# 4 Cone contraction and spectral gap.

example: The Perron-Frobenius theorem for positive matrices:

$A \in M_d(\mathbb{R})$, $A = (a_{ij})$ $a_{ij} > 0$

**Theorem 4.1**

$\lambda = r_{sp}(A) > 0$ and is a simple eigen value for A. Every other eigenvalue is of modulus strictly smaller.

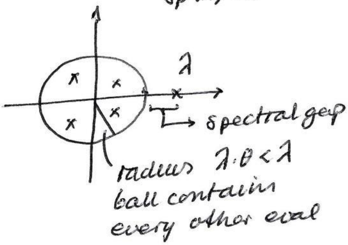
$\Updownarrow$

$\exists h \in \mathbb{R}^d$, (in fact $h \in ((\mathbb{R}_+^*)^d)$)
$\exists \ell \in \mathbb{R}^d$ (also $\ell \in ((\mathbb{R}_+^*)^d)$)
$\ell^t h = 1$, $\exists \lambda > 0$ s.t.
$\forall x \in \mathbb{R}^n$ (or $\mathbb{C}^n$)

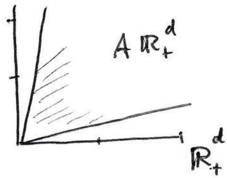
$$|\lambda^{-n} A^n x - h(\ell^t x)| \leq C\theta^n |x|$$

with $0 < \theta < 1$, $C < +\infty$.
The norm chosen is arbitrary
(but C depends upon the choice)

$sp(A) \subset \mathbb{C}$




Intuition:
$K = \mathbb{R}_+^d$ is a closed convex cone in $\mathbb{R}^d$, and $A : \mathbb{R}_+^d \to \mathbb{R}_+^d$ is a strict contraction (for the Hilbert metric)



Regularity of cones

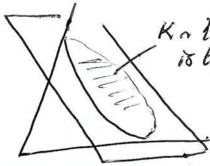
**Def 4.2** Let $K$ be an $\mathbb{R}$-cone in the Banach space $E$. We say that

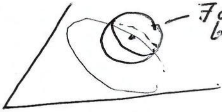
a) $K$ is outer regular iff $\exists\, l \in (E)^*,\ a < \infty$ s.t.

$\forall f \in K: \ \langle l, f\rangle \geq \frac{1}{a} \|f\|_E \|l\|_{E'}$

b) $K$ is inner regular iff

$\operatorname{int} K \neq \emptyset$




We will in our proofs use
more refined regularity
conditions
weaker

**Def 4.3** Let $K$ be an $\mathbb{R}$-cone in the Banach space $E$. We define the sectional aperture of $K$ (in $E$):

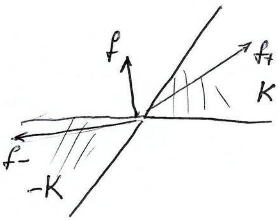
$$a = a(K,E) = \sup_{u,v \in K^*} \frac{|u|_E + |v|_E}{|u+v|_E} \in [1, +\infty]$$

**Def 4.4** $K$ $\mathbb{R}$-cone in $E$. We say that

a') $K$ is of bd sectional aperture iff $a(K,E) < +\infty$.

b') $K$ is regenerating in $E$ iff $\exists\, \gamma < \infty$ such that $\forall f \in E$ $\exists\, f_+, f_- \in K$: $f = f_+ - f_-$ and

$$|f_+|_E + |f_-|_E \leq \gamma \cdot |f|_E$$




Easy exercise: b) $\Rightarrow$ b')
inner reg $\Rightarrow$ regenerating

**Prop 4.5** $K$ is of $\alpha$-bd sectional aperture in $E$ iff $\forall x, y \in E$, $W = \mathrm{span}\{x, y\}$ there exists $m \in W'$ s.t. $\forall f \in K \cap W$:

(*) $|f|_E \leq m(f) \leq \alpha |f|_E$

Proof: (*) implies that for any $u, v \in W$:

$$|u| + |v| \leq m(u) + m(v) = m(u+v) \leq \alpha |u+v|$$

so $K$ is of $\alpha$-bd sect. aperture.

