3. The Hilbert metric & Birkhoff's inequality

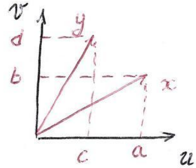
3.1 Cross-ratios [Bi-rapport]

Let E be a topological vector space of dimension ≥ 2.

Let $u, v \in E$ be linearly independent. Then any $x, y \in V = \operatorname{Span}\{u, v\}$ may be written uniquely in a unique way as:

$$x = au + bv$$
$$y = cu + dv$$

for some $a, b, c, d \in \mathbb{R}$.



Def 3.1 Suppose that x, y are non-zero and that no three vectors among u, v, x, y are parallel. We then define the cross-ratio of x, y w.r.t. u, v:

$$[x, y\,;\, u, v] := \frac{a \cdot d}{b \cdot c} \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$$

The cross-ratio is a projective invariant:

Prop 3.2 $[\lambda x, \mu y\,;\, \alpha u, \beta v] = [x, y\,;\, u, v] \quad \forall \lambda, \mu, \alpha, \beta \in \mathbb{R}^*$

proof: e.g. $[\lambda x, y\,;\, u, v] = \frac{(\lambda a)\, d}{(\lambda b) c} = \frac{ad}{bc}$

$$[x, y\,;\, \alpha u, v] = \frac{(a/\alpha)\, d}{b \cdot (c/\alpha)} = \frac{ad}{bc}.$$

The cross-ratio is a linear invariant:

Prop 3.3 Let $A \in L(E)$ be such that Au, Av are linearly independent. Then $[Ax, Ay\,;\, Au, Av] = [x, y\,;\, u, v]$

Proof: $Ax = a \cdot Au + b \cdot Av$
$Ay = c \cdot Au + d \cdot Av$
and the result follows

Prop 3.4 (Symmetries)

$[y, x\,;\, u, v] = [x, y\,;\, v, u] = 1/[x, y\,;\, u, v]$

If also x, y are lin. indep. then:

$$[u, v\,;\, x, y] = [x, y\,;\, u, v]$$

proof: $\frac{c \cdot b}{a \cdot d} = \left(\frac{a \cdot d}{b \cdot c}\right)^{-1}$

For the second, note that

$$dx - by = \underbrace{(ad - bc)}_{\neq 0} u$$
$$-cx + ay = \underbrace{(ad - bc)}_{\neq 0} v$$

so using 3.2:

$$[u, v\,;\, x, y] = \frac{d \cdot a}{(-c)(-b)} = \frac{ad}{bc}. \;/\!/$$

Remark (exc) one also has

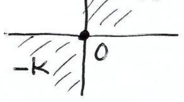
$$[u, y\,;\, x, v] = 1 - [x, y\,;\, u, v]$$

[The permutation group S_4 acts upon the cross-ratio. The orbit has 6 points]

Def 3.5 E real top v.sp.
A subset $K \subset E$ containing at least two points is called a proper, convex, closed cone (we call it an $\mathbb{R}_+$-cone for abbreviation) iff K is topol. closed and

1) $\mathbb{R}_+ K = K$
2) $K + K = K$
3) $K \cap (-K) = \{0\}$

<u>ex</u> to have in mind $\mathbb{R}_+^n$



Def 3.6 Given $x, y \in K^* = K \setminus \{0\}$ we set
$$\beta(x,y) = \inf\{t>0 : tx - y \in K\} \in (0, +\infty]$$
and then
$$d_K(x,y) = \log(\beta(x,y)\cdot\beta(y,x)) \in (-\infty, +\infty]$$

Prop 3.7 d_K defines a projective metric on K^*:

1) $d_K(x,y) \geq 0$ and $= 0$ iff $x \parallel y$
2) $d_K(\lambda x, y) = d(x, \lambda y) = d(x,y)$, $\lambda > 0$
3) $d_K(x,y) = d_K(y,x)$
4) $d_K(x,z) \leq d_K(x,y) + d_K(y,z)$

Proof: Let $t_1 = \beta(x,y)$, $t_2 = \beta(y,x)$
1) Both are >0 since K is closed.
Also
$u := t_1 x - y \in K$, $v = t_2 y - x \in K$
$\Rightarrow t_2 u + v = (t_1 t_2 - 1) x \in K$
$\Rightarrow t_1 t_2 - 1 \geq 0$ so $d_K(x,y) \geq 0$
If $= 0 \Rightarrow u = v = 0$ and $x \parallel y$
converse trivial.

