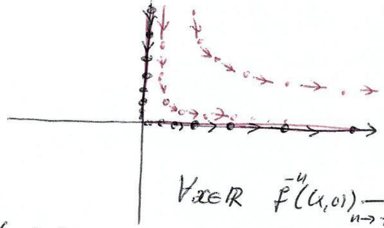
10. Hyperbolic Dynamics

Simplest example (linear)

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$|\lambda_1| > 1 > |\lambda_2| \quad (>0)$



$\forall x \in \mathbb{R} \quad f^{-n}((x,0)) \underset{n\to+\infty}{\longrightarrow} (0,0)$

$(0,0)$ is a "hyperbolic" fixed point.

$\forall y \in \mathbb{R} \quad f^{n}((0,y)) \underset{n\to+\infty}{\longrightarrow} (0,0)$

$W^u((0,0)) = \mathbb{R}\times\{0\}$ unstable manifold (submfld dim 1)

$W^s((0,0)) = \{0\}\times\mathbb{R}$ stable manifold (submfld dim 1)

More generally with $f(x) = Ax$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

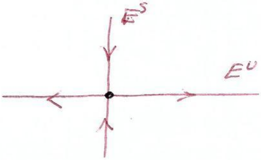
$\sigma(A) \cap S^1 = \emptyset$

$\sigma^u(A) = \sigma(A) \cap \{|\lambda| > 1\}$ unstable eigenvalues

$\sigma^s(A) = \sigma(A) \cap \{|\lambda| < 1\}$ stable eigenvalues

E^u and E^s associated eigenspaces.

$W^u(0) = E^u, \quad W^s(0) = E^s.$

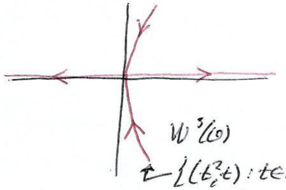


Non-linear example

$H(x,y) = (x+y^2, y)$, $H^{-1}(x,y) = (x-y^2, y)$

$f(x,y) = (2x, \frac{y}{2})$

$\tilde f(x,y) = H\circ f\circ H^{-1}(x,y) = (2x - \frac{7}{4}y^2, y/2)$



$\tilde f((t^2,t)) = ((\frac{t}{2})^2, (\frac{t}{2}))$

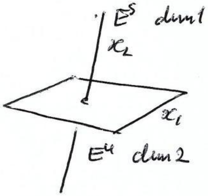
$\tilde f: W^s(0) \longrightarrow W^s(0)$ contracting map.

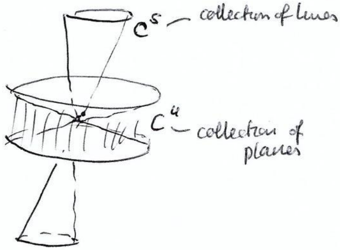
Def 1 Let $\mathcal{U} \subset \mathbb{R}^2$ be open and $f \in C^1(\mathcal{U}; \mathbb{R}^2)$.
A fixed point $p = f(p) \in \mathcal{U}$ is called hyperbolic iff

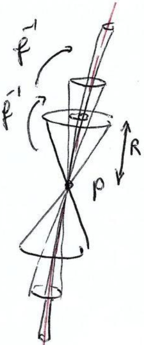
$$\sigma(f_*(p)) \cap S^1 = \emptyset$$

Intuitive construction of W^u_{loc}, W^s_{loc}.

Stable unstable cones







Stable cone (dim 1)

$$\mathcal{C}^s_R(p) = \{ (x_1, x_2) \in E^u \times E^s : |x_1| \le \sigma |x_2|,\ |x_2| \le R \}$$

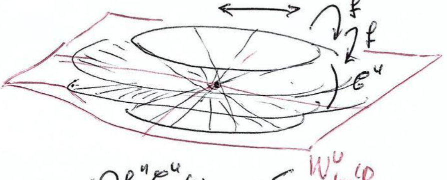
(p = origin here)

(Locally)

