9. Complex dynamics and dimensions

Def 1 Let $P \in \mathbb{C}[z]$, $d^\circ P \geq 2$.
The filled-in Julia set:
$$K_P = K(P) = \{z \in \mathbb{C} : P^n(z) \not\to \infty\}$$
The Julia set
$$J_P = J(P) = \partial K_P$$

<u>Rem</u> $K_P^{\circ\, c}$ is open. K_P is bd. So K_P and J_P are compact subsets of $\mathbb{C}$.

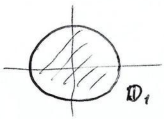
Lemma 2 J_P and K_P are completely P-invariant
$$J_P = P(J_P) = P^{-1}(J_P)$$
$$K_P = P(K_P) = P^{-1}(K_P)$$

Proof: Obvious for K_P. But also for $\mathring{K}_P$ because P being holomorphic is an open mapping. Then also ok for J_P.

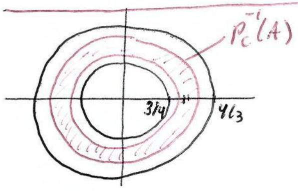
[Carleson - Gamelin: Complex Dynamics].

In the following we restrict our attention to the family of quadratic polynomials
$$P_c(z) = z^2 + c \quad \text{with } |c| < \tfrac{1}{6}$$

Trivial case: $c = 0$ $\quad |P_c(z)| = |z|^2$
$|P_c^n(z)| = |z|^{2^n}$ so $K(P_c) = \overline{\mathbb{D}_1}$ and $J(P_c) = S^1$. $\dim_H J(P_c) = 1$

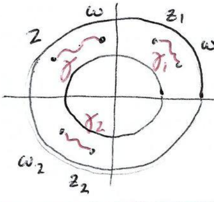


Lemma 3: Let $A = A_{3/4} = \{z : \frac{3}{4} < |z| < \frac{4}{3}\}$.
Then for $|c| < 1/6$:
$$P_c^{-1}(A) \subset\subset A \quad \text{and } J(P_c) \subset A$$



Proof: $|z| \leq \frac{3}{4} \Rightarrow |z|^2 + |c| \leq \frac{9}{16} + \frac{1}{6} < \frac{3}{4}$
$\Rightarrow z \in \mathring{K}(P_c)$
$|z| \geq \frac{4}{3} \Rightarrow |z|^2 - |c| \geq \frac{16}{9} - \frac{1}{6} > \frac{4}{3}$
$\ldots \Rightarrow z \in K(P_c)^c$

Def 4: Let $\Gamma_A(z, w)$, $z, w \in \bar{A}$ denote the set of rectifiable (or C^1) paths in $\bar{A}$ joining z and w. Set
$$d_A(z, w) = \inf\{\operatorname{len} \gamma : \gamma \in \Gamma_A(z, w)\}$$



<u>Lemma 5</u> For every $z, w \in A$
The pre-images ~~z,w and~~ may be paired z_1, w_1 and z_2, w_2 such that for $j = 1, 2$
$$d_A(z_j, w_j) \leq \frac{1}{\beta}\, d_A(z, w)$$
with $\beta = 3/2 > 1$

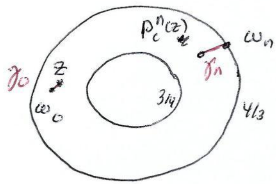
Proof For $z \in A$: $|P_c'(z)| = |2z| \geq \frac{3}{2} = \beta$
Now use local inverses to lift a path. //

Prop 6 $J(P_c) = \bigcap_{k \geq 0} P_c^{-k}(\bar{A})$

Proof: Let M denote the compact non-empty set on the RHS.

By Lemma 3, $J(P_c) \subset A$ so by invariance, Lemma 2, $J(P_c) \subset M$.