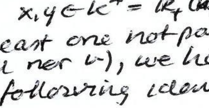
$u = t_1 x - y \in K$
$v = \frac{1}{t_1} y - x = -\frac{1}{t_1} u \in K$

Use
2) $\beta(\lambda x, y) = \frac{1}{\lambda}\beta(x,y)$ and
$\beta(y, \lambda x) = \lambda\beta(y,x)$
3) clear
4) $t_1 x - y$, $s_1 y - z \in K \Rightarrow$
$t_1 s_1 x - z \in K \Rightarrow$
$\beta(x,z) \leq t_1 s_1 = \beta(x,y)\beta(y,z)$
$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$ //

Given independent vectors $u, v \in E^*$ we write
$$\mathbb{R}_+(u,v) = \{tu + sv : t, s \geq 0\}$$
for the $\mathbb{R}_+$-cone generated by the two vectors.

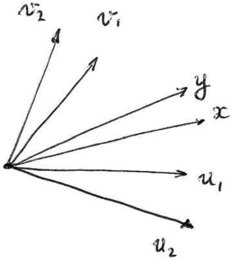
Prop 3.8 Given indep $u, v \in E^*$ and $x, y \in K^* = \mathbb{R}_+(u,v)^*$ (at least one not parallel to u nor v), we have the following identity:
$$d_K(x,y) = |\log|R[x,y;u,v]||$$
$$(\text{def}) =: d_{u,v}(x,y)$$

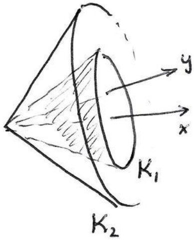
proof: $t_1 = \beta(x,y)$ $t_2 = \beta(y,x)$
Suppose both finite. Then
$u' = t_1 x - y \parallel u$ (or v)
$v' = t_2 y - x \parallel v$



(or exchanged)

$\Rightarrow t_2 u' + v' = (t_1 t_2 - 1) x$
$u' + t_1 v' = (t_1 t_2 - 1) y$
$\Rightarrow R[x,y;u,v] = R[x,y;u',v']$
$= \pm\frac{t_1 \cdot t_2}{1 \cdot 1}$ (or $\pm\frac{1 \cdot 1}{t_1 \cdot t_2}$)
and $d_K(x,y) = \log t_1 t_2$
by def 3.6. //





Lemma 3.9 [Birkhoff's inequality]

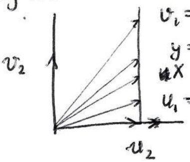
~~Suppose~~ Let $x, y \in \mathbb{R}_+[u_1, v_1] \subset \mathbb{R}_+[u_2, v_2]$ with x, y independent.

Suppose $d_{u_2,v_2}(u_1, v_1) \le \Delta \in [0, +\infty]$

Then

$$d_{u_2,v_2}(x,y) \le \tanh\frac{\Delta}{2}\, d_{u_1,v_1}(x,y)$$

Proof: In the basis (u_2, v_2) and scaling (positive factor) we may assume that



$v_1 = (1, b)$ $\quad a<c<d<b$

$y = (1, d)$

$x = (1, c)$

$u_1 = (1, a)$

$u_1 = u_2 + a \cdot v_2$ $\quad 0 \le a \ne b \le +\infty$

$v_1 = u_2 + b \cdot v_2$ $\quad$ (assume $a<b$)

$$\Delta = d_{u_2 v_2}(u_1, v_1) = \left|\log\left|\frac{1\cdot b}{1\cdot a}\right|\right| = \log\frac{b}{a}$$