$W^s_{loc}(p)$ is the graph of a map $B_{E^s}(0,R) \to E^u$

$W^s_{loc}(p) \xrightarrow[\text{along } E^u]{\text{proj}} E^s$ locally bijective

$$W^s_{loc}(p) = \bigcap_{n \ge 0} f^{-n} \mathcal{C}^s_R(p)$$



$W^u_{loc}(p)$ is the graph of a map $B_{E^u}(0,R) \to E^s$

$W^u_{loc}(p) \xrightarrow[\text{along } E^s]{\text{proj}} E^u$ locally bijective

$$\bigcap_{n \ge 0} f^n \mathcal{C}^u_R(p) =: W^u_{loc}(p)$$

$$\mathcal{C}^u_R(p) = \{ (x_1, x_2) \in E^u \times E^s : |x_2| \le \sigma |x_1|,\ |x_1| \le R \}$$

General setup

Banach spaces E^u and E^s

Let $E = E^u \times E^s$ with norm

$$\| (x_1, x_2) \|_E = \max \{ |x_1|_{E^u}, |x_2|_{E^s} \}$$

Let $R > 0$ and $D_R^u = \overline{B_{E^u}(0,R)}$, $D_R^s = \overline{B_{E^s}(0,R)}$

We will consider a family of "hyperbolic" maps:

Def 1 Let $0 < \theta < 1$, $0 < \alpha < \frac{1-\theta}{4}$ and let $\mathcal{F}$ denote the class of maps $f : D_R = D_R^u \times D_R^s \to E$ of the form:

$$f(x_1, x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} \Lambda_1 x_1 + \mathcal{H}_1(x_1,x_2) \\ \Lambda_2 x_2 + \mathcal{H}_2(x_1,x_2) \end{pmatrix}$$

where $\Lambda_1 \in GL(E^u)$, $\|\Lambda_1^{-1}\|_{L(E^u)} \le \theta$

$\Lambda_2 \in L(E^s)$, $\|\Lambda_2\|_{L(E^s)} \le \theta$

$$\sup_D \|\mathcal{H}_i\| \le \alpha \cdot R, \quad \sup_D \|\mathcal{H}_i'\| \le \alpha$$

$$i = 1,2$$

Here, eg. $\|\mathcal{H}_1'\| = \sup \{ |\partial_{x_1}\mathcal{H}_1 \cdot h_1 + \partial_{x_2}\mathcal{H}_1 \cdot h_2|_{E^u} ;$
$h = (h_1, h_2) \in D_1 \}$

so $|\mathcal{H}_1' \cdot h| \le \alpha \cdot \max(|h_1|, |h_2|)$.

We consider metric spaces consisting of (graphs of) Lipschitz maps:

Def 2

$$K^u = \{ \varphi_u : D^u \to D^s \mid Lip(\varphi) \le 1 \}$$

$$K^s = \{ \varphi_s : D^s \to D^u \mid Lip(\varphi) \le 1 \}$$

Both are complete metric spaces

We write

$$\mathcal{G}^u(\varphi_u) = \{ (x_1, \varphi_u(x_1)) : x_1 \in D^u \} \subset D$$

for the graph of $\varphi_u \in K^u$. (similarly for K^s)

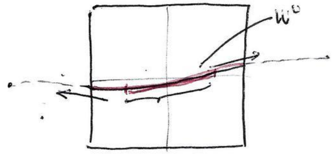
[unstable graph transform]

Thm 4 Let $\phi \in K^u$. To every $y_1 \in D^u$ there is a unique $x_1 = q_1(y_1, \phi)$ s.t.:

$$f_1(x_1, \phi(x_1)) = y_1 \qquad (*)$$

Define $\hat{\phi}(y_1) := f_2(x_1, \phi(x_1)), y_1 \in D^u$. Then $\hat{\phi} \in K^u$ and the map $\phi \overset{\Gamma}{\mapsto} \hat{\phi}$ (called the graph transform) is η- Lipschitz with

$$\eta = \frac{\theta(\theta+\alpha)}{1-\theta\alpha} < 1$$



Proof: Let $\varphi, \tilde{\varphi} \in K^u$, $x_1, \tilde{x}_1 \in D^u$. We have

$$|\varphi(x_1) - \tilde{\varphi}(\tilde{x}_1)| \le |\varphi(x_1) - \varphi(\tilde{x}_1)| + |\varphi(\tilde{x}_1) - \tilde{\varphi}(\tilde{x}_1)| \le |x_1 - \tilde{x}_1| + d_u(\varphi, \tilde{\varphi}) =: \Delta x_1 + \Delta\varphi$$

By the MVT

$$|R_i(x, \varphi(x)) - R_i(\tilde{x}, \tilde{\varphi}(\tilde{x}))| \le \alpha\,(|x - \tilde{x}| \vee |\varphi(x) - \tilde{\varphi}(\tilde{x})|) \le \alpha\,(|x - \tilde{x}| + d_u(\varphi, \tilde{\varphi})) \qquad (**)$$