For the other direction take $x, y \in K$ with $x, y$ lin indep. $W = W_{x,y} = \mathrm{Span}\{x, y\}$. Now: $W \cap K$ is isom to $\mathbb{R}^2_+$ and we may find $u_1, u_2 \in K$, normalized so $|u_1| = |u_2| = 1$ and s.t.

$$W \cap K = \mathbb{R}_+(u_1, u_2) = \{a u_1 + b u_2 : a, b \geq 0\}$$

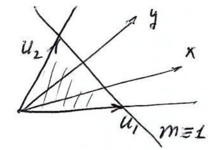
We now define the linear functional $m \in W'$ by declaring

$$m(a u_1 + b u_2) := a + b.$$

Setting $f = a u_1 + b u_2$ we have

$$|f|_E = |a u_1 + b u_2| \leq a + b = m(f)$$
$$= |a u_1| + |b u_2| \leq \alpha |a u_1 + b u_2|$$
$$= \alpha |f|_E$$ so indeed $\forall f \in K \cap W$

$$|f|_E \leq m(f) \leq \alpha |f|_E$$




**4.6 Remark**: The above Prop shows

$a) \Rightarrow a')$

outer reg $\Rightarrow$ bd sect. apert.

**Thm 4.7** Let $\alpha = \alpha(K; V)$ be the sectional aperture of $K$ in $V$. Then $\forall x, y \in K^*$:

$$\left| \frac{x}{|x|_E} - \frac{y}{|y|_E} \right|_E \leq 2 \alpha\, d_K(x, y)$$

proof: Given $x, y \in K^*$ lin indep. we construct $u_1, u_2$ and $m \in W'$ as above. Normalize $x, y$ so that $m(x) = m(y) = 1$

Set $u_t = \frac{1-t}{2} x + \frac{1+t}{2} y$, $t \in \mathbb{R}$.

Then $m(u_t) \equiv 1$, $\forall t \in \mathbb{R}$.

Define

$$J = \{t \in \mathbb{R} : u_t \in \mathbb{R}\} = [t_1, t_2] \supset [-1, 1]$$

One has $u_{t_1} = u_1$, $u_{-1} = x$, $u_1 = y$, $u_{t_2} = u_2$ (with this ordering) with $u_1, u_2$ as above.

When $t \in J$ the property (*) shows that

$$|u_t|_E \leq m(u_t) = 1 \leq \alpha |u_t|_E$$

so

$$|t(x-y)|_E = |t|\, |x-y|_E = |2u_t - x - y| \leq 4$$
$$\Rightarrow |t| \leq 4/|x-y|_E =: \rho \ (\geq 2)$$

Therefore $|t_1|, |t_2| \leq \rho$ which gives the bound for the Hilbert distance:

$$d_K(x, y) = \log \frac{t_2+1}{t_2-1} \frac{1-t_1}{1+t_1}$$
$$\geq \log \left(\frac{\rho+1}{\rho-1}\right)^2$$
$$= 2 \log \left(\frac{1+1/\rho}{1-1/\rho}\right) \Rightarrow$$

$$\frac{|x-y|_E}{4} = \frac{1}{\rho} \leq \tanh \frac{d}{4} \leq \frac{d}{4} \text{ so}$$

$$|x-y|_E \leq d_K(x, y).$$

Now since $1 = m(x) \leq \alpha |x|_E$

$$\left| \frac{x}{|x|} - \frac{y}{|y|} \right| = \left| \frac{x-y}{|x|} + y \frac{|y| - |x|}{|y||x|} \right|$$
$$\leq 2 \frac{|x-y|}{|x|} \leq 2 \alpha\, d_K(x, y)$$

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Thm 4.8 $K\subset E$ an $\mathbb{R}$-cone in a Banach space $E$ of $\mathcal{A}$-bd sectional aperture.

Let $T\in L(E)$, $T(K^*)\subset K^*$ and

$\Delta = \operatorname{diam}_K TK < +\infty$.

