Systèmes Dynamiques

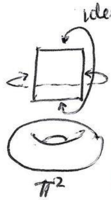
1. Intro

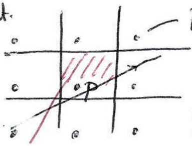
$f : M \to M$, $f^n : M \to M$ — mes. — topo — diff

(M, g) variété Riemannienne

$\mathbb{R}^n$, $I = [0,1]$, $S^1 = \mathbb{R}/\mathbb{Z}$

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$



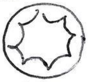


$\Gamma = \mathbb{Z}^2$ action de groupe discret additive sur $\mathbb{R}^2$

domaine fondamental

$S^2 \hookrightarrow \mathbb{R}^3$ plongée

← variété hyperbolique



$\circlearrowleft \Gamma$ groupe multipl

$(\mathbb{D}, g_{\mathbb{D}})$ disque de Poincaré

métrique très particulière

dyn① $\dim_H$ des ensembles de Julia

$f_c(z) = z^2 + c$ $J_c = \partial \{ z : f_c^n(z) \to \infty \}$ $\dim_H(J_c)$

Sys Dyn (temps discret)

$f \in C^0(M; M)$, $f \in C^r(M; M)$, $r \geq 1$

$f \in \mathrm{Diff}^r(M)$, $r \geq 1$

f^n, $n \geq 0$ (cas non-inversible)

f^n, $n \in \mathbb{Z}$ (cas inversible)

Sys Dyn (à temps continue)

$X \in \mathfrak{X}(M)$ champ de vecteurs C^r

(M compact)

$\phi_t = X^t \in \mathrm{Diff}^r(M)$

$\phi_t \circ \phi_s = \phi_{t+s}$, $\phi_0 \equiv \mathrm{Id}$.

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1. Sys Dyn à temps discret

$f \in C^0(M; M)$

Déf 1.1. $\mathrm{Fix}(f) = \{ x \in M : x = f(x) \}$

$\forall n \geq 1$: $\mathrm{Per}_n(f) = \mathrm{Fix}(f^n)$

points périodiques de période n.

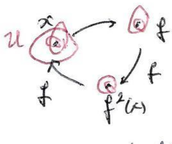
$\mathcal{P}_n(f) = \mathrm{Per}_n(f) \setminus \bigcup_{j=1}^{n-1} \mathrm{Per}_j(f)$

pts pér^s de période primitive n

— (notre déf!)

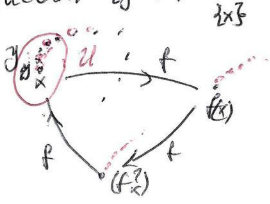
$x \in \mathrm{Per}_n(f)$ est dit attractif ssi

$\exists\, \mathcal{U} \in \mathcal{O}(x)$: $\bigcap_{k \geq 0} f^{nk}(\mathcal{U}) = \{x\}$



$x \in \mathrm{Per}_n(f)$ est dit répulsif ssi

$\exists\, \mathcal{U} \in \mathcal{O}(x)$: $\{ y : f^k(y) \in \mathcal{U}, \forall k \geq 0 \} = \{x\}$



Lorsque $f\in C^r(M;M)$, $r\geq 1$

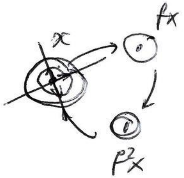
un pt per $x\in Per_n(f)$ est dit

stable si $\forall \lambda\in$ val pr (f^n_{*x}) : $|\lambda|<1$

instable si $\forall \lambda\in$ v.p. (f^n_{*x}) : $|\lambda|>1$

hyperbolique si $\forall \lambda\in$ vp... : $|\lambda|\neq 1$

$$f^n_{*x_0} = \frac{\partial}{\partial x} f^{(n)}(x)\Big|_{x=x_0}$$ — Jacobienne de $f^{(n)}$ en x_0



ex $f(x)=\begin{pmatrix}1/2 & 0\\ 0 & 1/3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

$f:\mathbb{R}^2\circlearrowleft$

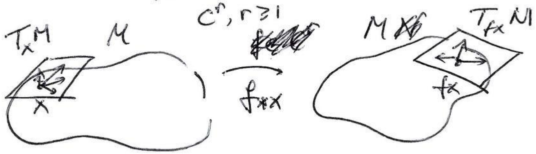
$\begin{pmatrix}0\\0\end{pmatrix}$ seul pt per de per. = 1

stable/attractif.