In order to solve $(*)$ for x_1 we write

$$\Lambda_1 x_1 + R_1(x_1, \phi(x_1)) = y_1 \quad \text{or}$$

$$x_1 = \Lambda_1^{-1}(y_1 - R_1(x_1, \phi(x_1)))$$

so x_1 should be a fixed pt of

$$G(x) = \Lambda_1^{-1}(y_1 - R_1(x, \phi(x)))$$

As $\|\Lambda_1^{-1}\| \le \theta$ we get by the estimate on R_1: $|G(x) - G(x')| \le \theta\alpha|x - x'|$ (with $\theta\alpha < 1$). Also $|G(x)| \le \theta(1+\alpha)R < R$, so G is a strict Lipschitz contraction of D^u, whence has a unique fixed point

$$x_1 = q_1(y_1, \phi)$$

We set $\Gamma(\phi)(x_1) := f_2(x_1, \phi(x_1)) =: y_2$

Since $|f_2(x_1, \varphi(x_1))| \le (\theta + \alpha)R < R \Rightarrow y_2 \in D^s$

Now, take also $\tilde{\varphi} \in K^u$, $\tilde{y}_1 \in D^u$ and construct $\tilde{x}_1$ and $\tilde{y}_2$ as above.

Abbreviate:

$$\Delta x_1 = |x_1 - \tilde{x}_1|, \quad \Delta y_1 = |y_1 - \tilde{y}_1|, \quad \Delta\varphi = d_u(\varphi, \tilde{\varphi})$$

$$\Delta y_2 = |y_2 - \tilde{y}_2|$$

By the fixed pt equation

$$\Delta x_1 = |G(x_1, y_1, \varphi) - G(\tilde{x}_1, \tilde{y}_1, \tilde{\varphi})| \le \theta\Delta y_1 + \theta\alpha(\Delta x_1 + \Delta\varphi) \Rightarrow$$

$$\Delta x_1 \le \frac{\theta}{1-\theta\alpha}(\Delta y_1 + \alpha\Delta\varphi).$$

Using $(**)$ on the stable coordinate:

$$\Delta y_2 = |f_2(x_1, \varphi(x_1)) - f_2(\tilde{x}_1, \tilde{\varphi}(\tilde{x}_1))|$$

$$\le \theta\,|\varphi(x_1) - \tilde{\varphi}(\tilde{x}_1)| + \alpha \cdot (\Delta x_1 + \Delta\varphi)$$

$$\le (\theta + \alpha)(\Delta x_1 + \Delta\varphi)$$

$$\le (\theta+\alpha)\left(\frac{\theta}{1-\theta\alpha}(\Delta y_1 + \alpha\Delta\varphi) + \Delta\varphi\right)$$

$$= \frac{\theta(\theta+\alpha)}{1-\theta\alpha}\Delta y_1 + \frac{\theta+\alpha}{1-\theta\alpha}\Delta\varphi$$

From this we may read two properties. For $y_1 = \tilde{y}_1$:

$$\Delta y_2 \le \frac{\theta+\alpha}{1-\theta\alpha}\Delta\varphi \Rightarrow$$

$$d_u(\Gamma(\varphi), \Gamma(\tilde{\varphi})) \le \frac{\theta+\alpha}{1-\theta\alpha} d_u(\varphi, \tilde{\varphi})$$

and for $\varphi = \tilde{\varphi}$:

$$\Delta y_2 \le \frac{\theta(\theta+\alpha)}{1-\theta\alpha}\Delta y_1 \Rightarrow$$

$$\mathrm{Lip}(\Gamma(\varphi)) \le \frac{\theta(\theta+\alpha)}{1-\theta\alpha} < 1$$

Thus $\Gamma(\varphi) \in K^u$ and

$$\varphi \longmapsto \Gamma(\varphi)$$

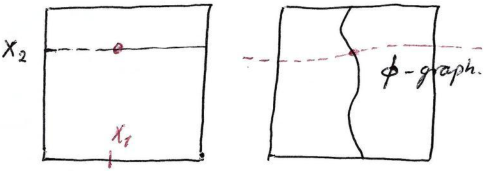
is $\eta = \frac{\theta+\alpha}{1-\theta\alpha} < 1$ Lipschitz //

