Chap 2 Meas preserving Dyn Sys

Def and Prop 2.1

$T : M \to M$ $\quad (M, \mathcal{B}, \mu)$ proba

is measure preserving iff

1) $\forall A \in \mathcal{B} : \mu(A) = \mu(T^{-1}A)$ iff
2) $\forall f \in L^1(\mu) : \int f \, d\mu = \int f \circ T \, d\mu$

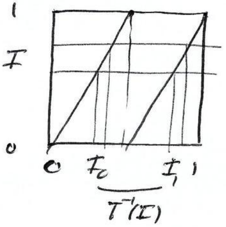
$2) \Rightarrow 1)$ $\quad f = \mathbb{1}_A : \int \mathbb{1}_A \, d\mu =$

$\int f \, d\mu =$ ~~[illegible]~~ $\int f \circ T \, d\mu =$

$\int \mathbb{1}_A \circ T \, d\mu = \int \mathbb{1}_{T^{-1}A} \, d\mu$ //

$1) \Rightarrow 2)$ ok for indicator fct, ok for linear comb, Leb. ~~Dom~~ Monotone conv //

ex $T(x) = 2x \mod \mathbb{Z}$ (circle doubling map)



$\lambda =$ Lebesgue

$|I_0| = |I_1| = \frac{1}{2}|I|$

~~[illegible]~~

$\lambda(I) = \lambda(T^{-1}I)$

intervals generate $\mathcal{B} \Rightarrow \lambda(A) = \lambda(T^{-1}A)$

so T is λ-preserving

Notation: (symmetric difference)

$A \triangle B := (A \cup B) \setminus (A \cap B)$

Given a measure μ

$A \equiv B \ (\overset{\mu}{\text{mod}}\ 0) \Leftrightarrow$

$\mu(A \triangle B) = 0 \Leftrightarrow$

$\mathbb{1}_A = \mathbb{1}_B$ in $L^1(\mu)$

Definition 2.2

$(M, \mathcal{B}, \mu, T)$ meas. pr. tr. proba

is said to be ergodic iff

$\forall A \in \mathcal{B}$: $\quad \mathbb{1}_A = \mathbb{1}_A \circ T$ in $L^1(\mu)$

$\mu(A \triangle T^{-1}A) = 0 \Rightarrow$

$\mu(A) = 0$ or 1.

---

ex: $T(x) = 2x \mod \mathbb{Z}$

$\mu = \delta_0$ is invariant and ergodic.

$T(0) = 0$

$\mu(A) = \mu(T^{-1}A) = \begin{cases} 1 & \text{iff } A \ni 0 \\ 0 & \text{if not} \end{cases}$

ex $T(x) = 2x \mod \mathbb{Z}$

$\mu =$ Leb. is invar and ergodic (exo class).

Prop 2.3 ~~Remark~~: If a meas fct f is a.s. T-invariant and μ is ergodic then f is a.s. constant

proof: If f is not a.s. constant one may find $v \in \mathbb{R}$ so that eq

$\mu(f > v) > 0$ and

$\mu(f \le v) > 0$

But $\{f > v\}$ and $\{f \le v\}$ are T-invar mod 0, so one has to have meas 1, the other 0.

Strong mixing

Def 2.5
$(X, \mathcal{B}, \mu, T)$ proba mesure preserving transformation is said to be strongly mixing provided that

$\forall f \in L^\infty(\mu), g \in L^{\infty}(\mu)$

$$\lim_{n\to\infty} \int_X f\circ T^n \cdot g \, d\mu = \int_X f d\mu \int_X g d\mu$$

(alternative def with L^2) ~~equivalent~~ / or L^p, L^q $\frac{1}{p}+\frac{1}{q}=1$)

Prop: 2.6
strong mixing $\Rightarrow$ ergodicity

Suppose $1\!\!1_A = 1\!\!1_A \circ T$ in $L^1(\mu)$, $A \in \mathcal{B}$

Then also $1\!\!1_A = 1\!\!1_A \circ T$ in $L^\infty(\mu)$

$1\!\!1_A = \dots = \underbrace{1\!\!1_A \circ T^n}_{= 1\!\!1_A}$ $\forall n$ in $L^\infty(\mu)$

Thus

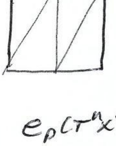
$$\lim_{n\to\infty} \int 1\!\!1_A \circ T^n \, 1\!\!1_A = \int 1\!\!1_A \cdot 1\!\!1_A$$

$= \int 1\!\!1_A \cdot \int 1\!\!1_A$ so

$\mu(A) = \mu(A)^2 \Rightarrow \mu(A) = 0$ or 1

so T is μ-ergodic.

ex: $T(x) = 2x \mod 1$, $\mu =$ Leb. is strongly mixing whence ergodic



$e_p(x) = \exp(2\pi i \, p \cdot x)$

$e_p(Tx) = \exp(2\pi i \, p \cdot 2x)$
$= e_{2p}(x)$

e_1 e_2 e_4

$e_p(T^n x) = e_{2^n p}(x)$.

