7 Decay of correlations for thermodynamic Gibbs like measures $T:\Omega\to\Omega$ cont. and CLT Central limit Theorem.

$\mathcal{L}$ positive Ruelle transfer operator acting upon $X = Lip(\Omega,d) \subset C(\Omega,d)$ with spectral gap: $\exists!\, P\in\mathbb{R}$

$$h\in X,\ h>0,\quad \mu\in M_+^1(\Omega)\quad \mu(h)=1$$

$$\mathcal{L}h = e^P h,\quad \mu\mathcal{L} = e^P\mu \text{ and } 0<\eta<1:$$

$$\forall \phi\in X:\ \| e^{-nP}\mathcal{L}^n\phi - h\langle\mu,\phi\rangle\|_X \le C\eta^{n-1}\|\phi\|_X$$

$$\mathcal{L}(A\circ T\, B) = A\cdot\mathcal{L}B$$

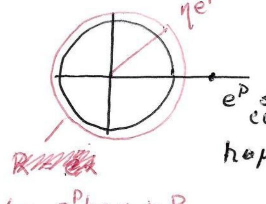
Then $d\nu = h\,d\mu$ is T-invariant mixing (whence ergodic).

Lemma 1: The operator $R = e^{-P}\mathcal{L} - h\otimes\mu$ verifies $\|R^n\|_{L(X)} \le C\eta^{n-1}$

proof $\|R^n\phi\|_X = \|e^{-nP}\mathcal{L}^n - h\langle\mu,\phi\rangle\|_X \le C\eta^{n-1}\|\phi\|_X$

Prop 2 For $A\in L^\infty$, $B\in X$:
$$|\nu(A\circ T^n B) - \nu(A)\nu(B)| \le C\eta^{n-1}|A|_\infty|B|_X$$

$$|\mu(A\circ T^n Bh) - \mu(Ah\cdot[\mu(Bh)])| =$$
$$|e^{-nP}\mu(A\,\mathcal{L}^n[Bh]) - \#\quad| =$$
$$|\mu(A\,R^n(Bh))| \le |A|_\infty \tilde{C}\eta^{n-1}\|Bh\|_X \le C\eta^{n-1}|A|_\infty\|B\|_X$$



e^P simple eval.

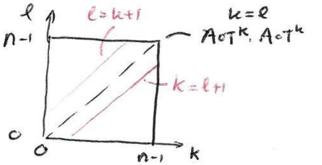
$h\otimes\mu$ projection

$$\mathcal{L} = e^P h\otimes\mu + R$$

Coroll 3 Suppose $\mathbb{E}(A)=0$ i.e $\nu(A)=\mu(Ah)=0$. Then
$$|\nu(A\circ T^n A)| \le C\eta^{n-1}|A|_\infty|A|_X = O(\eta^n)$$

Coroll 4 Suppose $\mathbb{E}A=0$. Then $\sigma_A^2 = \lim_n \frac{1}{n}\nu((S_nA)^2)$
$$= \nu(A^2) + 2\sum_{k\ge1}\nu(A\circ T^k A) \ge 0$$
exists

proof:
$$\frac{1}{n}\nu\Big(\sum_{k,\ell=0}^{n-1} A\circ T^k A\circ T^\ell\Big) =$$
$$\frac{1}{n}\Big[n\,\nu(A^2) + 2\sum_{p=1}^{n-1}\frac{(n-p)}{n}\nu(A\circ T^p A)\Big]$$
$$\xrightarrow[n\to\infty]{} \nu(A^2) + \lim_n 2\sum_{p=1}^{n-1}\Big(1-\frac{p}{n}\Big)\times \nu(A\circ T^p A)$$
$$= \nu(A^2) + 2\sum \nu(A\circ T^k A)$$



ex: If A is a co-bdry $A = b\circ T - b$, $b\in X$

$S_nA = b\circ T^n - b$

$\frac{1}{n}\nu((S_nA)^2) \le \frac{1}{n}\text{const} \xrightarrow[n]{} 0 = \sigma_A^2$

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5 ~~Formalism~~ Thermodynamic ~~al~~ Formalism

Perturbation theory

$\sigma : \Sigma_a^+ \to \Sigma_a^+$ ~~$(\Sigma_a^+(\theta))$~~

$X = \mathrm{Lip}(\Sigma_a^+, d_\theta)$ (complex valued)

For $g \in X$ we define the Ruelle transfer operator $\mathcal{L} = \mathcal{L}(g)$

$$\mathcal{L}_g \phi(y) = \sum_{x : \sigma x = y} e^{g(x)} \phi(x)$$

Ruelle-Perron-Frobenius thm.