Now let $z \in M$, $n \geq 1$. Then $P_c^n(z) \in M$ as well



Pick $\omega_n \in \partial D_{4/3} \subset (K(P_c))^c$ ~~and $\gamma_n \in \Gamma_A(\omega_n$, ... lifting~~ and $\gamma_n \in \Gamma_A(\omega_n, P_c^n(z))$ with $\mathrm{len}(\gamma_n) \leq 1$.
Lifting by P_c^{-1} along the orbit of z we obtain $\omega_0 \in (K(P_c))^c$ with $d(z, \omega_0) \leq \frac{1}{\beta^n}$.
Letting $n \to \infty$ we conclude $z \in \mathrm{Cl}((K(P_c))^c)$

Taking instead $\omega_n \in \partial D_{3/4} \subset (\mathring{K}(P_c))$ we see that $z \in \mathrm{Cl}(\mathring{K}(P_c)) = K(P_c)$
Then $z \in K(P_c) \cap \overline{K(P_c)^c} = \partial K_{P_c} = J(P_c)$.

Thm 7 ($\Omega = J(P_c)$, $d = d_A$) $\circlearrowleft P_c$ is uniformly expanding and uniformly mixing

Proof Clear from Lemma 5 and Prop 6

Conformality and distortion

$z \mapsto az$ maps circles to circles.
$\Rightarrow$ A holomorphic map takes round balls into almost round balls.
(necessary for estimating $\dim_H$ using dynamical balls)

Lemma 7: [Köebe distortion]
Let $\phi : \mathbb{D}_1 \xrightarrow{C^\omega} \mathbb{C}$ be univalent (= injective), $\phi(0)=0$, $\phi'(0)=1$. Then

$$\forall z \in \mathbb{D}_1 : \frac{|z|}{(1+|z|)^2} \le |\phi(z)| \le \frac{|z|}{(1-|z|)^2}$$

[Carleson-Gamelin: Complex Dynamics].

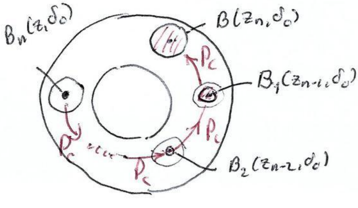
By scaling we also have

Coroll 8 Let $\delta > 0$, $\lambda \in (0,+\infty)$
Let $\phi : \mathbb{D}_\delta \xrightarrow{C^\omega} \mathbb{C}$ be univalent
$\phi(0)=0$, $|\phi'(0)| = \frac{1}{\lambda}$. Then

$$\forall z \in \mathbb{D}_\delta : \frac{\delta}{\lambda} \frac{|z|/\delta}{(1+|z|/\delta)^2} \le |\phi(z)| \le \frac{\delta}{\lambda} \frac{|z|/\delta}{(1-|z|/\delta)^2}$$

Recall also:
$\phi : \mathbb{D}_r \xrightarrow{C^\omega} \mathbb{D}_\delta \Rightarrow |\phi'(0)| \le \frac{\delta}{r}$
$\phi(0)=0$
by Schwarz' Lemma.

Bowen balls: on $J(P_c) \subset A$
Given $z \in J(P_c)$ let $z_n = P_c^n(z)$, $n \ge 0$ be its forward orbit. Let
$0 < \delta_0 \le \operatorname{dist}(J(P_c), A^c)$



As usual we set

$$B_n(z;\delta_0) = \bigcap_{k=0}^{n} P_c^{-k}(B(P_c^k z, \delta_0))$$

we have $\forall n \ge 0$
$B_n(z;\delta_0) \subset B(z, \frac{\delta_0}{\beta^n})$

Def 9 Given $z \in J(P_c)$, $0 < r < \delta_0$ we define

$$\underline{n}(z,r;\delta_0) = \max\{k \ge 0 : B(z,r) \subset B_n(z;\delta_0)\}$$

Clearly $n = \underline{n}(z,r;\delta_0)$ is the unique value such that
$B(z,r) \subset B_n(z,\delta_0)$ but
$B(z,r) \not\subset B_{n+1}(z,\delta_0)$