Thm 5 [stable graph transform]
Let $\phi \in K^s$. Then to every $x_2 \in D^s$ there are unique points $x_1 \in D^u$, $y_2 \in D^s$ such that

$$y_2 = f_2(x_1, x_2)$$
$$\phi(y_2) = f_1(x_1, x_2)$$

We define $\Gamma\phi(x_2) := x_1$.
Then $\Gamma: K^s \to K^s$ is η-Lipschitz with

$$\eta = \frac{\theta}{1-2\alpha\theta} < 1$$



proof: We should solve:

$$f_1(x_1, x_2) = \phi(f_2(x_1, x_2)) \text{ or}$$

$$\Lambda_1 x_1 + \mathcal{H}_1(x) = \phi(\Lambda_2 x_2 + \mathcal{H}_2(x)) \Leftrightarrow$$

$$x_1 = \Lambda_1^{-1}(\phi(\Lambda_2 x_2 + \mathcal{H}_2) - \mathcal{H}_1)$$

Define (with $\mathcal{H}_i = \mathcal{H}_i(x_1, x_2)$)

$$G(x_1, x_2, \phi) := \Lambda_1^{-1}(\phi(\Lambda_2 x_2 + \mathcal{H}_2) - \mathcal{H}_1)$$

$$|G(x_1, x_2, \phi)| \le \theta(R + \alpha R) = \theta(1+\alpha)R < R.$$

Also varying x_1, x_2, ϕ:

$$\Delta G \le \theta\, 2\alpha\, \Delta x_1 + (\theta^2 + 2\alpha\theta)\Delta x_2 + \theta \Delta\phi$$

Being a contraction wrt x_1 $\exists!$ fixed point $x_1 = G(x_1, x_2, \phi)$. Then

$$(**) \qquad \Delta x_1 \le \frac{\theta(\theta+2\alpha)}{1-2\alpha\theta}\Delta x_2 + \frac{\theta}{1-2\alpha\theta}\Delta\phi$$

For the constants in (**)

$$\frac{\theta(\theta+2\alpha)}{1-2\alpha\theta} < 1 \text{ since}$$

$$\Leftrightarrow \theta^2 + 2\alpha\theta < 1 - 2\alpha\theta \Rightarrow$$

$$4\alpha\theta < 4\alpha < 1-\theta < 1-\theta^2$$

Also $\frac{\theta}{1-2\alpha\theta} < 1$ since

$$2\alpha\theta < 2\alpha < 1-\theta$$

Defining $\Gamma\phi(x_2) := x_1$ we obtain

$$\mathrm{Lip}(x_2 \to \Gamma\phi(x_2)) \le \frac{\theta(\theta+2\alpha)}{1-2\alpha\theta} < 1$$

$$|\Gamma\phi(x_2)| \le R$$

so that $\Gamma \in K^s$ as well as

$$d_s(\Gamma\phi, \Gamma\tilde\phi) \le \frac{\theta}{1-2\alpha\theta} d_u(\phi, \tilde\phi) =: \eta\, d_u(\phi, \tilde\phi)$$

with $\eta < 1$.

Thm 6 Let $f \in \mathcal{F}$, $f \in C^r$, $r \geq 1$. Then f has a unique fixed pt $p = f(p) \in D$. The fixed pt is hyperbolic and if $\varphi_s \in \mathcal{K}^s$, $\varphi_u \in \mathcal{K}^u$ are the fixed pts of the s/u graph transforms then

$$W^s_{loc}(p) = \mathcal{G}^s(\varphi_s)$$

$$W^u_{loc}(p) = \mathcal{G}^u(\varphi_u)$$

Both are C^r-submanifolds

Proof: $p = f(p) \in D$ is the unique fixed pt of

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \Lambda_1^{-1}(x_1 - \mathcal{X}_1(x)) \\ \Lambda_2 x_2 + \mathcal{X}_2(x) \end{pmatrix}$$

(check that this is a $(\theta + 2\alpha) < 1$ Lipschitz contraction of D)

One has $|p_1| \leq \frac{\alpha\theta}{1-\theta} R$ $\quad |p_2| \leq \frac{\alpha}{1-\theta} R$

Let $\varphi_u = \Gamma_u(\varphi_u) \in \mathcal{K}^u$ be the unique fixed point for the unstable graph transf.
For $y_1 \in D^u$ there is $x_1 = g_1(y_1, \varphi_u)$ such that

$$f(x_1, \varphi_u(x_1)) = (y_1, \varphi_u(y_1)) \quad \leftarrow \text{since} = \Gamma_u(\varphi_u)$$