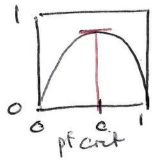
Remarque : Pour un pt per $x\in Per_n(f)$

x stable $\Rightarrow$ x attractif

x instable $\Rightarrow$ x repulsif.



ex



$f(x)=ax(1-x)$

$0<a<4$

$Crit(f)=\{\frac{1}{2}\}$

$Reg(f) = \{x\in M : f_{*x}\in L(T_xM; T_{fx}M)$ est inversible$\}$

$Crit(f) = M\setminus Reg(f)$

(M,g) variété riem.

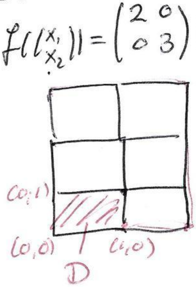
$f\in C^r(M;M)$, $r\geq 1$ est dite uniformément dilatante ssi $\exists \lambda>1$:

$\forall v\in T_xM$: $\|f_{*x}v\|_{T_{fx}M}\geq \lambda\|v\|_{T_xM}$

ex :

ex :

$f\left(\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right)=\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$



f est uniformément dilatante avec plein de pts per.

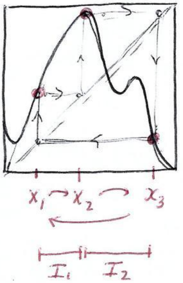
Thm de Šarkovskii (1964)

version simplifiée [Lee-Yorke '75]

Soit $f : [0,1] \circlearrowleft$ continue.

$\mathcal{P}_3(f) \neq \emptyset \Rightarrow \forall n \geq 1 \; \mathcal{P}_n(f) \neq \emptyset$

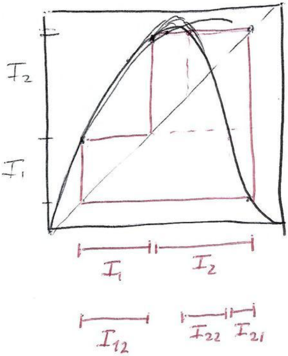
(c.à.d: si f admet une orbite periodique de per. primit. 3 alors f admet des orbites per. de toutes per. primit.)



(x_1, x_2, x_3)

$(1, *, 2)$

1 et 2.



Propriétés:

~~Lemme~~: I, J ints compacts.

$f : I \to \mathbb{R}$ continue.

(A) $fI \supset I \Rightarrow \exists p \in I : p = f(p)$

(B) $fI \supset J \Rightarrow \exists I' \subset I : fI' = J$ (surjection)

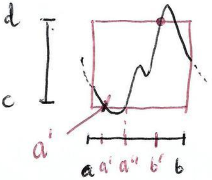
Démo (A): $I = [a,b]$. $g(x) = x - f(x)$

$\exists a' \in I : f(a') = a$, $g(a') \leq 0$

$\exists b' \in I : f(b') = b$, $g(b') \geq 0$

(TVI) $\Rightarrow \exists p \in [a', b'] \subset I : g(p) = 0$

(B)



cas $\exists a' < b'$: $f(a') = c$, $f(b') = d$:

$a' = \min \{x \geq a : fx = c\}$

$b' = \min \{x > a' : fx = d\}$

$a'' = \max \{x < b' : fx = c\}$

Alors:

$f[a'', b'] = [c, d]$

Avec l'ordre $x_1 < x_2 < x_3$ on aura:

$fI_1 \supset I_2$

$fI_2 \supset I_1 \cup I_2$

Par prop (A) on a:

$\exists p \in I_2 : p = f(p) \in \mathcal{P}_1(f)$

Par prop (B):

