6. Positive transfer operators and spectral gap.

(Ω, d) compact metric sp.
$D = \operatorname{diam} D < +\infty$.

Def 6.1 A d-fold covering cont. (d loc. bijective/homeo) map $T:\Omega\to\Omega$ is said to be uniformly expanding and uniformly mixing if $\forall$ couple $y, y'\in\Omega$ we may label the pre-images

$T^{-1}(y) = \{x_i\}_{i=1,\dots,d}$,
$T^{-1}(y') = \{x_i'\}_{i=1,\dots,d}$

such that $\forall i$: $d(x_i, x_i') \le \frac{1}{\beta} d(y,y')$ with $\beta>1$ indep of y, y'.

$X := \mathrm{Lip}(\Omega, d)$, $\varphi\in X$

$$\|\varphi\| = \sup_\Omega|\varphi| + \mathrm{Lip}(\varphi) \qquad \mathrm{Lip}(\varphi) = \sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{d(x,y)}$$

Recall the family of log-Lipsch cones:

$$a>0:\ K_a = \{\varphi:\Omega\to\mathbb{R}_+ : \forall x,y\in\Omega: \varphi_x \le \varphi_y e^{a\, d(x,y)}\}$$

(which is inner and outer regular)

Let $g\in X$ and define the transfer operator

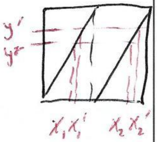
$$\boxed{\mathcal{L}_g\varphi(y) = \sum_{x\in\Omega : Tx=y} e^{g(x)}\varphi(x)}$$

Prop 6.2 $\mathcal{L}_g$ has a spectral gap on X.

proof: As in chap 5 we see that for $\varphi\in K_a$

$$\mathcal{L}_g\varphi(y) = \sum_{i=1}^{d} e^{g(x_i)}\varphi(x_i) \le \sum_{i=1}^{d} e^{g(x_i')}\varphi(x_i') \times e^{(\mathrm{Lip}\, g + a)/\beta\, d(y,y')}$$

Ex



$T(x) = 2x \mod \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$

so choosing a large enough (s.t. $\sigma = (\frac{\mathrm{Lip}\, g}{a}+1)\frac{1}{\beta}<1$)

$\mathcal{L}_g\varphi \in K_{\sigma a}$

and $\Delta = \operatorname{diam}_{K_a} K_{\sigma a}^* < +\infty$

By our sp gap thm $\mathcal{L}_g\in L(X)$ has a sp gap: $\exists\lambda>0$, $h\in X$, $\ell\in X'$, $\theta = \tanh\frac{\Delta}{4} < 1$, $C<\infty$:

$$\boxed{\forall\varphi\in X:\ \|\lambda^{-n}\mathcal{L}_g^n\varphi - h\langle\ell,\varphi\rangle\|_X \le C\theta^{n-1}\|\varphi\|_X}$$

Thm 6.3 [Riesz representation]
Let (Ω, d) be a compact metric sp. Then $(C^0(\Omega))'$ (bd lin functs) is isomorphic to the space of finite signed Borel measures on (Ω, d): $M(\Omega)$

$$M(\Omega) \simeq (C^0(\Omega))'$$

$$d\mu \leftrightarrow \hat{\mu}$$

$$\forall \phi \in C^0(\Omega): \hat{\mu}(\phi) = \int_\Omega \phi \, d\mu$$

Lemma 6.4: The linear functional ℓ in Prop 6.2 extends to a positive bd ~~positive~~ linear functional on $C^0(\Omega)$ whence ~~a proba measure~~ gives rise to ~~define~~ a proba measure on Ω, $(\langle \ell, 1\!\!1\rangle = 1)$ $\mu_g(\phi) = \int \phi \, d\mu_g := \langle \ell, \phi \rangle$, $\phi \in X$.

proof: $1\!\!1 \in C^0(\Omega) \cap K_a$ ($a>0$) as in Prop 6.2

The operator $\mathcal{L}_g$ is positive: $\mathcal{L}_g : C^0_+(\Omega) \to C^0_+(\Omega)$. It follows: $\forall \phi \in C^0(\Omega)$

$$-|\phi|_0 1\!\!1 \le \phi \le |\phi|_0 1\!\!1 \Rightarrow$$

$$-|\phi|_0 \mathcal{L}_g^n 1\!\!1 \le \mathcal{L}_g^n \phi \le |\phi|_0 \mathcal{L}_g^n 1\!\!1$$