Thm (spectral gap)

1) $\exists!\ P = P(g) \in \mathbb{R}$, $h = h(g) \in X_+$, $\mu = \mu(g) \in M_+^1(\Sigma_a^+)$ (proba meas) such that $\langle \mu, h \rangle = 1$ and

$\mathcal{L}_g h = h e^P$, $\mu \mathcal{L} = \mu e^P$

There are consts $C < \infty$, $0 < \theta < 1$:

$$\| e^{-nP} \mathcal{L}_g^n \phi - h \int \phi \, d\mu \| \le C \theta^{n-1} \|\phi\|_X$$

for every $\phi \in X$.

2) The measure $d\nu = h \, d\mu$ is a σ-invar. proba measure which is mixing (whence ergodic).

The quantity $P(g)$ is called the "pressure" of g.

Given $A \in X$ the map:

$$t \in \mathbb{C} \longmapsto \mathcal{L}_t = \mathcal{L}(g + tA) \in L(X)$$

is analytic: $\mathcal{L}_g(e^{tA}\phi)$

$$\mathcal{L}_t \phi(y) = \sum_{x : \sigma x = y} e^{g(x) + tA(x)} \phi(x)$$

$$= \sum e^{g(x)} \Big(\sum_{k \ge 0} \frac{t^k}{k!} A(x)^k \Big) \phi(x)$$

($L(X)$-norm convergent power-series)

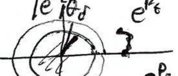
Thm [Kato: Perturbation theory for Linear Operators chap VII §1].

Let $t \in \mathcal{O}(0) \subset \mathbb{C} \to \mathcal{L}_t \in L(X)$ be an analytic family of linear operators. Suppose that λ_0 is a simple eval for $\mathcal{L}_0$ with right/left evector $h_0 \in X$, $\mu_0 \in X'$ $(\mu_0(h_0) = 1)$. Then there is $\delta > 0$ s.t. for $|t| < \delta$ $\mathcal{L}_t$ has a simple eval λ_t and right/left evec^s h_t, μ_t (with $\mu_t(h_t) = 0\ \forall t$) such that λ_t, h_t, μ_t $|t| < \delta$ are analytic.

Furthermore: A spectral gap ~~depends continuously upon $t \in \mathcal{O}(0)$~~ persists under small enough pert.

We will apply this to $t \mapsto \mathcal{L}_t = \mathcal{L}[g + tA]$

Coroll^s $\exists \delta > 0$ s.t. for $|t| < \delta$, $\mathcal{L}_t$ has a simple eval $e^{P(t)}$ and right/left evecs $h_t \in X$/$\mu_t \in X'$, $\langle \mu_t, h_t \rangle = 1$ and $t \mapsto P(t)$, h_t, μ_t are analytic (for $|t| < \delta$)



$\forall |t| < \delta$: $0 < \theta_\delta < 1$

$$\| e^{-nP_t} \mathcal{L}_t^n \phi - h_t \langle \mu_t, \phi \rangle \|_X \le C_\delta \theta_\delta^{n-1} \|\phi\|_X$$

NB: P_t may be complex valued here.

$$\mathcal{L}_s\phi_y = \sum_{x:Tx=y} e^{g(x)+sA(x)}\phi(x)$$

$$= \mathcal{L}_0(e^{sA}\phi)$$

$\mathcal{L}_s$ Analytic in $s \in \mathbb{C}$

$$\mathcal{L}_s h_s = e^{P(s)} h_s$$
$$\mu_s \mathcal{L}_s = e^{P(s)} h_s$$

For a simple eval $P(s)$, $h(s)$, $\mu(s) \in X'$ depends analytically upon s

$$\mathcal{L}_s(B\circ T\cdot\phi) = \sum_{x:Tx=y} e^{g+sA} B\circ T(x)\phi(x)$$

$$= B(y)\,\mathcal{L}_s\phi\,(y)$$

$$\mathcal{L}_s(B\circ T\cdot\phi) = B\cdot\mathcal{L}_s\phi$$

$$\mathcal{L}_s^n\phi = \mathcal{L}_0(e^{sA}\mathcal{L}_0(e^{sA}\cdots\mathcal{L}_0(e^{sA}\phi))\cdots))$$
$$= \mathcal{L}_0^n(e^{s(A+A\circ T+\dots+A\circ T^{n-1})}\phi)$$
$$= \mathcal{L}_0^n(e^{sS_nA}\phi)$$

$$\frac{d}{ds}\mathcal{L}_s^n\phi = \mathcal{L}_0^n(e^{sS_nA}\cdot S_nA\cdot\phi) = \mathcal{L}_s^n(S_nA\,\phi)$$