Thus $f(\mathcal{G}^u(\varphi_u)) \supset \mathcal{G}^u(\varphi_u)$.
As $\Delta x_1 \leq \underbrace{\frac{\theta}{1-\theta\alpha}}_{<1} \Delta y_1$ (see unstable gr. transf.)

we see that $\bigcap_{k \geq 0} f^{-k} \mathcal{G}^u(\varphi_u) = \{p\}$ which is f invariant whence equals the fixed point from above

Thus $\mathcal{G}^u(\varphi_u) \subset W^u_{loc}(p)$

To get equality take some $y = (y_1, y_2) \in D$ and assume that ~~there~~ there is $z = (z_1, z_2) \in D$ s.t. $f^n(z_1, z_2) = (y_1, y_2)$ and $f^j(z) \in D$ $\forall\, 0 \leq j \leq n$. Let $\tilde{\varphi}(x_1) := z_2$
Then by the unst. graph transform contraction

$$|y_2 - \varphi_u(y_1)| \leq d_u(\Gamma_u^n \tilde{\varphi}, \Gamma_u^n \varphi_u) \leq \eta^n \cdot 2R$$

If true $\forall n \geq 0$ then $y_2 = \varphi_u(y_1)$ so $\mathcal{G}^u(\varphi_u) = W^u_{loc}(p)$.

Let $\varphi_s = \Gamma_s(\varphi_s) \in \mathcal{K}^s$ be the unique fixed pt for the stable graph transform
Then for every $x_2 \in D^s$ there is $y_2 \in D^s$, $x_1 = \varphi_s(x_2)$:

$$f(\varphi_s(x_2), x_2) = (\varphi_s(y_2), y_2)$$

Thus $f\, \mathcal{G}^s(\varphi_s) \subset \mathcal{G}^s(\varphi_s)$
One also finds $\Delta y_2 \leq \underbrace{(\theta + \alpha)}_{<1} \Delta x_2$ so a strict contraction

$$\bigcap f^n \mathcal{G}^s(\varphi_s) = \{p\}$$

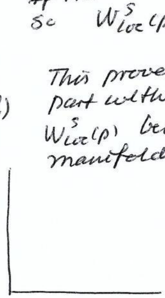
(again the fixed pt $p = f(p)$)

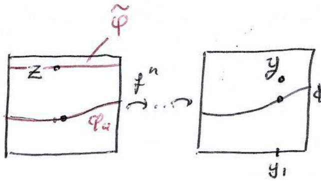
If $x = (x_1, x_2) \in D$ and $f^j x \in D$ for $0 \leq j \leq n$, $f^n x = (z_1, z_2)$ then take $\tilde{\varphi}(x_2) := z_1$ and use the stable graph tr:

$$|x_1 - \varphi_s(x_2)| \leq d_s(\Gamma_s^n \tilde{\varphi}_s, \Gamma_s^n \varphi_s) \leq 2R \cdot \eta^n$$

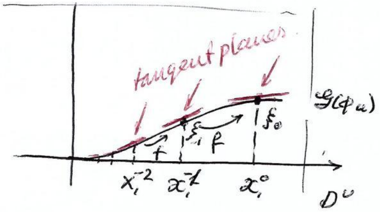
If true $\forall n \geq 0$: $x_1 = \varphi_s(x_2)$ so $W^s_{loc}(p) = \mathcal{G}^s(\varphi_s)$

This proves the first part with $W^u_{loc}(p)$ and $W^s_{loc}(p)$ being Lipschitz manifolds.





Proof (sketch) of W^s_{loc}, W^u_{loc} being C^r-manifolds:



To show differentiability of φ_u at $x_1^0 \in D^u$, consider the sequence of pre-images, i.e.

$$f(x^{-n-1}, \varphi_u(x^{-n-1})) = (x_1^{-n}, \varphi_u(x^{-n})),\ n \geq 0$$

$$f(\xi_{-n-1}) = \xi_{-n}$$

One shows that (projectively)

$$f_{*\xi_{0-1}} \circ \dots \circ f_{*\xi_{0-n}}(\mathcal{H}_{\xi_{-n}}) \to \mathcal{H}^*_{\xi_0}$$

with e.g. $\mathcal{H}_{\xi_{-n}} = E^u \times \{0\}$ being the horizontal plane at $T_{\xi_{-n}}D$ converges as $n \to +\infty$. and depends continuously upon x_0. The limit is the graph of a linear map.

