5. A family of Log-Lipschitz cones

(Ω, d) a bounded metric space.
$D = \operatorname{diam} \Omega < +\infty$.

$X = \mathrm{Lip}_{\mathbb{R}}(\Omega, d)$ with norm:

$$\|\phi\| = \sup_{\Omega} |\phi| + \mathrm{Lip}(\phi)$$

with

$$\mathrm{Lip}(\varphi) = \sup \frac{|\varphi(x) - \varphi(y)|}{d(x,y)}$$

5.1 We define for every $a > 0$ the convex, closed and proper cone (an $\mathbb{R}$-cone)

$$K_a = \{ \varphi : \Omega \to \mathbb{R}_+ \mid \forall x, y \in \Omega : \varphi_x \le e^{a d(x,y)} \varphi_y \}$$

Remark: When $\varphi \in C_a^*$, then $\varphi > 0$ everywhere and (exo)

$$\varphi \in C_a^* \Leftrightarrow \mathrm{Lip}(\log \phi) \le a.$$

Prop 5.2 K_a is inner regular in X and is of bounded sectional aperture. (in fact even outer regular).

Proof: Let $0 < r < 1$ and $h \in X$ with $\|h\| \le r$. Then for $x, y \in \Omega$:

$$\begin{aligned} 1 + h_x &\le 1 + h_y + r\, d(x,y) \\ &\le (1 + h_y)\left(1 + \frac{r\, d(x,y)}{1-r}\right) \\ &\le (1+h_y) \exp\left(\frac{r}{1-r} d(x,y)\right) \end{aligned}$$

Thus if $\frac{r}{1-r} \le a$ ($\Leftrightarrow r \le \frac{a}{1+a}$) then $B_X(\mathbb{1}, r) \subset K_a$.

Given $f \in K_a$ and $x, y \in \Omega$.
Assume $0 < f(y) \le f(x)$. Then
$f(y) \ge f(x) e^{-a d(x,y)}$ so
$f(x) - f(y) \le f(x)(1 - e^{-ad}) \le f(x)\, a\, d(x,y)$
$\Rightarrow \mathrm{Lip}(f) \le a \cdot |f|_0$. Thus,
for $f, g \in K_a$: $\|f\| + \|g\| \le (1+a)(|f|_0 + |g|_0)$
$\le 2(1+a)|f+g|_0 \le 2(1+a)\|f+g\|$
So K_a is of $\mathcal{A} = 2(1+a)$ - bd sect. apert.

To see outer regularity pick $x_0 \in \Omega$ and set $\langle \ell, \phi \rangle = \varphi(x_0)$, $\varphi \in X$. Then $\|\ell\|_{X'} = 1$ and for $\varphi \in K_a$ we have

$$\begin{aligned} \|\varphi\| &= \sup \varphi + \mathrm{Lip}(\varphi) \\ &\le (1 + a) \sup \varphi \\ &\le (1+a) e^{2aD} \varphi(x_0) \\ &= (1+a) e^{2aD} \langle \ell, \phi \rangle \end{aligned}$$

not optimal

so outer regular with aperture $\mathcal{A} = (1+a) e^{2aD}$

Prop 5.3 For $0 < b < a$ one has

$$\Delta = \operatorname{diam}_{K_a} K_b \le 2 \log \frac{a+b}{a-b} + 2bD$$

Proof Let $f, g \in K_b^*$. For $x, y \in \Omega$ ($d = d(x,y)$) we have

$$e^{-b \cdot d} \le \frac{f_x}{f_y}, \frac{g_x}{g_y} \le e^{b \cdot d}$$

$t_1 = \beta(f, g)$ verifies

$t_1 f_x - g_x \le e^{ad}(t_1 f_y - g_y)$ or

$$t_1 \ge \frac{g_y}{f_y} \cdot \frac{1 - g_x/g_y\, e^{-ad}}{1 - f_x/f_y\, e^{-ad}} \le \frac{g_y}{f_y} \frac{1 - e^{-(a+b)d}}{1 - e^{-(a-b)d}}$$

t_1 the min verifying this so

$$t_1 = \sup_{d > 0}(\mathrm{RHS}) \le \sup_y \frac{g_y}{f_y} \frac{a+b}{a-b}$$

Similarly $t_2 \le \sup_x \frac{f_x}{g_x} \frac{a+b}{a-b}$

so

$$t_1 t_2 \le \sup_{x,y} \frac{f_x g_y}{f_y g_x} \cdot \left(\frac{a+b}{a-b}\right)^2$$

$$\le e^{2bD} \left(\frac{a+b}{a-b}\right)^2$$ and

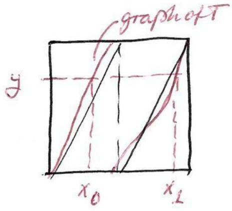
$$d_{K_a}(f, g) \le 2bD + 2 \log\left(\frac{a+b}{a-b}\right) \quad //$$