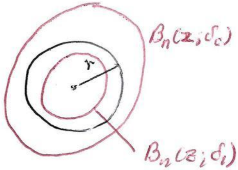
Key geometric estimate

Lemma 10. Let $z\in J(P_c)$, $0<r<\delta_0$.
Set $n=\underline{n}(z,r;\delta_0)$ and $\lambda_n(z)=|(P_c^n)'(z)|$.
We have $1<\beta\le|P_c'|\le\alpha<+\infty$.

setting $\delta_1=\frac{2}{3}\delta_0\frac{\alpha}{(1+\alpha)^2}$ we have

$$B_n(z;\delta_1)\subset B(z,r)\subset B_n(z;\delta_0)$$

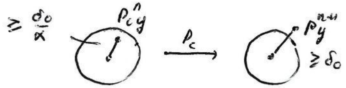
$$\delta_1/\lambda_n(z)\le r\le\delta_0/\lambda_n(z)$$



Proof: By Schwarz' Lemma $\lambda_n\le\frac{\delta_0}{r}$
(since $P_c^n B(z,r)\subset P_c^n B_n(z;\delta_0)=B(z_n;\delta_0)$)

As $B(z,r)\not\subset B_{n+1}(z;\delta_0)$ there is
$y\in B(z,r)$ s.t. $d(P_c^{n+1}y,\ z_{n+1})\ge\delta_0$
It follows that (M.V.T.)

$$\delta_0>d(P_c^n y,z_n)\ge\frac{\delta_0}{\alpha}$$



We have $P_c^n: B_n(z;\delta_0)\xrightarrow[\text{biyec.}]{C^\omega}B(z_n;\delta_0)$
so there is a univalent
holomorphic inverse $\psi_n: B(z_n;\delta_0)\xrightarrow{C^\omega}B_n(z;\delta_0)$
with $\psi_n(z_n)=z$. and

$$|\psi_n'(z_n)|=\frac{1}{|(P_c^n)'(z)|}=\frac{1}{\lambda_n(z)}$$

By the lower Koebe estimate (Coroll 8)

$$r>d(y,z)=d(\psi_n(P_c^n y),\psi_n(z_n))$$

(*) $$\ge\frac{\delta_0}{\lambda_n}\cdot\frac{1/\alpha}{(1+1/\alpha)^2}=\frac{\delta_0}{\lambda_n}\frac{\alpha}{(1+\alpha)^2}>\frac{\delta_1}{\lambda_n}$$

Let $w\in B(z_n,\delta_1)$ and $t=\frac{\delta_1}{\delta_0}=\frac{2}{3}\underbrace{\frac{\alpha}{(1+\alpha)^2}}_{\le 1/4}\le\frac{1}{6}$
Since $\frac{d(w,z_n)}{\delta_0}\le t$ we get by
the upper Koebe estimate

$$d(\psi_n w,z)\le\frac{\delta_0}{\lambda_n}\frac{t}{(1-t)^2}\le\frac{\delta_0}{\lambda_n}t\,\frac{3}{2}$$

$$=\frac{\delta_0}{\lambda_n}\frac{\alpha}{(1+\alpha)^2}<r \quad\text{by } (*)$$

Thus $B_n(z;\delta_1)\subset B(z,r)$ //

Summing up:

$(J(P_c), d) \circlearrowleft P_c$

is a uniformly expanding, unif mixing dyn. syst.

~~Def~~ .

Let $X = Lip_{\mathbb{R}}(J(P_c))$. and $g \in X$.
We define a Ruelle transfer operator

$$\mathcal{L}_g \phi(\omega) = \sum_{z:\, P_c(z)=\omega} e^{g(z)} \phi(z), \quad \omega \in J(P_c)$$

It has a spectral gap with associated pressure $P(g)$ and Gibbs measure ν_g.