$$\mathcal{H}^*_{\xi_0} = \{(y_1, A_{\xi_0} y_1) : y_1 \in E^u\}$$

Consider the "non-linearity"

$$N_{\varphi_u}(A, x, r) := \sup_{0<|h|<r} \frac{1}{|h|} |\Delta_h \varphi(x) - A_x h|$$

with $\Delta_h \varphi(x) = \varphi(x+h) - \varphi(x)$
$(x, x+h) \in D^u$

One uses the dynamics to establish $(\tilde{\theta}<1)$

$$N_\varphi(A, x_0, r) \leq \tilde{\theta} N_\varphi(A, x_{-1}, \tilde{\theta} r) + \varepsilon_{x_{-1}}(\theta r)$$

with $|\varepsilon| \leq 2$ uniformly

$$\Rightarrow N_\varphi(A, x_0, r) \leq \varepsilon_{x_{-1}}(\theta r) + \tilde{\theta}\varepsilon_{x_{-2}}(\tilde{\theta}^2 r) + \theta^2 \varepsilon_{x_{-3}}(\theta^3 r) + \dots$$

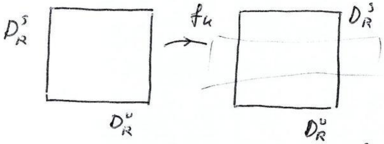
$$\Rightarrow N_\varphi(A, x_0, r) = \varepsilon(r)$$

$\Rightarrow \varphi$ is diff at x_0 with $\varphi'(x_0) = A_{x_0}$.

For higher order derivatives one iterates so called k-jets (k'th order Taylor expansions) and estimates the remainder as above. //

A sequence of hyperbolic maps

$(f_k)_{k\in\mathbb{Z}}$, $f_k \in \mathcal{F}$



$D_k = D_R^u \times D_R^s$ $\quad$ $D_{k+1} = D_R^u \times D_R^s$

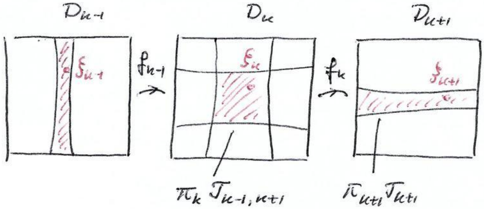
$J_{k,n}$: set of (k,n) trajectories:

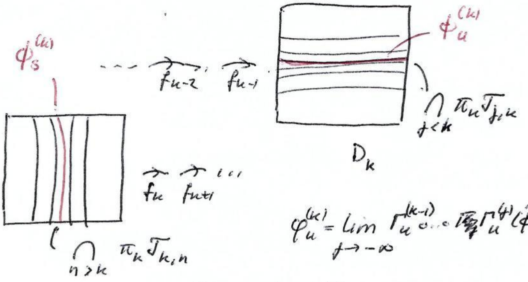
$\underline{\xi} = (\xi_k, \dots, \xi_n)$ each $\xi_j \in D_j$

$f_k \xi_k = \xi_{k+1}, \dots, f_{n-1}\xi_{n-1} = \xi_n$.

$\pi_j : J_{k,n} \to D_n$ projection onto j^{th} ~~last~~ point in seq.

~~π_k : $J_{k,n} \to D_k$ onto first.~~



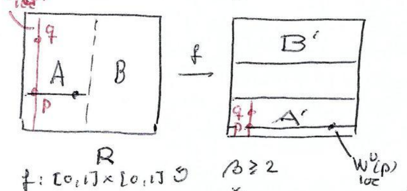


$$\varphi_u^{(k)} = \lim_{j\to-\infty} \Gamma_u^{(k-1)} \circ \dots \circ \Gamma_u^{(j)}(\tilde\varphi)$$

$$\phi_s^{(k)} = \lim_{n\to+\infty} \Gamma_s^{(k)} \circ \Gamma_s^{(k+1)} \circ \dots \Gamma_s^{(n)}(\tilde\varphi)$$

e.g. vertical diam of $\pi_k J_{j,k} \le 2R\cdot\eta^{k-j}$
by graph transform contraction

Ex (cont.)