Lemma 6: $\mathcal{L}_s^n\phi = \mathcal{L}_0^n(e^{sS_nA}\phi)$

$$\frac{d}{ds}\mathcal{L}_s^n\phi = \mathcal{L}_s^n((S_nA)\cdot\phi)$$

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before

$\boxed{\text{Prop 6.7: } P'(s) = \nu_s(A)}$

proof: $e^{P(s)} = \mu_s(\mathcal{L}_s h_s)$

$$P'(s)e^{P(s)} = \mu_s'(\mathcal{L}_s h_s) + \mu_s(\mathcal{L}_s A h_s) + \mu_s(\mathcal{L}_s h_s')$$
$$= e^{P(s)}(\mu_s'(h_s) + \mu_s(h_s')) + e^{P(s)}\nu_s(A)$$
$$= e^{P(s)}\frac{d}{ds}(\underbrace{\mu_s(h_s)}_{\equiv 1}) + e^{P(s)}\nu_s(A)$$

Remark: For real s, μ_s and ν_s extends as proba measures on Ω. This need not be the case for complex s. We always have $\mu_s, \nu_s \in X'$ (analytic in s $\in$ neighb. of $\mathbb{R}$)

$P(s)$ pressure of $g+sA$.

Assume $P'(0) = \mathbb{E}(A) = \nu_0(A) = 0 = \mu_0(Ah_0)$ (and $P(0)=0$)

$$P'(s) = \mu_s(Ah_s)$$
$$= \mu_s(\mathcal{L}_s^n A \mathcal{L}_s^n h_s) \cdot e^{-2nP(s)}$$

<u>Calculation of $P''(0)$:</u>

$$P''(0) = \underbrace{-2nP'(0)^2}_{=0} + \mu_0'(\mathcal{L}_0^n A \mathcal{L}_0^n h_0) + \mu_0(\mathcal{L}_0^n A \mathcal{L}_0^n h_0')$$
$$+ \mu_0(\mathcal{L}_0^n (S_n A) A \mathcal{L}_0^n h_0) +$$
$$\mu_0(\mathcal{L}_0^n A \cdot \mathcal{L}_0^n (S_n A) h_0) \quad A \circ T^n$$

$$\|\mathcal{L}_0^n A \mathcal{L}_0^n h_0\|_{L(X)} \leq C|\mu_0(Ah_0)| + \text{rest}$$
$$(h_0\mu + R_0^n) \quad (h_0\mu_0 + R_0^n) \leq 0 \quad + R \cdot \mathcal{O}(\eta^n)$$

so $\forall n \geq 1$

$$P''(0) = \mathcal{O}(\eta^n) + \nu((S_n A)\cdot A) + \nu(A \circ T^n (S_n A))$$
$$= \mathcal{O}(\eta^n) + \sum_{k=0}^{n-1} \nu(A \circ T^k \cdot A) + \sum_{k=0}^{n-1} \nu(A \circ T^n \cdot A \circ T^k)$$
$$= \mathcal{O}(\eta^n) + \nu(A^2) + 2\sum_{k=1}^{n-1} \nu(A \circ T^k \cdot A) + \underbrace{\nu(A \circ T^n \cdot A)}_{= \mathcal{O}(\eta^n)}$$
$$\lim_{n\to\infty} = \nu(A^2) + 2\sum_{k=1}^{n-1} \nu(A \circ T^k \cdot A)$$

$$= \lim_{n\to\infty} \frac{1}{n} \nu((S_n A)^2) = \sigma_A^2$$

More generally:

<u>Theorem</u> 8 Given $A \in X$ $\mathcal{L}_s = \mathcal{L}_{g+sA}$ and $P(s) := P(g+sA)$ the pressure $\log \rho_{sp}(\mathcal{L}_{g+sA})$ we have:

$$P'(s) = \mathbb{E}_s(A) = \nu_s(A)$$
$$P''(s) = \sigma_A(s)^2 = \lim \frac{1}{n} \nu_s((S_n(A - \mathbb{E}A))^2)$$
$$= \nu_s((A - \mathbb{E}A)^2) + 2\sum_{k \geq 1} \nu_s((A - \mathbb{E}A) \circ T^k \cdot (A - \mathbb{E}A))$$

Calculation of $P''(0)$

Let $P(s) := P(g+sA)$.
Assume first $P'(0) = \mu_0(Ah_0) = 0$
as well as $P(0) = 0$

We have for $s \in \mathcal{O}(0)$:

$$P'(s) = \mu_s(A h_s)$$
$$= \mu_s(\mathcal{L}_s^n A \mathcal{L}_s^n h_s)\, e^{-2nP(s)}$$

A priori estimate:

$$\|\mathcal{L}_0^n A \overbrace{\mathcal{L}_0^n}^{h_0\mu_0 + R_0^n}\|_{L(X)} \le$$
$$/\ |\underbrace{\mu_0(A h_0)}_{=0}| + \mathcal{O}(q^n)$$

