8 Conformal repellers and dimensions

(locally compact) (Ω, d) metric space (e.g. $\mathbb{R}^n$)

$B(x,r)$ open ball centered at $x\in\Omega$ of radius $r>0$ (r-ball)

$J \subset \Omega$ a measurable subset.

Def 1 Given $\delta>0$ let $N(\delta, J)$ denote the minimal number of δ-balls necessary to cover J. Then the upper box dimension of J:

$$\overline{\dim}_B J = \limsup_{\delta\to 0^+} \frac{\log N(\delta, J)}{\log 1/\delta}$$

(lower box dim defined similarly with liminf; when limit exists just box-dim).

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exs $\quad \operatorname{Card} J < +\infty \Rightarrow \dim_B J = 0$

ex $\quad J \subset \mathbb{R}^n$ bounded, $\operatorname{int} J \neq \emptyset \Rightarrow \dim_B J = n$

ex $\quad 1/3$-Cantor set

Ω_0, Ω_1, Ω_2, $\vdots$

$$N\left(r_n = \frac{1}{3^n}, J\right) = 2^n \qquad \dim_B J = \frac{\log 2}{\log 3}$$

$$J = \bigcap_{n\geq 1} \Omega_n$$

ex $\quad \dim_B(\mathbb{Q}\cap[0,1]) = 1$

$$\dim_B\left(\{0\}\cup\left\{\tfrac{1}{n} : n\geq 1\right\}\right) = \frac{1}{2}$$

Def 2 For $\delta>0$ let δ-cover(J) denote the collection of countable open covers of J with sets of diam $<\delta$. Define for $s\geq 0$:

$$M_s(\delta, J) = \inf\left\{\sum |U_i|^s : (U_i)_{i\geq 1}\ \delta\text{-cover}(J)\right\}$$

and

$$M_s(J) = \lim_{\delta\to 0^+}\uparrow M_s(\delta, J) \in [0, +\infty]$$

called the s-dim Hausdorff measure of J. The Hausdorff dim of J:

$$\dim_H(J) = \inf\{s\geq 0 : M_s(J) < \infty\}$$

Rem: $\quad \dim_H J \leq \underline{\dim}_B J \leq \overline{\dim}_B J$.

Prop 3 $\quad \exists!\, s^* \in [0, +\infty]$:

$\forall\, 0\leq s < s^* : M_s(J) = +\infty$

$\forall\, s^* < s < \infty : M_s(J) = 0$

$s^* = \dim_H J$.

$M_{s^*}(J)$ is called the Hausdorff measure of J.

$M_{s^*}(J) \in [0, +\infty]$

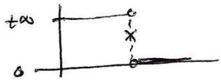
proof $\quad$ For $0\leq s_1 < s_2 < +\infty$, and $\delta > 0$, $|U| = \operatorname{diam} U < \delta$:

$$|U|^{s_2} = |U|^{s_2 - s_1}|U|^{s_1} \leq \delta^{s_2-s_1}|U|^{s_1}$$

$$\Rightarrow M_{s_2}(\delta, J) \leq \delta^{s_2 - s_1} M_{s_1}(\delta, J)$$

So $M_{s_2}(J) > 0 \Rightarrow M_{s_1}(J) = +\infty$

$M_{s_1}(J) < +\infty \Rightarrow M_{s_2}(J) = 0$



ex $\quad \dim_H(\mathbb{Q}) = 0 \quad M_0(\mathbb{Q}) = +\infty$.

Def 4 Let μ be a positive measure on (Ω, d). We define its (Hausdorff) dimension

$$\dim_H \mu = \inf\{\dim_H \underset{\text{meas.}}{A} : \mu(A^c) = 0\}$$

Rem: $\quad \dim_H \mu \leq \dim_H \operatorname{supp}\mu$

Rem: The infimum is attained: let $(A_n)_{n\geq 1}$ be s.t. $\mu(A_n^c) = 0$, $\dim_H A_n \to \dim_H \mu$. Then $\bigcap A_n$ has full measure and dimension $\dim_H \mu$.

