}(V)$ is a union of disjoint open sets $\{\tilde{V}_\alpha\}_{\alpha \in A}$ such that $\phi : \tilde{V}_\alpha \xrightarrow{C^\omega} V$ is bi-holomorphic

in fact this is a universal cov map

Thm 4 A Riemann covering map $\phi : \mathbb{D} \xrightarrow{C^\omega} \mathcal{U}$ defines a unique (up to const) hyperbolic metric on $\mathcal{U}$.

Exceptions $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$-lattice do not admit a hyperbolic covering map. $\mathbb{C}/\mathbb{Z}$, $(\mathbb{Z} + \tau\mathbb{Z})$

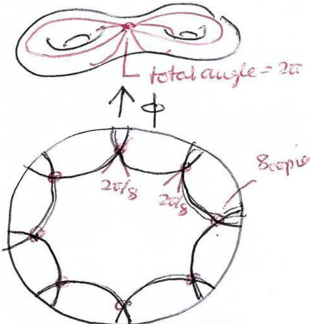
Example Riemann surface



$$\pi^1(\mathcal{C}) \longrightarrow \underbrace{\mathrm{Aut}^\omega(\mathbb{D})}_{\text{deck-transformations}}$$

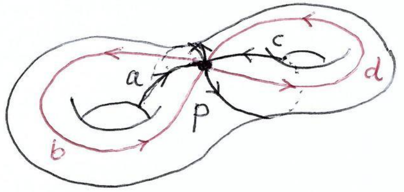
Poincaré–Koebe & uniformisation

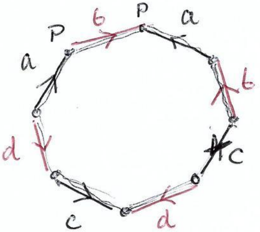
Example: Double-doughnut Riemann surface



Thm 5 Every Riemann surface is conformally equivalent to either one of the exceptions or $\Gamma \backslash \mathbb{D}$ where Γ is a Fuchsian group. a discrete subgr of $\mathrm{Aut}^\omega(\mathbb{D})$ without torsion ($g^n = id$, $n \geq 2 \Rightarrow g = id$)

The genus 2 Riemann surface, S



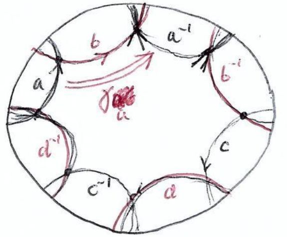


$\pi_1(S,p)$ generated by 4 closed orbits

a, b, c, d

$\psi \uparrow$ (universal) covering map

C^ω provided sum of angles $= 2\pi$ so e.g symmetric case each angle $= \pi/4$

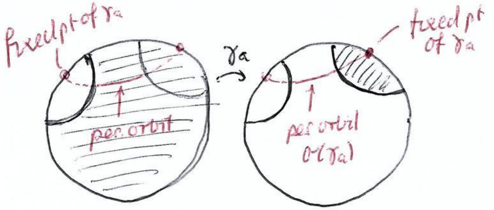


deck-transformations

$\gamma_a, \gamma_b, \gamma_c, \gamma_d \in Aut^\omega(\mathbb{D})$

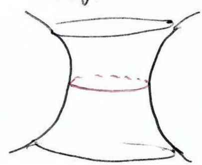
$\Gamma = \langle \gamma_a, \gamma_a^{-1}, \gamma_b, \gamma_b^{-1}, \gamma_c, \gamma_c^{-1}, \gamma_d, \gamma_d^{-1} \rangle$

$\gamma_d^{-1}\gamma_c^{-1}\gamma_d\gamma_c\gamma_b^{-1}\gamma_a^{-1}\gamma_b\gamma_a = Id.$



For S equipped with ds, the Liouville measure $d\mu$ is invariant under the geodesic flow. We may normalise $d\mu$ to be a probability measure. (assumed from now on)

Rem : To each $\gamma \in \Gamma$, $\gamma \neq id$ corresponds a unique periodic orbit $O(\gamma)$ and a unique element $\alpha \in \pi^1(S)$ $\alpha(\gamma)$



One has

$$|\mathrm{tr}\,\gamma| = 2\cosh\frac{\mathrm{len}(O(\gamma))}{2}$$

Thm [Birkhoff for flows]

Let $\phi : \mathbb{R}\times X \to X$ be a measurable measure preserving flow on the probability space $(X, \mathcal{B}, \mu)$

Writing $\phi_t(x) := \phi(t,x)$ μ-preserving means: $\forall f \in L^1(X);\ t \in \mathbb{R}$

$$\int f\circ\phi_t(x)\, d\mu(x) = \int f(y)\, d\mu(y)$$

Then for any $f \in L^1(X)$ the following limits exist for μ-a.e. x:

$$f^+(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \phi_t(x)\,dt$$

$$f^-(x) = \lim_{T\to+\infty} \frac{1}{T}\int_{-T}^0 \phi_t(x)\,dt$$

Furthermore, for μ-a.e x:

$$f^+(x) = f^-(x).$$

Remark: We do not need μ ergodic in order to reach this conclusion: that $f^+ = f^-$ a.e.

proof: We admit the first part. It relies on using the discrete Birkhoff thm on the time one map $\phi_{t=1}$ and the partially integrated functions $x \mapsto \int_0^1 \phi_t(x)\,dt$.