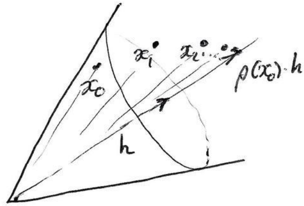
Then

1.) $K$ contains a unique $T$-invariant half-line $\mathbb{R}_+h$, $h\in K$, $|h|=1$

2) Set $Th=\lambda h$, $\lambda>0$. There are const^s $M, C<\infty$, $0<\theta<1$ and a fct $\rho: K\to\mathbb{R}_+$:

$\forall x\in K, n\geq 1$ $|\lambda^{-n}T^n x - h\rho(x)|_E \leq C\theta^{n-1}|x|_E$

$\rho(x)\leq M|x|_E$. $\rho(x)=0, x\in K \Leftrightarrow x=0$




Proof: Pick $e_0\in K$, $|e_0|_E=1$. Set $e_{n+1}=\frac{Te_n}{|Te_n|_E}$, $n\geq 0$

Then for $n+m>n\geq 1$:

$$d_K(e_{n+m}, e_n) = d_K(Te_{n+m-1}, Te_{n-1})$$
$$\leq \theta\cdot d(e_{n+m-1}, e_{n-1}) \leq \dots \leq$$

Birkhoff $\theta=\tanh\frac{\Delta}{4}$

$$\theta^{n-1} d(e_{m+1}, e_1) \leq \theta^{n-1}\Delta.$$

Since $K$ is of $\mathcal{A}$-bd sect. aperture:

$$|e_{n+m}-e_n|_E \leq 2\mathcal{A}\Delta\cdot\theta^{n-1}, \quad (*)$$

The sequence $(e_n)_{n\geq 0}$ is thus Cauchy, whence converges

$$h := \lim_n e_n \in K, \quad |h|=1.$$

We have by cont of $T$:

$$Th = \lim_n Te_n = \lim_n e_{n+1}|Te_n|$$
$$= h\cdot|Th| =: \lambda h, \quad \lambda>0$$

since $Th\in K^*$

So indeed $\mathbb{R}_+h$ is $T$-invariant ($h$ is an eigenvector of $T$).

If $k\in K^*$ is another $T$-eigenvector then $d_K(h,k) = d_K(Th, Tk)\leq\theta d_K(h,k)$ forces $h\parallel k$ whence unicity of $\mathbb{R}_+h$.

For 2, take $x\in K^*$, $x_0:=x$, and define $x_n = \lambda^{-n}T^n x_0 = \lambda^{-1}Tx_{n-1}$, $n\geq 1$

$\mathcal{A}$ bd-sect. aperture yields by (*), letting $m\to\infty$

$$\left|h - \frac{x_n}{|x_n|_E}\right|_E \leq 2\mathcal{A}\Delta\cdot\theta^{n-1}$$

or equivalently $\forall n\geq 1$

$$|x_n - |x_n|_E h|_E \leq 2\Delta\mathcal{A}\theta^{n-1}|x_n|_E$$

(valid also when $x=0$). We have

$$|x_{n+1}-x_n|_E = |(\lambda^{-1}T-\mathbb{1})x_n|_E$$

since $Th=\lambda h$

$$= |(\lambda^{-1}T-\mathbb{1})(x_n - |x_n|_E h)|_E$$
$$\leq \|\lambda^{-1}T-\mathbb{1}\|\cdot 2\Delta\mathcal{A}\cdot\theta^{n-1}|x_n|_E$$
$$=: q\cdot\theta^{n-1}|x_n|_E.$$

In particular

$$|x_{n+1}|_E \leq (1+q\theta^{n-1})|x_n|_E \Rightarrow$$
$$|x_n|_E \leq \prod_{n\geq 1}(1+q\theta^{n-1})\,\|\lambda^{-1}T\|\cdot|x|_E$$
$$=: M|x|_E \quad M<\infty \text{ indep of } x.$$

Thus,

$|x_{n+1}-x_n| \leq Mq\theta^{n-1}|x|$ whence

$|x_{n+m}-x_n| \leq \frac{Mq}{1-\theta}\theta^{n-1}|x|$ showing that $(x_n)_{n\geq 0}$ is Cauchy.