In particular if either p or q are non-zero $(e_p \circ T^n, e_q) = 0$ for n large

$$\lim_{n\to\infty} \Big(\sum_{\text{finite}} a_k e_k \circ T^n, \sum_{\text{finite}} b_\ell e_\ell\Big) = a_0 b_0$$

So ok for trigo polyn's, dense in L^2

Alternative:
For $p \neq 0$

$\int e_p \circ T^n g \, dx = \int e_{2^n p} g \, dx \xrightarrow[n\to\infty]{} 0$

Riem Leb.

So for $P = \sum_{\text{finite}} a_p e_p$

$\int P \circ T^n \cdot g \, dx \xrightarrow[n]{} a_0 \cdot \int g \, dx$

For general $f \in L^\infty, g \in L^1$ for $\varepsilon > 0$ find P st $\|P - f\|_2 < \varepsilon$. Then

$|\int P\circ T^n g - f\circ T^n g| \le$

$|\int (P - f)\circ T^n \cdot g| \le$

$|\int |P - f|| \, |g|_\infty \le \varepsilon |g|_\infty$

and $\int P \circ T^n g \xrightarrow[n]{} 0$

Thm 2.4 [Birkhoff] I.

Let $(X, \mathcal{B}, \mu, T)$ (proba.) be a meas. preserving transformation.
Then $\forall f \in L^1(\mu)$: the sequence $(\frac{1}{n} S_n f)_{n \geq 1}$ converge ~~a.s~~ μ-a.s (and in $L^1(\mu)$) towards a fct $\bar{f} \in L^1(\mu)$ which is T-invariant.

(admitted: ~~Ergth course~~)

$S_n f = f + f\circ T + \dots + f\circ T^{n-1}$

Thm [B.] 2.4 II.
If in addition μ is T ergodic then $\bar{f} = \int f \, d\mu$ a.e.
(i.e is ~~a.s~~ constant a.s.)

ex: $T(x) = 2x \mod 1$, λ = Leb.

$f(x) = \cos(2\pi x)$ $\quad \int f \, d\lambda = 0$

$S_n f(x) = \sum_{k=0}^{n-1} \cos(2\pi \cdot 2^k x)$

$S_n f(0) = \sum_0^{n-1} 1 = n$

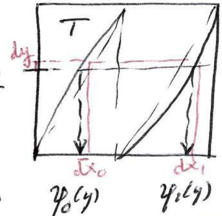
$\frac{1}{n} S_n f(0) = 1 \neq 0 = \int f \, d\lambda$.

Nevertheless $\frac{1}{n} S_n f(x) \to 0$ Leb-a.s. x

Later on we will see that
$\frac{1}{\sqrt{n}} S_n f(X) \xrightarrow{law} N(0, \sigma^2)$
$X \sim U([0,1])$ law μ.

$f(x) \in L^1([0,1])$ any fct
For Leb a.e $x \in [0,1]$
$\frac{1}{n} \sum_{k=0}^{n-1} f\circ T^k(x) \longrightarrow \int_0^1 f(s)\,ds$

Iterating a fct by
$T(x) = T_\mu(x) = 2x + \mu \cdot \sin(2\pi x) \mod 1$
is "bad" ($\mu \neq 0$, small)



pre-images.

Use duality to cast to iterations by contracting pre-images! "Transfer Operator"

Given $f \in L^\infty([0,1])$, $\phi \in L^1([0,1])$
write
$$\int_0^1 f\circ T(x)\,\phi(x)\,dx = \int_0^1 f(y)\Big[\sum_{x:Tx=y}\frac{\phi(x)}{|T'(x)|}\Big]dy = \int_0^1 f\cdot(\mathcal{L}\phi)\,dy$$
$$\mathcal{L}\phi(y) = \sum_{x:Tx=y}\frac{\phi(x)}{|T'(x)|} = \sum_{k=0}^{1} |\psi_k'(y)|\cdot(\phi\circ\psi_k(y))$$

Idea [Ruelle] Replace study of $f\circ T^n$ by the study of $\mathcal{L}^n\phi$
(non-lin. dynamics in finite dim ⇝ linear dynamics in ∞-dim).

Def 2.7 Given $T\in C^r(M;M)$, $r\geq 1$, M compact, unif expanding
We ~~say that~~ define for $g\in C^r(M)$ $g>0$
the Ruelle transfer operator
$$\mathcal{L}_{\cancel{g}T,g}(y) := \sum_{x:Tx=y} g(x)\,\phi(x),$$

with $\phi\in X$ some Banach space
$X =$ ~~L...~~ $L^\infty(M)$, $C^0(M)$, $C^r(M)$, ... $r>0$
$Lip(M)$