$\exists I_{12} \subset I_1 : fI_{12} = I_2$

$\exists I_{21} \subset I_2 : fI_{21} = I_1$

$\exists I_{22} \subset I_2 : fI_{22} = I_2$

Matrice de transition

$$A = \begin{array}{c|cc} & & \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ \hline & 1 & 2 \end{array}$$

source $\nearrow$; $1 \; 2 \leftarrow$ arrivé

On a par iteration

$\exists I_{121} \in I : fI_{121} = I_{21}$

$I_{121} \xrightarrow{f} I_{21} \xrightarrow{f} I_1$

D'où (A) $\Rightarrow \exists p \in I_{121} : f^2(p) = p$ mais $f(p) \neq p$

$\exists I_{\underbrace{12..21}_{n}} \in I : fI_{\underbrace{12..21}_{n}} = I_{\underbrace{2..21}_{n}}$

$fI_{\underbrace{2..21}_{n}} = fI_{\underbrace{2..21}_{n-1}}$

$\exists p \in I_{12..21} : f^n p = p$, $\forall 0 < q < n$ $f^q p \neq p$ //

$\Sigma_d^+ = \prod_{k\geq 0} \{0,1,\dots,d-1\}$ $\qquad \Sigma_d = \prod_{k\in\mathbb{Z}} \{0,1,\dots,d-1\}$

$\sigma^+ : \Sigma_d^+ \circlearrowleft$ $\qquad \cancel{\sigma : \Sigma_d} \qquad \sigma : \Sigma_d \circlearrowleft$

$\cancel{\sigma((\xi_k)_{k\geq 0})}$

$\xi \doteq (\xi_k)_{k\geq 0} \mapsto (\xi_{k+1})_{k\geq 0} = \sigma(\xi)$

$\sigma(\xi)_k = \xi_{k+1} \qquad (\xi_0, \xi_1, \dots) \mapsto (\xi_1, \xi_2, \dots)$

Pt fixe $\quad \sigma(\xi) = \xi \Rightarrow \xi_k = \xi_{k+1} \Rightarrow \xi = (\bar{s}) \quad \bar{s} = (s,s,s,\dots)$

Pt per $\quad \sigma^n(\xi) = \xi \Rightarrow \xi_k = \xi_{k+n} \Rightarrow \xi = (\overline{s_0 s_1 \dots s_{n-1}})$

$\# Per_n(\sigma) = d^n$

$A \in M_d(\{0,1\}) \qquad A = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$ $\qquad$ $011101011\dots \in \Sigma_A^+$
$010011\dots \notin \Sigma_A^+$

$\Sigma_A^+ = \{\xi \in \Sigma_d^+ : A_{\xi_k \xi_{k+1}} \equiv 1 \quad \forall k \geq 0\} \qquad \sigma^+ : \Sigma_A^+ \circlearrowleft$

$\xi \in Per_n(\sigma)$ ssi $\xi_k = \xi_{k+n} \ \forall k$ et $A_{\xi_k \xi_{k+1}} \equiv 1 \ \forall k$

ssi $\xi = (\overline{s_0 s_1 \dots s_n})$ et $A_{s_0 s_1} \cdots A_{s_{n-1} s_n} A_{s_n s_0} = 1$

$$\# Per_n(\sigma) = \sum_{0 \leq s_0, \dots, s_n \leq d-1} A_{s_0 s_1} \cdots A_{s_n s_0} = \sum_{s_0} (A^n)_{s_0 s_0}$$

$$= \operatorname{tr} A^n \qquad sp(A) = \{\lambda_1, \lambda_2\} \quad \frac{1 \pm \sqrt{5}}{2}$$

$\lambda(\lambda-1)-1=0$
$\lambda^2-\lambda-1=0$
$\lambda = \frac{1\pm\sqrt{5}}{2}$

Fct génératrice $\qquad \zeta_A(z) = \exp\left(-\sum \frac{z^k}{k} \# Per_k(\sigma)\right)$
$|z| < \delta^r$ avec z petit

ex de Fct ζ - dynamique

$$= \exp\left(-\sum \frac{z^k}{k} \operatorname{tr} A^k\right)$$
$$= \exp\left(-\sum \frac{z^k}{k} \sum_{j=1}^{d} \lambda_j^k\right)$$
$$= \prod_{j=1}^{d} \exp\left(-\sum \frac{z^k}{k} \lambda_j^k\right)$$
$$= \prod \exp(\ln(1 - z\lambda_j))$$
$$= \prod (1 - z\lambda_j) = \det(1 - zA)$$