For $\phi \in X = \mathrm{Lip}(\Omega, d)$ taking limits

$$-|\phi|_0 h \langle \ell, 1\!\!1\rangle \le h \langle \ell, \phi\rangle \le |\phi|_0 h \langle \ell, 1\!\!1\rangle$$

$$\Rightarrow |\langle \ell, \phi\rangle| \le |\phi|_0 \langle \ell, 1\!\!1\rangle$$

normalise so that $\langle \ell, 1\!\!1\rangle \equiv 1$

Then

$$|\langle \ell, \phi\rangle| \le |\phi|_0$$
$$\langle \ell, 1\!\!1\rangle = 1$$

ℓ is thus uniformly bd on X for the C^0-norm. X is dense in $C^0(\Omega) \Rightarrow$ ℓ extends (uniquely) to bd lin functional $\hat{\mu}$ on $C^0(\Omega)$ with

$$\hat{\mu}(\phi) := \langle \ell, \phi \rangle \quad \forall \phi \in X (\subset C^0(\Omega))$$

By Riesz $\exists!\, \mu \in M^1_+(\Omega)$

$$\hat{\mu}(\phi) = \int_\Omega \phi \, d\mu$$

$\mu(\phi) \ge 0$, $\phi \ge 0$ and

$$\mu(1\!\!1) = \int_\Omega d\mu = 1$$

Rem: The above measure is called a g-measure.

Rem: As X is dense in $C^0(\Omega)$ we also have: $\forall \phi \in C^0(\Omega)$

$$\lambda_g^{-n} \mathcal{L}_g^n \phi \to h_g \langle \ell_g, \phi \rangle = h_g \int_\Omega \phi \, d\mu_g$$

Def 6.5 (Bowen balls)

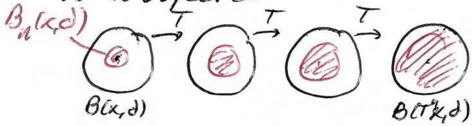
Let $\delta>0$ be an injectivity radius for T, i.e. $\forall x\in\Omega$ $T: B(x,\delta)\to TB(x,\delta)$ is a bijection. Define for $n\geq 1, x\in\Omega$

$$B_n(x,\delta) = \bigcap_{k=0}^{n} T^{-k}B(T^k x,\delta) = \{x'\in\Omega : d(T^k x, T^k x')<\delta,\ \forall 0\leq k\leq n\}.$$

One verifies that by uniform expansion

$$T^n: B_n(x,\delta) \to B(T^n x,\delta)$$

is a bijection



Lemma 6.6 Let $g\in Lip(\Omega,d)$. Then $\forall x', x \in B_n(x,\delta)$ $x\in\Omega, n\geq 1$

$$|S_n g(x') - S_n g(x)| \leq Lip(g)\frac{\beta\delta}{\beta-1}$$

proof: For $0\leq k\leq n$:

$$d(T^k x, T^k x') \leq \frac{1}{\beta} d(T^{k+1}x, T^{k+1}x') \leq \dots \leq \frac{1}{\beta^{n-k}} d(T^n x, T^n x') \leq \frac{\delta}{\beta^{n-k}}$$

So $|S_n g(x') - S_n g(x)| \leq$

$$\sum_0^n |g\circ T^k(x') - g\circ T^k(x)| \leq \sum_{k\geq 0} Lip\, g \cdot \frac{\delta}{\beta^{n-k}} = Lip\, g \frac{\delta}{1-\frac{1}{\beta}} \ //$$

Lemma 6.7 Given h_g, ℓ_g and $\mu_g\in M_+^1(\Omega)$ from spectral gap and Lemma 6.4 the measure

$$d\nu_g = h_g\, d\mu_g$$

is T-invariant and mixing (whence ergodic)

proof: Let $\phi\in X$. Then

$$\int \phi\circ T\, d\nu_g = \langle \ell_g, \phi\circ T\cdot h_g\rangle = \lambda^{-1}\langle \ell_g, L_g(\phi\circ T\cdot h_g)\rangle = \lambda^{-1}\langle \ell_g, \phi\cdot L_g h_g\rangle = \langle \ell_g, \phi h_g\rangle = \int\phi\, d\nu_g$$

Let $a,b\in X$. Then for $n\geq 1$

$$\int a\circ T^n b\, d\nu_g = \langle \ell_g, a\circ T^n\cdot b h_g\rangle = \lambda^{-n}\langle \ell_g, L_g^n(a\circ T^n b h_g)\rangle = \lambda^{-n}\langle \ell_g, a\cdot L_g^n(b h_g)\rangle \xrightarrow[n\to\infty]{}$$