Remark For Birkhoff's contraction factor we get

$$\theta = \tanh \frac{\Delta}{4} = \frac{e^{\Delta/2} - 1}{e^{\Delta/2} + 1} \le \frac{(a+b) e^{bD} - (a-b)}{(a+b) e^{bD} + (a-b)}$$

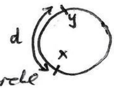
example: Uniformly expanding map on S^1

$T: S^1 \xrightarrow{C^2} S^1$, $T'(x) \geq \Lambda > 1 \ \forall x$ (or $T'(x) \leq -\Lambda < -1 \ \forall x$)



We define the Ruelle transfer operator for $f \in L^1(S^1)$:

$$\mathcal{L} f(y) = \sum_{\substack{x \in S^1 \\ T(x) = y}} \frac{1}{|T'(x)|} f(x)$$



We first consider $\mathcal{L}$ acting upon the subspace

$X = Lip(S^1, d)$ $\quad d(x,y) =$ min dist of x, y on the circle

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Lemma 5.4 Let $M = \sup_x \left|\frac{T''(x)}{T'(x)}\right| < \infty$. Then for any $a > \frac{M}{\Lambda - 1}$ the operator $\mathcal{L}$ is a strict contraction of the log-Lipschitz cone K_a and thus has a spectral gap on X.

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proof: Given y, y' at distance $\leq 1/2$ we may pair all preimages x_i, x_i', $i = 1, .., d_T \leftarrow$ degree of T such that $d(x_i, x_i') \leq \frac{1}{\Lambda} d(y, y')$

We have

$$\left|\log \frac{DT(x_i)}{DT(x_i')}\right| = \left|\int_{x_i}^{x_i'} \frac{D^2T(t)}{DT(t)} dt\right| \leq M |x_i - x_i'| \leq \frac{M}{\Lambda} d(y, y')$$

For $f \in K_a = \{\varphi : S^1 \to \mathbb{R}_+ : \varphi_x \leq e^{a d(x,y)} \varphi_y \ \forall x, y \in S^1\}$:

$$\mathcal{L} f(y') = \sum_i \frac{1}{|DT(x_i')|} f(x_i') \leq \sum_i \frac{1}{|DT(x_i)|} f(x_i) e^{(\frac{M}{\Lambda} + \frac{a}{\Lambda}) d(y, y')}$$
$$= \mathcal{L} f(y) e^{(\frac{M}{\Lambda} + \frac{a}{\Lambda}) d(y, y')}$$

Setting $b = \frac{1}{\Lambda}(M + a)$ we have $b < a$ provided $a > \frac{M}{1 - \Lambda}$ and the result follows from our estimates on $K_b \hookrightarrow K_a$.

Lemma[5.5] $\exists!\, h \in X = \mathrm{Lip}(S^1)$ ~~st~~: $dm = h\,dx$ ~~...~~ is a T-invariant probability.
One has $\forall f, g \in X;\ n \geq 1$

$$\left| \int f \circ T^n g\, dm - \int f\, dm \int g\, dm \right| \leq C \cdot \|f\| \cdot \|g\| \cdot \theta^{n-1} \quad (\theta < 1)$$

Proof: Recall the defining equation for $\mathcal{L}$:

$$\int f \circ T\, \phi\, dx = \int f \cdot (\mathcal{L}\phi)\, dy$$

$$\mathcal{L}\phi(y) = \sum_{x : Tx = y} \frac{1}{|T'(x)|} \phi(x)$$

Jacobian wrt Leb meas.

$\mathcal{L}$ contracts K_a (a large enough) strictly and has a spectral gap. $\exists h \in K_a^*$: $\mathcal{L}h = \lambda h$, $\lambda > 0$ unique when normalized.

Then

$$\int f \circ T \cdot h\, dx = \int f \cdot (\mathcal{L}h)\, dy = \int f \cdot \lambda h\, dy$$

Taking $f \equiv 1$: $\int h\, dx = \lambda \underbrace{\int h\, dy}_{>0} \Rightarrow \boxed{\lambda = 1}$

Normalize h s.t. $\int h\, dx = 1$. Then $\boxed{dm = h\, dx}$ is a probability and

$$\int f \circ T\, h\, dx = \int f \cdot \mathcal{L}h\, dy = \int f\, h\, dy$$

or $\int f \circ T\, dm = \int f\, dm$

Thus, m is T-invariant.

If $\tilde{h}$, $d\tilde{m} = \tilde{h}\, dx$ is such a meas. then

$$\int f \circ T\, \tilde{h}\, dx = \int f\, \mathcal{L}\tilde{h}\, dx = \int f\, \tilde{h}\, dx$$

for $f \in X$ implies $\mathcal{L}\tilde{h} = \tilde{h}$, but $\lambda = 1$ is a simple eigenvalue.