$f : [0,1]\times[0,1] \circlearrowleft$ $\beta \geq 2$

$$f((x_1,x_2)) = \begin{cases} (2x_1, \frac{x_2}{\beta}), & 0 \leq x_1 \leq 1/2 \leftarrow A \\ (2x_1 - 1, \frac{x_2 - 1 + \beta}{\beta}), & \frac{1}{2} < x_1 \leq 1 \leftarrow B \end{cases}$$

$\Omega = \bigcap f^n R = [0,1] \times$ Cantor set.

By tracking the symbols of an orbit
$$s_k = \begin{cases} \text{"A"} & \text{if } f^k(x) \in A \\ \text{"B"} & \text{if } f^k(x) \in B \end{cases} \quad k \in \mathbb{Z}$$

we get an injective map

$$x \in R \overset{I}{\mapsto} \underline{s}(x) = (s_k)_{k\in\mathbb{Z}} \in \Sigma_2$$

One has a semi-conjugacy

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\sigma \text{ shift}} & \Sigma_2 \\ I \uparrow\downarrow & & \downarrow\uparrow I \\ \Omega & \xrightarrow{f} & \Omega \end{array}$$

$$W^u(p) = \{ y : d(f^k p, f^k y) \xrightarrow[k\to-\infty]{} 0 \}$$
$$W^s(p) = \{ y : d(f^k p, f^k y) \xrightarrow[k\to+\infty]{} 0 \}$$

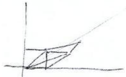
unstable / stable sets
here collection of horizontal / vertical lines

Ex

$f : \mathbb{T}^2 \longrightarrow \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

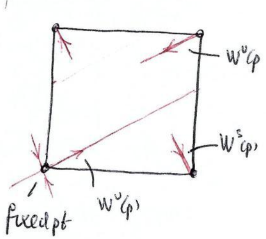
$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mod 1$

$\lambda^u = \frac{1+\sqrt{5}}{2}$ $\lambda^s = \frac{1-\sqrt{5}}{2} = -\frac{1}{\gamma}$

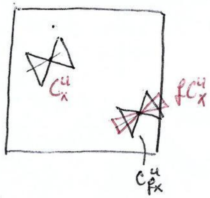


$\lambda^u = \gamma$ $\lambda^s = -\frac{1}{\gamma}$

$e^u = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}$ $e^s = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}$

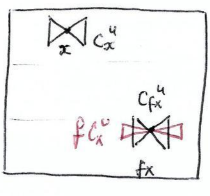


unstable cones $\rightsquigarrow$ local unstable manifolds

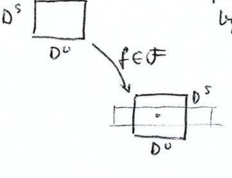


Similar for stable cones

Unstable cones



In local coors



A finite number of charts suffices to describe the map by elements in $\mathcal{F}$ (elementary hyperbolic maps)

Def Let M be a compact connected Riemannian manifold. Let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$.
A hyperbolic ~~structure~~ continuous of (M,f) is a splitting of the tangent bundle into subbundles:

$$T_xM = E^u_x \oplus E^s_x$$

$\dim d_u$ $\quad$ $\dim d_s$

such that there are constants $C < \infty$, $\theta \in (0,1)$: $\forall n \geq 1$

$$\| f^n_* |_{E^s} \| \leq C\theta^n$$

$$\| f^{-n}_* |_{E^u} \| \leq C\theta^n$$

Rem One may change norms (Riem. metric)

1) so that $C=1$.

2) Each M admits a finite cover ~~of~~ with rectangles (R_i) and transition maps $f_{ij} \in \mathcal{F}(R_i, R_j)$

$\Downarrow$

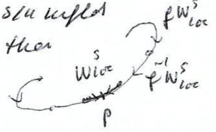
~~(M,f) admits~~ s

$\forall p \in M$, there are stable unstable manifolds W^s, W^u associated.

$\forall x \in W^s(p)$: $d(f^n x, f^n p) \xrightarrow[n \to +\infty]{} 0$

$\forall x \in W^u(p)$: $d(f^n x, f^n p) \xrightarrow[n \to -\infty]{} 0$

First one finds local s/u mfld $W^s_{loc}(p)$, $W^u_{loc}(p)$ and then



$$W^s(p) := \bigcup_{n \geq 0} f^{-n} W^s_{loc}(p)$$

$$W^u(p) := \bigcup_{n \geq 0} f^{n} W^u_{loc}(p)$$