(by spectral gap)

$$\langle \ell_g, a\cdot h_g\rangle\langle \ell_g, b\cdot h_g\rangle = \int a\, d\nu_g \cdot \int b\, d\nu_g$$

Since $X\subset L^1\cap L^2(d\nu_g)$ is dense in $L^1(d\nu_g)$ thm 5.6 implies that T is strongly mixing with respect to ν_g (and thus ergodic as well)

Thm 6.8 (Gibbs measure)
The ergodic measure ν_g is the unique T-invariant measure that satisfies: $\exists\, 0<C_1<C_2<+\infty$: $\forall x\in\Omega$, $n\geq 1$

$$C_1 \leq \frac{\nu_g(B_n(x,\delta))}{\lambda^{-n} e^{S_n g(x)}} \leq C_2 \quad (1)$$

we also write

$$\nu_g(B_n(x,\delta)) \asymp \lambda^{-n} e^{S_n g(x)}$$

ratio of two sides bd unif from above and below in $(0,+\infty)$

Proof: Let $N\in\mathbb{N}$ be such that

$$\frac{\operatorname{diam}\Omega}{\beta^N} < \delta \quad \text{(injectivity rad.)}$$

Given $x_0=x\in\Omega$, $x_{n+1}=T(x_n)$ consider $y\in\Omega$. Since $d(y,x_N)\leq D=\operatorname{diam}\Omega$ there is $x'_{N-1}\in T^{-1}(y)$ st. $d(x'_{N-1},x_{N-1})\leq \frac{D}{\beta}$

$\vdots$

$x'_0\in T^{-1}(x'_1)$ s.t. $d(x'_0,x_0)\leq \frac{D}{\beta^N}<\delta$.

Thus $B(x,\delta)\cap T^{-N}(y)\neq\emptyset$ so

(*) $\quad T^N B(x,\delta)=\Omega \quad \forall x\in\Omega$.

Iterating $\mathcal{L}_g\phi(y)=\sum_{x:Tx=y} e^{g(x)}\phi(x)$ we have

$$\mathcal{L}_g^n\phi(y)=\sum_{x:T^nx=y} e^{S_ng(x)}\phi(x)$$

whence also

$$\mathcal{L}_g^n(e^{-S_ng}\phi)=\sum_{x:T^nx=y}\phi(x)$$

With $n=N$, $\phi=\mathbb{1}_{B(x,\delta)}$; by (*):

$$\mathcal{L}_g^N(e^{-S_Ng}\mathbb{1}_{B(x,\delta)})\geq \mathbb{1}$$

Applying μ_g $(\mu_g\mathcal{L}_g=\lambda_g\mu_g)$:

$$\lambda_g^N\mu_g(e^{-S_Ng}\mathbb{1}_{B(x,\delta)})\geq 1$$

Thus $\mu_g(B(x,\delta))\asymp 1$ (N is fixed) and as $0<\min h\leq\max h<\infty$

$\nu_g(B(x,\delta))\asymp 1$ all $x\in\Omega$.

For $x'\in B_n(x,\delta)$ we have ~~for~~ $(n\geq 1)$

$$|S_ng(x')-S_ng(x)|\leq \operatorname{Lip} g\,\frac{\beta\delta}{\beta-1}$$

Therefore as $T^n B_n(x,\delta)\xrightarrow{\text{bij}} B(T^nx,\delta)$

$$\mathcal{L}_g^n(e^{-S_ng}\mathbb{1}_{B_n(x,\delta)})(y)=\sum_{x':T^nx'=y}\mathbb{1}_{B_n(x,\delta)}(x')=\mathbb{1}_{B(T^nx,\delta)}(y)$$

$$\begin{aligned}\Rightarrow \nu_g(B_n)&\asymp\mu_g(B_n)=\mu_g(\mathbb{1}_{B_n})\\ &\asymp e^{S_ng(x)}\mu_g(e^{-S_ng}\mathbb{1}_{B_n})\\ &=\lambda_g^{-n}e^{S_ng(x)}\mu_g(\mathcal{L}_g^n(e^{-S_ng}\mathbb{1}_{B_n}))\\ &=\lambda_g^{-n}e^{S_ng(x)}\mu_g(\mathbb{1}_{B(T^nx,\delta)})\\ &\asymp\lambda_g^{-n}e^{S_ng(x)}\end{aligned}$$