~~Assume $0 \le a < b \le \infty$~~

$x = u_2 + c \cdot v_2$ $\quad a \le c \ne d \le b$

$y = u_2 + d \cdot v_2$ $\quad$ (assume $c<d$)

(I) $$d_{u_2,v_2}(x,y) = \left|\log\frac{d}{c}\right| = \int_c^d \frac{1}{t}\,dt$$

We write x, y as lin combinations of u_1, v_1:

$$x = \frac{b-c}{b-a} u_1 + \frac{c-a}{b-a} v_1$$

$$y = \frac{b-d}{b-a} u_1 + \frac{d-a}{b-a} v_1$$

so the Hilbert dist

$$d_{u_1,v_1}(x,y) = \left|\log\frac{(b-c)(d-a)}{(c-a)(b-d)}\right| =$$

$$\left|\log\frac{d-a}{b-d} - \log\frac{c-a}{b-c}\right| =$$

$$\int_c^d \frac{b-a}{(b-t)(t-a)}\,dt \quad \text{(II)}$$

In order to bound the ratios of integrands we write

$$\eta = \sup_{a \le t \le b} \frac{1}{t}\,\frac{(b-t)(t-a)}{b-a}$$

$$\begin{pmatrix}\text{max}\\ \text{for}\\ t=\sqrt{ab}\end{pmatrix} = \frac{(b-\sqrt{ab})(\sqrt{ab}-a)}{\sqrt{ab}\,(b-a)}$$

$$= \frac{(\sqrt{b}-\sqrt{a})^2}{b-a} = \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}$$

$$= \frac{\sqrt{b/a}-1}{\sqrt{b/a}+1} = \frac{e^{\Delta/2}-1}{e^{\Delta/2}+1} = \tanh\frac{\Delta}{4}$$

Finally:

$$d_{u_2,v_2}(x,y) = \int_c^d \frac{1}{t}\,dt$$

$$\le \int_c^d \eta \cdot \frac{b-a}{(b-t)(t-a)}\,dt$$

$$= \eta \cdot d_{u_1,v_1}(x,y)$$

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Thm 3.10 [Garett Birkhoff 56-57]
(son of George David Birkhoff)

Let $T: V \to V$ be a cont lin map.

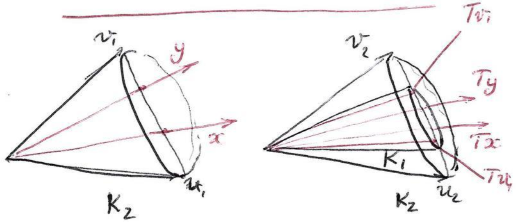
Let $K_1 \subset K_2 \subset V$ be embedded proper closed cones, s.t.

$$T(K_2^*) \subset K_1^*$$

Set $\Delta = \operatorname{diam}_{K_2} K_1 = \sup_{x,y \in K_1^*} d_{K_2}(x,y) \le +\infty$

Then for any $x, y \in K_2^*$:

$$d_{K_2}(Tx, Ty) \le \tanh \tfrac{\Delta}{4}\, d_{K_2}(x,y)$$



Proof: We may assume that $Tx \nparallel Ty$ (or else the LHS is zero)

Then $\operatorname{Span}\{x,y\} \cap K_2 = \mathbb{R}_+(u_1, v_1)$
and $\operatorname{Span}\{Tx, Ty\} \cap K_2 = \mathbb{R}_+(u_2, v_2)$ } in fact we only need that these are closed (not K_1, K_2)

We have (using The Birkhoff lemma)

$$d_{K_2}(Tx, Ty) = d_{u_2, v_2}(Tx, Ty) \le$$

$$\left(\tanh \tfrac{1}{4} d_{u_2, v_2}(Tu_1, Tv_1)\right) d_{Tu_1, Tv_1}(Tx, Ty) \le$$

$$\tanh \tfrac{\Delta}{4} \cdot d_{u_1, v_1}(x,y) =$$

$$\tanh \tfrac{\Delta}{4} \cdot d_{K_2}(x,y) \quad //$$