Def 5 Let $\mu \in M^1_+(\Omega)$. $J \subset \Omega$

Let $s \geq 0$

We say that μ is upper (respectively lower) Ahlfors s-regular (on J) if $\exists\, C < +\infty$ st. for every $x \in J$ and $r > 0$:

$$\mu(B(x,r)) \leq C r^s \quad \text{(upper)}$$

respectively: (for $0 < r < 1$):

$$\frac{1}{C} r^s \leq \mu(B(x,r)) \quad \text{(lower)}$$

Thm 6

1. μ upper Ahlfors s-reg on J, $\mu(J) > 0$ $\Rightarrow$ $\dim_H J \geq s$.

2. μ lower Ahlf. s-reg on J $\Rightarrow$ $\dim_H J \leq \overline{\dim}_B J \leq s$

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proof: 1) Let $(U_i)_{i \geq 1}$ be a δ-cover(J). Pick $x_i \in U_i \cap J$. Then

$$1 \leq \sum_{i \geq 1} \mu(U_i) \leq \sum_i \mu(B(x_i, |U_i|)) \leq \sum_i C |U_i|^s$$

$\Rightarrow$ $\mathcal{M}_s(\delta, J) \geq \frac{1}{C} > 0$ indep of δ.

$\Rightarrow$ $\dim_H J \geq s$.

2) Let $0 < \delta \leq 1$ and let $(x_k)_{k=1}^{N(\delta)}$ be a maximally δ-separated set in J., i.e: $\forall i \neq j : d(x_i, x_j) \geq \delta$.

Then

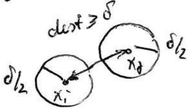
$$\forall i \neq j : B(x_i, \delta/2) \cap B(x_j, \delta/2) = \emptyset$$

but $J \subset \bigcup B(x_i, \delta)$

$$\Rightarrow \quad 1 \geq \sum_{i=1}^{N(\delta)} \mu(B(x_i, \delta/2)) \geq N(\delta) \frac{1}{C} \left(\frac{\delta}{2}\right)^s$$

$$\Rightarrow \quad N(\delta) \leq C \cdot 2^s \cdot \delta^{-s}$$

$$\Rightarrow \quad \overline{\dim}_B J \leq s.$$



Def 7 A <u>Besicovitch</u> cover
1. of $J \subset \Omega$ is a collection $\mathcal{A}$ of open balls centered at points in J such that <u>every</u> $x \in J$ is the center of at least one ball in $\mathcal{A}$.

Let $\mathcal{A}_\delta = \{B \in \mathcal{A} \mid |B| < \delta\}$

2. $\mathcal{A}$ is said to be a fine Besicovitch cover provided $\mathcal{A}_\delta$ is a Bes.-cover $\forall \delta > 0$.

<u>Rem</u>: An equivalent def of a fine Bes-cover is that

$\forall x \in J : \exists r_n \to 0^+ : B(x, r_n) \in \mathcal{A}$.

covered by 3 disjointed families

Thm 8 [Besicovitch covering lemma]
$\Omega = \mathbb{R}^n$, d = Euclidean dist.

$\exists\, b_n \in \mathbb{N}$ (dep. only upon the dim of the Eucl. sp)
such that if $\mathcal{A}$ is a Bes-cover of J then $\exists$ a countable subcover $\mathcal{C}$ of $\mathcal{A}$ for which <u>any</u> $x \in \mathbb{R}^n$ is contained in at most b_n balls in $\mathcal{C}$.

$\Updownarrow$

$\exists$ countable families
$C_1, \ldots, C_{b_n} \subset \mathcal{A}$ such that each C_i consists of disjoint balls and

$$J \subset C_1 \cup \ldots \cup C_{b_n} =: \mathcal{C}$$

<u>Rem</u> More generally a metric space (Ω, d) is said to have the Besicovitch property if $\exists\, b_\Omega < \infty$ for which the conclusion in Thm 8 holds.