Now the functions f^+, f^- are ϕ_t invariant (obvious) and verifies (part of the proof of Birkhoff) that if E is measurable and ϕ_t invariant then $\int_E f^+ = \int_E f = \int_E f^-$.

So take $\alpha < \beta$ and consider

$$E = E_{\alpha,\beta} = \{x : f^-(x) \le \alpha < \beta \le f^+(x)\}$$

This is invariant and thus

$$\int_E f = \int_E f^- \le \alpha\mu(E) \le \beta\mu(E) \le \int_E f^+ = \int_E f$$

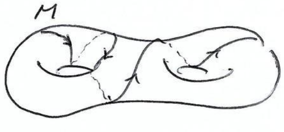
which implies $\mu(E) = 0$.

Taking eg $\alpha, \beta \in \mathbb{Q}$ and then reversing roles we find $f^+ = f^-$ a.e.

Hopf's theorem

Thm Let (M,g) be a hyperbolic Riemann surface which is compact and of curvature equal to -1 everywhere (and has no boundary).

Then the geodesic flow on M is ergodic with respect to the Liouville measure.



Proof: Let μ = Liouville meas.

It suffice to show that for a continuous fct f on M

$f^+(x)$ and $f^-(x)$

are constant a.e.

First note that f^+ and f^- are both Φ_t invariant.

Also by uniform continuity of f we have that for $x \in M$, $y \in W^s(x)$: $d(\varphi^t x, \varphi^t y) \to 0$

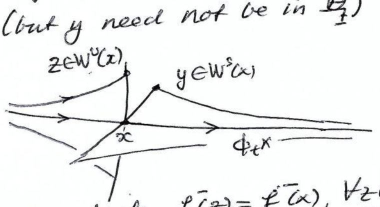
$\Rightarrow d(f(\varphi^t x), f(\varphi^t y)) \to 0$.

Let $E \subset M$ be a measurable set of full measure s.t. f_+ and f_- exists on B_I and are equal there.

By the above $\forall y \in W^s(x)$, $x \in E$

$f^+(y) = f^+(x)$

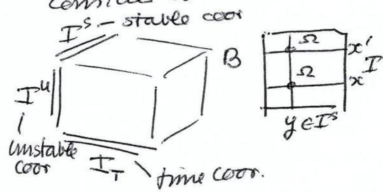
(but y need not be in B_I)



Similarly $f^-(z) = f^-(x)$, $\forall z \in W^u(x)$

(but z need not be in B_I)

Going to "ideal" coordinates we consider a "box"



The set B_I has full measure in B and has the form

$B_I = I_T \times \Omega$,

with $\Omega \subset I^u \times I^s$ of full measure.

By Fubini $\exists A^u \subset I^u$ s.t. $\forall x \in A^u$ the set

$I_x^s = \{ y \in I^s : (x,y) \in \Omega \}$

has full measure in I_s

For $x, x' \in A^u$, $I_x^s \cap I_{x'}^s$ then also has full measure, in particular there is an element $y \in I_x^s \cap I_{x'}^s$

since $(x,y), (x',y) \in \Omega$:

$f^-(x,y) = f^+(x,y) = f^+(x',y) = f^-(x',y)$

Thus f^- has the same const value on I_s^x $\forall x \in A^u$ whence on $I_T \times \{(x,y) : x \in A^u, y \in I_x^s\}$ which has full measure in B

//

Additional remarks...

(M, μ) probability space

g^t a flow on M preserving μ.

$g^t_* \mu = \mu, \quad t \in \mathbb{R}$.

Birkhoff: Let $f \in L^1(M, \mu)$ and define

$\Omega_+ = \Omega_+(f) = \{x \in M : \lim_{T \to +\infty} \frac{1}{T} \int_0^T f \circ g^t(x) dt =: f_+^*(x) \text{ exists}\}$

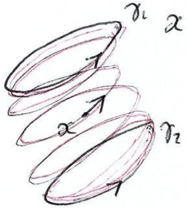
$\Omega_- = \Omega_-(f) \quad \{x \in M : \lim_{T \to +\infty} \frac{1}{T} \int_0^T f \circ g^{-t}(x) dt =: f_-^*(x) \text{ exists}\}$

$\Omega_\pm(f) = \{x \in \Omega_+ \cap \Omega_- : f_+^*(x) = f_-^*(x)\}$

Then $\Omega_+(f), \Omega_-(f)$ and $\Omega_\pm(f)$ are all measurable subsets of M and have full μ measure

One may also describe $x \in \Omega_\pm$ as the points for which the following double limit exists:

$$\boxed{f_\pm^*(x) = \lim_{T_1, T_2 \to +\infty} \frac{1}{T_1 + T_2} \int_{-T_1}^{T_2} f \circ g^t(x) dt}$$



x example of point $\in (\Omega_+ \cap \Omega_-) \setminus \Omega_\pm$

the forward orbit of x accumulates upon γ_1 (per. orbit)

the backward orbit of x accum. upon γ_2 (per. orbit) distinct from γ_1