The calculation $\mathcal{L}^n(\phi) \xrightarrow[n]{} h \langle l, \phi \rangle$

$$\int \phi\, dx = \int \mathbb{1} \circ T^n \phi\, dx = \int \mathbb{1}\, (\mathcal{L}^n \phi)\, dx \to$$

$\int h\, dy \cdot \langle l, \phi \rangle = \langle l, \phi \rangle$ shows that

$$\boxed{\langle l, \phi \rangle = \int \phi\, dx}$$

For the correlation decay:
Recall: $\forall \phi \in X,\ n \geq 1$:

$$\|\mathcal{L}^n \phi - h \langle l, \phi \rangle\| \leq C \theta^{n-1} \|\phi\|$$

$\lambda = 1$

Note also that if $g, h \in X$ then $\|g \cdot h\| \leq \|g\| \cdot \|h\|$ (Banach algebra)

$$\left| \int f \circ T^n g\, h\, dx - \int f \cdot h\, dy \int g\, h\, dx \right| =$$

$$\left| \int f\, \mathcal{L}^n(gh)\, dy - \quad '' \quad \right| =$$

$$\left| \int f \cdot \left(\mathcal{L}^n(gh) - \left(\int gh\, dx\right) h \right) dy \right| =$$

$$\left| \int f \cdot \left(\mathcal{L}^n(gh) - h \cdot \langle l, gh \rangle \right) dy \right| \leq$$

$$\int |f|\, dy \cdot C \cdot \theta^{n-1} \|g \cdot h\| \leq$$

$$\tilde{C} \cdot \|f\| \cdot \|g\| \cdot \theta^{n-1} \quad //$$

Strong mixing and approximation

Thm 5.6 $(X, \mathcal{B}, \mu, T)$ a PMPT. Let $\mathcal{S} \subset L^1 \cap L^2$ be dense in L^1. Suppose that for all $A, B \in \mathcal{S}$:

$$\lim_{n\to+\infty} \int A \circ T^n B = \int A \int B$$

Then for any $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} \le 1$ we have for all $f \in L^p$, $g \in L^q$

$$\lim_{n\to+\infty} \int f \circ T^n g = \int f \int g \quad (*)$$

(all integrals w.r.t μ)

Proof: We first show that if $f, g \in L^\infty(\mu)$ then $(*)$ holds. Let $\varepsilon > 0$ and pick $A, B \in \mathcal{S}$ so that $|f - A|_2 \le \varepsilon$, $|g - B|_2 \le \varepsilon$.

Then:

$$\left|\int f \circ T^n g - \int A \circ T^n B\right| =$$
$$\left|\int (f\circ T^n - A \circ T^n) g + \int A \circ T^n (g - B)\right| \le$$
$$|f\circ T^n - A \circ T^n|_2 |g|_2 + |A \circ T^n|_2 |g - B|_2 =$$
$$|f - A|_2 |g|_2 + |A|_2 |g - B|_2 \le$$
$$\varepsilon \cdot |g|_2 + (|f|_2 + \varepsilon) \cdot \varepsilon =$$
$$\varepsilon (|f|_2 + |g|_2 + \varepsilon).$$

Similarly

$$\left|\int f \int g - \int A \int B\right| \le \varepsilon (|f|_2 + |g|_2 + \varepsilon)$$

Now using the mixing property for A and B we deduce

$$\limsup_{n\to\infty} \left|\int f \circ T^n g - \int f \cdot \int g\right| \le 2\varepsilon (|f|_2 + |g|_2 + \varepsilon)$$

and $\varepsilon > 0$ was arbitrary.

In the general case as in the theorem, let $a, b \in L^\infty$ be such that

$$|f - a|_p \le \varepsilon, \quad |g - b|_q \le \varepsilon$$

Repeating the above argument using Hölder-integral inequality we deduce again

$$\limsup \left|\int f\circ T^n g - \int f \int g\right| \le 2\varepsilon (|f|_p + |g|_q + \varepsilon)$$

and $\varepsilon > 0$ was arbitrary. //

Thm 5.7

Let $T: S^1 \to S^1$ be a uniformly expanding, C^2-map of S^1. (i.e $\exists \lambda > 1$: $|T'(x)| \ge \lambda$ $\forall x \in S^1$).

Then

a) There exists a unique T-invariant measure dm which is abs cont with respect to Lebesgue. ie $dm = h \cdot dx$

Furthermore

b) m is T-mixing, whence ergodic.

c) For Lebesgue a.e $x \in S^1$ and $\phi \in L^1(S^1, d\mathrm{Leb})$

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi \circ T^k(x) \xrightarrow[n\to\infty]{} \int \phi \, dm$$

proof: Our previous construction of $h \in \mathcal{K}_a$, $h > 0$, $dm = h\,dx$ yields a probability verifying a) and b) for functions in X. But X is dense in $L^2(dm)$ so thm 5.6 yields mixing + ergodicity. (Note that $\inf h > 0$ so $L^2(dm) \cong L^2(d\mathrm{Leb})$)

c): Note that m-a.e is the same as Leb-a.e. ($0 < \inf h \le \sup h < +\infty$). So c) follows from Birkhoff.

For any $f \in X$ we have for Leb a.e x

$$\frac{1}{n}\sum f \circ T^n(x) \to \int f \, dm$$

and as X is dense in C^0 this identifies uniquely the measure m //