In terms of characteristic functions the conclusion in thm 8 may be written

$$\mathbb{1}_J \leq \sum_{B \in \mathcal{C}} \mathbb{1}_B \leq b_n \cdot \mathbb{1}_\Omega$$

Def 9 $\mu \in M_+^1(\Omega)$. Define for $x \in \Omega$ and $\delta < 1$:

$$\underline{d}_\mu(x,\delta) = \inf_{0<r\le\delta} \frac{\log \mu B(x,r)}{\log r} \quad \text{and}$$

$$\underline{d}_\mu(x) = \lim_{\delta\to 0^+}\uparrow \underline{d}_\mu(x,\delta).$$

We call $\underline{d}_\mu(x)$ the (lower) pointwise dimension at x. We set

$$\underline{d}^*_\mu := \mu\text{-}\operatorname{ess\,sup}_{x\in\Omega} \underline{d}_\mu(x) = \sup\{s>0 : \mu(s<\underline{d}_\mu)>0\}$$

Rem For $0<r<1$ one has

$$\frac{\log \mu B(x,r)}{\log r} \ge s \Leftrightarrow \mu B(x,r) \le r^s$$

- For every $s \le \underline{d}_\mu(x,\delta)$, $0<r\le\delta$:

  $$\mu B(x,r) \le r^s$$

  smallest number for which this holds

- If $\mu B(x,r) = 0$ for some $r>0$ then $\underline{d}_\mu(x) = +\infty$. But such points have measure zero and do not appear in the μ-ensup.

Thm 10 When (Ω,d) has the Besicovitch property:

$$\dim_H \mu = \underline{d}^*_\mu$$

proof Let $s < \underline{d}^*_\mu$:

$$0 < \mu\{x : s < \underline{d}_\mu(x)\} = \mu\{\bigcup_{n\ge1}\{x : s<\underline{d}_\mu(x,\tfrac{1}{n})\}\}$$

$$\Rightarrow \exists n \in \mathbb{N} : c = \mu(D) > 0,\ D = \{x : s < \underline{d}_\mu(x,\tfrac{1}{n})\}$$

Now let $A \subset \Omega$ be meas and s.t. $\mu A^c = 0$.
Let $(U_i)_{i\ge1}$ be a $\frac{1}{n}$-cover of A.
It also covers $A\cap D$ and $\mu(A\cap D) = c > 0$.
~~Then~~ Let $\boxed{I = \{i\ge1 : U_i \cap A \cap D \neq \emptyset\}}$ and
for each $i \in I$ pick $x_i \in U_i \cap A \cap D$.
Since $|U_i| \le \frac{1}{n}$ $\therefore$ $0 < c = \mu(A\cap D) \le$

$$\sum_{i\in I}\mu(U_i\cap A\cap D) \le \sum_{i\in I}\mu(U_i) \le \sum_I \mu(B(x_i,|U_i|))$$

$$\le \sum_I |U_i|^s \le \sum_{i\ge1}|U_i|^s \quad \text{so } \dim_H(A) \ge s.$$

Let $s > \underline{d}^*_\mu$. Then $\{\underline{d}_\mu(x) \ge s\}$ has zero measure, and for a.e. $x\in\Omega$:

$$A = \{x : \underline{d}_\mu(x) < s\}$$

has full measure.

For every $x \in A$ there must be a sequence $r_n \to 0^+$ s.t.

$$\mu(B(x,r_n)) \ge r_n^s$$

The cover $\mathcal{A} = \{B(x, r_n(x)) : x\in A, n\ge1\}$ is thus a fine Besicovitch cover of A, whence by the Besic thm. admits a countable subcover $\mathcal{C} \subset \mathcal{A}$ having the Besic.-property. (any $x\in\Omega$ lies in at most b_Ω distinct balls in $\mathcal{C}$). Thus

$$\sum_{i : B(x_i,r_i)\in\mathcal{C}} |B(x_i,r_i)|^s \le \sum_i 2^s r_i^s \le$$

$$2^s \sum_i \mu(B(x_i,r_i)) \le$$

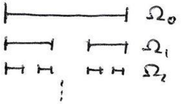
$$2^s \cdot b_\Omega \cdot \mu(\Omega) < +\infty \Rightarrow$$

$$M_s(A) \le 2^s b_\Omega \quad \Rightarrow$$

$$\dim_H A \le s. \quad //$$

Bowen's formula

Ex: The $\frac{1}{3}$-Cantor set.