$h_0\mu_0 + R_0^n$

$$\|\mathcal{L}_0^n A \mathcal{L}_0^n\|_{L(X)} = \mathcal{O}(q^n)$$

$$P''(0) = \underbrace{-2nP'(0)^2}_{=0} + \overbrace{\mu_0'(\mathcal{L}_0^n A \mathcal{L}_0^n h_0)}^{\mathcal{O}(q^n)} + \underbrace{\mu_0(\mathcal{L}_0^n A \mathcal{L}_0^n h_0')}_{\mathcal{O}(q^n)}$$

$A \circ T^n$

$$+ \mu_0(\mathcal{L}_0^n A \mathcal{L}_0^n (S_n A) h_0)$$
$$+ \mu_0(\mathcal{L}_0^n (S_n A) A \mathcal{L}_0^n h_0)$$

$$= \mathcal{O}(q^n) + \nu_0((S_n A)\cdot A) + \nu_0(A \circ T^n (S_n A))$$
$$= \mathcal{O}(q^n) + \nu_0(A^2) + 2\sum_{k=1}^{n-1} \nu(A \circ T^k \cdot A)$$
$$\xrightarrow[n\to\infty]{} \sigma_A^2 = \nu_0(A^2) + 2\sum_{k=1}^{\infty} \nu(A \circ T^k \cdot A)$$

Theorem 8
Let $g, A \in X$ and let $P(s) = P(g+sA)$ be the pressure of $g+sA$.
Then P is analytic in $s \in \mathbb{R}$ and we have the formulae:

$$P'(0) = \nu_0(A) = \mathbb{E}_0(A)$$
$$P''(0) = \sigma_A^2 =$$
$$\lim_{n\to\infty} \frac{1}{n}\, \mathbb{E}_0((S_n(A - \mathbb{E}A))^2) =$$
$$\mathbb{E}_0((A - \mathbb{E}A)^2) +$$
$$2\sum_{k\ge 1} \mathbb{E}_0((A - \mathbb{E}A)\circ T^k \cdot (A - \mathbb{E}A))$$

$$\sigma_A^2 \ge 0$$

Theorem 9 (Central limit theorem)
Let $g, A \in X$ and let ν_g be the Gibbs measure associated.

Suppose $\nu_g(A) = 0$.
Then

$$\frac{1}{\sqrt{n}} S_n A \xrightarrow[n]{\text{Law}} N(0, \sigma_A^2)$$

where

$$\sigma_A^2 = \lim_n \frac{1}{n} \nu_g((S_n A)^2)$$
$$= \nu_g(A^2) + 2 \sum_{k \geq 1} \nu_g(A \circ T^k \cdot A) \geq 0$$

<u>proof</u> convergence in law $\Leftrightarrow$ convergence ptwise of char function (Levy's thm).

We will show that $\forall t \in \mathbb{R}$

$$\varphi_n(t) = \nu_g(e^{i \frac{t}{\sqrt{n}} S_n A}) \xrightarrow[n \to \infty]{} e^{-\frac{t^2}{2} \sigma_A^2}$$

Let as before $P(s) = P(g + sA)$.
We have (assume $P(0) = 0$)

$$\varphi_n(t) = \mu_0(\mathcal{L}^n_{i\frac{t}{\sqrt{n}}} h_0)$$

Recall for $|s| \leq \varepsilon_0$, $\mathcal{L}_s$ is analytic and has a uniform spectral gap:

$$\mathcal{L}_s^n = e^{nP(s)} h_s \mu_s + \underbrace{R_s^n}_{= O(\eta^n)}$$

and $s \mapsto P(s), h_s, \mu_s$ are analytic.

$$P(s) = 0 + sP'(0) + \frac{s^2}{2} P''(0) + O(s^3)$$
$$= 0 + 0 + \frac{s^2}{2} \sigma_A^2 + O(s^3)$$

Setting $s = it/\sqrt{n}$:

$$nP(\tfrac{it}{\sqrt{n}}) = n \cdot (-\tfrac{t^2}{2n}) \sigma_A^2 + O((\tfrac{t}{\sqrt{n}})^3)$$
$$\xrightarrow[n]{} -\frac{t^2}{2} \sigma_A^2 + 0$$

since for fixed t we have $|s| < \varepsilon_0$ for n large enough

Also

$$h_{\frac{it}{\sqrt{n}}} \xrightarrow[n]{} h_0, \quad \mu_{\frac{it}{\sqrt{n}}} \to \mu_0$$

and

$$\varphi_n(t) \to e^{-t^2/2 \, \sigma_A^2}$$

since

$$\varphi_n(t) = e^{nP(\frac{it}{\sqrt{n}})} \mu_0(h_{it/\sqrt{n}}) \mu_{it/\sqrt{n}}(h_0) + O(\eta^n)$$
$$\longrightarrow e^{-t^2/2 \, \sigma_A^2} \times 1 \times 1 + 0$$