Since $\nu_g(B_n(z,\delta_1)) \asymp \nu_g(B_n(z,\delta_0))$
we obtain from our geom. estimate for any $z \in J$, $r \in (0,\delta_0)$

$$\nu_g(B(z,r)) \asymp \nu_g(B_n(z,\delta_0)) \asymp e^{-nP(g) + S_n g(z)}$$

At the same time

$$r \asymp \frac{1}{\lambda_n(z)} = \frac{1}{|(P_c^n)'(z)|}$$

By expansion $\underline{n}(z,r;\delta_0) \xrightarrow[r \to 0^+]{} \infty$

We denote

$$\Lambda_g = \int \log|P_c'|\, d\nu_g$$

the Lyapunov exponent of ν_g.

$$\nu_g(g) = \int g\, d\nu_g$$

the average of g wrt ν_g

~~Sta~~.

Thm 11 The Hausdorf dim of ν_g is given by

$$\dim_H \mu = \frac{P(g) - \int g\, d\nu_g}{\int \log|P_c'|\, d\nu_g}$$

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proof: $\mathbb{C}$ has the Besicovitch property.
Thus by Besicovitch' local dimension thm

$$\dim_H \mu = d_\mu^*$$

with d_μ^* being the essential sup of $d_\mu(z) = \lim_{\delta \to 0^+} d_\mu(z,\delta)$ and

$$d_\mu(z,\delta) = \inf_{0<r<\delta} \frac{\log \mu B(z,r)}{\log r}$$

In our case when $0<r<\delta_0$ is small we have with $n = \underline{n}$

$$\log \mu B(z,r) \asymp \log \mu B_n(z,\delta_0) \asymp -nP(g) + S_n g(z) = -n\left(P(g) - \frac{1}{n} S_n g(z)\right)$$

and

$$\log r \asymp -\log \lambda_n(z) = -(S_n \log|P_c'|)(z)$$

Thus

$$\frac{\log \mu B(z,r)}{\log r} = \frac{P(g) - \frac{1}{n} S_n g(z)}{\frac{1}{n} S_n \log|P_c'|}$$

but by Birkhoff for ν_g-a.e. $z \in J(P_c)$:

$$\frac{1}{n} S_n g(z) \xrightarrow[n]{} \nu_g(g)$$

$$\frac{1}{n} S_n \log|P_c'| \xrightarrow[n]{} \int \log|P_c'|\, d\nu_g$$

Special case : $g = -s \log |P_c'|$

$$\mathcal{L}_s \phi(w) = \sum_{z : P_c z = w} \frac{1}{|P_c'(z)|^s} \phi(z)$$

Pressure $\mathcal{P}(s)$. Recall

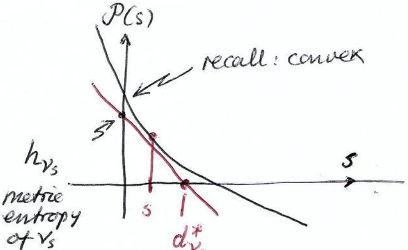
$$\mathcal{P}'(s) = \mathbb{E}_{\nu_s}(-\log |P_c'|) = -\Lambda_s$$
$$\in [-\log \alpha, -\log \beta] \subset \mathbb{R}_-^*$$

since $1 < \beta \leq |P_c'| \leq \alpha$

The local dimension of ν_g :

$$\underline{d}^*_{\nu_g} = \dim_H \nu_g = \frac{P(s) \overbrace{- \int g \, d\nu_g}^{s\Lambda_s}}{\Lambda_s}$$

$$= \frac{\mathcal{P}(s)}{-\mathcal{P}'(s)} + s.$$



By convexity $\max \dim_H \nu_s = s^*$
where $P(s^*) = 0$ (Bowen)

In fact when $P(s^*) = 0$ then $\nu_{s^*}(B(x,r))$

$$\nu_{s^*}(B_n(z, \delta_0)) \asymp e^{-n \cdot 0 + S_n g(-s^* \log |P_c'|)(z)}$$
$$= \frac{1}{|(P_c^n)'(z)|^{s^*}} \asymp r^s$$

So ν_{s^*} is s^*-Ahlfors regular.