$\Omega = \bigcap_{k\geq 0} \Omega_k$ compact, totally disconnected without isolated pts.

in fact

$\forall x \in \Omega$

$$\frac{r}{2} \leq |\operatorname{diam} B_\Omega(x,r)| \leq 2r$$

the constant here depends upon the construction of the Cantor set. (here for $\frac{1}{3}$)

Given $x \in \Omega$, $0 < r < \frac{1}{3}\delta_0$ we define

$$\underline{n}(x,r;\delta_0) = \max\{k \geq 0 : B(x,r) \subset B_k(x,\delta_0)\}$$

One has for $n = \underline{n}(x,r;\delta_0)$:

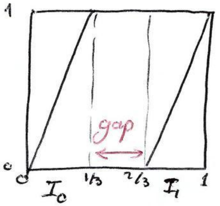
$$B_n(x,\tfrac{\delta_0}{3}) \subset B(x,r) \subset B_n(x,\delta_0)$$

and $B(x,r) \not\subset B_{n+1}(x,\delta_0)$

[There is thus $y \in B(x,r)$ for which $T^n y \in B(T^n x, \delta_0)$ but $T^{n+1} y \notin B(T^{n+1} x, \delta_0)$. so $d(T^n y, T^n x) \geq \frac{\delta_0}{3}$ (slope $T' = 3$)

If $y \in B_n(\delta_0/3)$ then this is impossible]

Dynamical System (partially defined)
$3x \bmod 1$ but removing the middle part



$T: \underbrace{I_0 \sqcup I_1}_{D(T)} \to [0,1]$ piecewise affine

$|T'(x)| = 3$, $x \in D(T)$

$\Omega = \bigcap_{k\geq 0} T^{-k} D(T) \circlearrowleft T$-inv.

$T: (\Omega, d) \circlearrowleft$ is uniformly expanding and uniformly mixing

Pick $0 < \delta_0 < 1/3$. ← the gap

Then each $B_n(x,\delta_0)$ is necessarily a subset of either I_0 or I_1.

So $T^n : B_n(x,\delta_0) \to B(T^n x, \delta_0)$ is affine, slope 3^n.

Write $\lambda_n(x) = |(T^n)'(x)| = 3^n$

Let $\psi_0 : I \to I_0$; $\psi_1 : I \to I_1$ be the inverse branches of T

$\psi_0(y) = y/3$, $\psi_1(y) = 2+y/3$.

For $s \geq 0$: Define the transfer op:

$$\mathcal{L}_s \varphi(y) = \sum_{x \in D(T)} \frac{1}{|T'(x)|^s} \varphi(x) = \sum \frac{1}{\lambda(x)^s} \varphi(x)$$

Acting upon $X = Lip(\Omega, d)$ it has a spectral gap and by chap 6 there is an associated Gibbs measure coming from $d\nu_s = h_s d\mu_s$.

Here $\mathcal{L}_s \mathbb{1} = \sum \frac{1}{T'(x)^s} \mathbb{1} = 2 \cdot \frac{1}{3^s} \mathbb{1}$

so $h = \mathbb{1}$, $P(s) = \log \lambda(s) = \log 2 - s \log 3$

We have $\mathcal{L}_s^n \varphi(y) = \sum_{x \in D(T^n) : T^n x = y} \frac{1}{\lambda_n(x)^s} \varphi(x)$

so by Gibbs mean-property, $n = \underline{n}(x,r;\delta_0)$

$$\begin{aligned}\nu_s(B(x,r)) &\asymp \nu_s(B_n(x,\delta_0)) \\ &\asymp e^{-nP(s)} (1/\lambda_n(x))^s \\ &\asymp e^{-nP(s)} |B_n(x,\delta_0)|^s \\ &\asymp e^{-nP(s)} r^s\end{aligned}$$

$\Rightarrow \nu_s$ is s-Ahlfors regular iff $P(s) = 0$

Thm [Bowen 75] $s = \dim_H \Omega$ is the unique value st $P(s) = 0$

Here $\log 2 - s \log 3 = 0 \Leftrightarrow s = \log 2 / \log 3$