Suppose $\tilde\nu_g$ is a T-invariant measure satisfying (1) (possibly for other constants). Then $\nu_g(B_n(x,\delta))\asymp\tilde\nu_g(B_n(x,\delta))$ and as $\{B_n(x,\delta): x\in\Omega, n\geq 1\}$ forms a base for the topology $\tilde\nu_g\ll\nu_g$ so (Radon-Nikodym) $d\tilde\nu_g=f\,d\nu_g$ with $f\in L^1(d\nu_g)$, $\int f\,d\nu_g=1$

Let $a\in L^\infty(d\nu_g)$. Then

$$\int a\circ T\, f\,d\nu_g=\int a\circ T\,d\tilde\nu_g=\int a\,d\tilde\nu_g=\int a f\,d\nu_g$$

so also (iterate)

$$\int \tfrac1n S_na\cdot f\,d\nu_g=\int a\, f\,d\nu_g$$

Birkhoff $\Rightarrow$ $\frac1n S_na\xrightarrow[n]{}\int a\,d\nu_g$ point-wise for ν_g-a.e. point

As $|\frac1n S_na|\leq\|a\|_\infty$ dominated convergence shows:

$$\int a\,d\nu_g\int f\,d\nu_g=\int a\, f\,d\nu_g \text{ or}$$

$$\int a(1-f)\,d\nu_g=0 \quad \forall a\in L^\infty$$

$\Rightarrow f=1 \quad \nu_g$-a.e. //

Analytic perturbation theory

[Kato: Perturbation theory for linear operators, chap VII §1]

$X = Lip(\Omega, d)$ is a Banach algebra

$\|a \cdot b\|_X \le \|a\|_X \cdot \|b\|_X$, $a, b \in X$.

$T: \Omega \to \Omega$ uniformly expanding and mixing.

Given $g, A \in X$, $t \in \mathbb{C}$ we consider the operator:

$$\mathcal{L}_t \phi(y) := \sum_{x: Tx = y} e^{g(x) + tA(x)} \phi(x)$$

Then the map $t \in \mathbb{C} \mapsto \mathcal{L}_t \in L(X)$ is analytic since

$$\mathcal{L}_t \phi(y) = \sum_x e^{g(x)} \Big(\sum_{k \ge 0} \frac{t^k}{k!} A(x)^k\Big) \phi(x)$$

$$= \sum_k \frac{t^k}{k!} \sum_x e^{g(x)} (A(x))^k \phi(x)$$

$$= \sum \frac{t^k}{k!} \mathcal{L}_g (A^k \phi) \quad \text{and}$$

$$\|\mathcal{L}_g (A^k \phi)\| \le \|A\|^k \|\mathcal{L}_g\| \|\phi\|_X$$

so the entire series ~~con~~ is norm convergent $\forall t \in \mathbb{C}$.

Thm 6.9 [Kato] Let $\mathcal{L}_t \in L(X)$ be an analytic family of operators and suppose that λ_0 is a simple and isolated eigenvalue of $\mathcal{L}_0$ with eigenprojection $h_0 \otimes l_0$, $h_0 \in X$, $l_0 \in X'$, $\langle l_0, h_0 \rangle = 1$

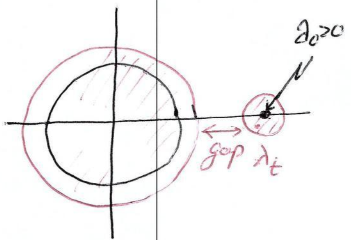
Then there is $\delta > 0$ and analytic functions

$t \in \mathbb{D}_{\mathbb{C}}(0, \delta) \mapsto h_t \in X$, $l_t \in X'$, $\lambda_t \in \mathbb{C}$

such that λ_t is a simple and isolated eigenvalue for $\mathcal{L}_t$ with eigenprojection $h_t \otimes l_t$, $\langle l_t, h_t \rangle \equiv 1$

Furthermore, if $\mathcal{L}_0$ has a sp gap (our case) then taking $\delta > 0$ small enough the spectral gap persists, i.e.: $\exists C < \infty$, $\theta < 1$: $\forall t \in \mathbb{D}_{\mathbb{C}}(0, \delta)$, $\phi \in X$:

$$\|\lambda_t^{-n} \mathcal{L}_t^n \phi - h_t \langle l_t, \phi \rangle\|_X \le C \theta^{n-1} \|\phi\|_X$$



Remark: In general, λ_t need not be real (for t complex) and l_t ~~need~~ may not extend to a measure (it stays a distribution on Lipschitz fcts)