Set $\rho(x) := \lim_n |x_n| \leq M|x|$

$$|x_n - \rho(x)h| \leq |x_n - |x_n|h| + ||x_n| - \rho(x)|$$
$$\leq \left(2\Delta\mathcal{A} + \frac{q}{1-\theta}\right) M\theta^{n-1}|x|$$
$$=: C\theta^{n-1}|x|, \quad C \text{ indep de } x.$$

Since we also have

$$|x_{n+1}|_E \geq (1-q\theta^{n-1})|x_n|_E$$

pick $N$ s.t. $q\theta^{n-1}\leq 1/2$, $n\geq N$

Then

$$\rho(x) \geq \Big(\overbrace{\prod_{n\geq N}(1-q\theta^{n-1})}^{>0}\Big)|x_N|_E$$

implying $\rho(x)>0$ si $x\neq 0$.

**Thm 4.9** (Spectral gap).

Let $K$ be an $\mathbb{R}$-cone in a real B-sp $E$, of bd sectional aperture and regenerating in $E$.
Let $T \in L(E)$, $TK^* \subset K^*$ with
$$\Delta = \operatorname{diam}_K TK < +\infty.$$
Then $\exists!\, \lambda>0$, $h \in K^*$, $|h|=1$, $\ell \in E'$
s.t. $\langle \ell, h\rangle = 1$ and
$$\forall x\in E,\ n\geq 1 \quad |\lambda^{-n}T^n x - h\langle \ell, x\rangle| \leq C\theta^{n-1}|x|_E$$
with $C<\infty$, $0<\theta<1$. One also has.
$$\forall x \in K^*: \ \langle \ell, x\rangle > 0.$$

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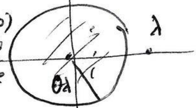
Rem: One may take $\theta = \tanh \frac{\Delta}{4}$

Interpretation: $\lambda>0$ is a simple eigenvalue of $T$ with spectral projection $h\otimes \ell$. (1-dim)

The difference $T-\lambda h\otimes\ell$ has spectral radius $\leq \theta\cdot\lambda < \lambda$.

(Const $\gamma<\infty$)




Proof: Being regenerating given $x \in E$ there are $x_+, x_- \in K$ with
$$|x_+|+|x_-| \leq \gamma|x|, \quad x = x_+ - x_-.$$
By the previous thm:
$$|\lambda^{-n}T^n x_+ - h\rho(x_+)| \leq C\theta^{n-1}|x_+|$$
$$|\lambda^{-n}T^n x_- - h\rho(x_-)| \leq C\theta^{n-1}|x_-|$$
Set $\rho(x) := \rho(x_+) - \rho(x_-)$. We have
$$|\lambda^{-n}T^n x - h\rho(x)| \leq C\theta^{n-1}(|x_+|+|x_-|)$$
$$\leq C\theta^{n-1}\gamma|x|$$
Thus, $\lim_n \lambda^{-n}T^n x = h\cdot\rho(x)$
showing that $\rho(x)$ is indep. of the choice of decomposition.
Also $|\rho(x)| \leq M|x_+| + M|x_-| \leq M\cdot\gamma|x|$

Given $x, y \in E$, $a, b \in \mathbb{R}$:
$$\lambda^{-n}T^n(ax+by) = a\cdot\lambda^{-n}T^n x + b\lambda^{-n}T^n y$$
$$\downarrow n \qquad\qquad \downarrow n \qquad\qquad \downarrow n$$
$$h\cdot\rho(ax+by) = h\cdot a\rho(x) + b\rho(y)\cdot h$$
implying linearity of $x \mapsto \rho(x)$

thus we may write
$$\rho(x) =: \langle \ell, x\rangle$$
with $\ell \in E'$, $|\ell|_{E'} \leq M\cdot\gamma$.

For $x \in K^*$, $\rho(x)>0$ implying the last claim //