Transitive topological Markov chains of given entropy and period with or without measure of maximal entropy

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Abstract

We show that, for every positive real number h and every positive integer p, there exist oriented graphs G, G' (with countably many vertices) that are strongly connected, of period p, of Gurevich entropy h, such that G is positive recurrent (thus the topological Markov chain on G admits a measure of maximal entropy) and G' is transient (thus the topological Markov chain on G' admits no measure of maximal entropy).

1 Vere-Jones classification of graphs

In this paper, all the graphs are oriented, have a finite or countable set of vertices and, if u, v are two vertices, there is at most one arrow $u \to v$. A *path* of length n in the graph G is a sequence of vertices $(u_0, u_1, \ldots, u_n)$ such that $u_i \to u_{i+1}$ in G for all $i \in [\![0, n-1]\!]$. This path is called a *loop* if $u_0 = u_n$.

Definition 1 Let G be an oriented graph and let u, v be two vertices in G. We define the following quantities.

- $p^G_{uv}(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u$ and $u_n = v$; $R_{uv}(G)$ is the radius of convergence of the series $\sum p^G_{uv}(n) z^n$.
- $f^G_{uv}(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u$, $u_n = v$ and $u_i \neq v$ for all $0 < i < n$; $L_{uv}(G)$ is the radius of convergence of the series $\sum f^G_{uv}(n) z^n$.

Definition 2 Let G be an oriented graph and V its set of vertices. The graph G is *strongly connected* if for all $u, v \in V$, there exists a path from u to v in G. The *period* of a strongly connected graph G is the greatest common divisor of $(p^G_{uu}(n))_{u \in V, n \geq 0}$. The graph G is *aperiodic* if its period is 1.

Proposition 3 (Vere-Jones [8]) *Let G be an oriented graph. If G is strongly connected, $R_{uv}(G)$ does not depend on u and v; it is denoted by $R(G)$.*

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If there is no confusion, $R(G)$ and $L_{uv}(G)$ will be written R and L_{uv}.

In [8] Vere-Jones gives a classification of strongly connected graphs as transient, null recurrent or positive recurrent. These definitions are lines 1 and 2 in Table 1. The other lines of Table 1 state properties of the series $\sum p_{uv}^G(n)z^n$, which give alternative definitions (lines 3 and 4 are in [8], the last line is Proposition 4).

	transient	null recurrent	positive recurrent
$\sum_{n>0} f_{uu}^G(n)R^n$	< 1	1	1
$\sum_{n>0} nf_{uu}^G(n)R^n$	$\leq +\infty$	$+\infty$	$< +\infty$
$\sum_{n\geq 0} p_{uv}^G(n)R^n$	$< +\infty$	$+\infty$	$+\infty$
$\lim_{n\to+\infty} p_{uv}^G(n)R^n$	0	0	$\lambda_{uv} > 0$
	$R = L_{uu}$	$R = L_{uu}$	$R \leq L_{uu}$

Table 1: properties of the series associated to a transient, null recurrent or positive recurrent graph G (G is strongly connected); these properties do not depend on the vertices u, v.

Proposition 4 (Salama [7]) *Let G be a strongly connected oriented graph. If G is transient or null recurrent, then $R = L_{uu}$ for all vertices u. Equivalently, if there exists a vertex u such that $R < L_{uu}$, then G is positive recurrent.*

2 Topological Markov chains and Gurevich entropy

Let G be an oriented graph and V its set of vertices. We define Γ_G as the set of two-sided infinite paths in G, that is,

$$\Gamma_G := \{(v_n)_{n\in\mathbb{Z}} \mid \forall n \in \mathbb{Z}, v_n \to v_{n+1} \text{ in } G\} \subset V^{\mathbb{Z}}.$$

The map σ is the shift on Γ_G. The *topological Markov chain* on the graph G is the dynamical system (Γ_G, σ).

The set V is endowed with the discrete topology and Γ_G is endowed with the induced topology of $V^{\mathbb{Z}}$. The space Γ_G is not compact unless G is finite.

The topological Markov chain (Γ_G, σ) is transitive if and only if the graph G is strongly connected. It is topologically mixing if and only if the graph G is strongly connected and aperiodic.

If G is a finite graph, Γ_G is compact and the topological entropy $h_{top}(\Gamma_G, \sigma)$ is well defined (see e.g. [2] for the definition of the topological entropy). If G is a countable graph, the *Gurevich entropy* [3] of the graph G (or of the topological Markov chain Γ_G) is given by

$$h(G) := \sup\{h_{top}(\Gamma_H, \sigma) \mid H \subset G, H \text{ finite}\}.$$

This entropy can also be computed in a combinatorial way, as the exponential growth of the number of paths with fixed endpoints.

Proposition 5 (Gurevich [4]) *Let G be a strongly connected oriented graph. Then for all vertices u, v*

$$h(G) = \lim_{n \to +\infty} \frac{1}{n} \log p_{uv}^G(n) = -\log R(G).$$

Moreover, the variational principle is still valid for topological Markov chains.

Theorem 6 (Gurevich [3]) *Let G be an oriented graph. Then*

$$h(G) = \sup\{h_\mu(\Gamma_G) \mid \mu \ \sigma\text{-invariant probability measure}\}.$$

In this variational principle, the supremum is not necessarily reached. The next theorem gives a necessary and sufficient condition for the existence of a measure of maximal entropy (that is, a probability measure μ such that $h(G) = h_\mu(\Gamma_G)$) when the graph is strongly connected.

Theorem 7 (Gurevich [4]) *Let G be a strongly connected oriented graph of finite positive entropy. Then the topological Markov chain on G admits a measure of maximal entropy if and only if the graph G is positive recurrent. Moreover, such a measure is unique if it exists.*

3 Construction of graphs of given entropy and given period that are either positive recurrent or transient

Lemma 8 *Let $\beta \in (1, +\infty)$. There exist a sequence of non negative integers $(a(n))_{n \geq 1}$ and positive constants c, M such that*

- $a(1) = 1$,
- $\sum_{n \geq 1} a(n) \frac{1}{\beta^n} = 1$,
- $\forall n \geq 2,\ c \cdot \beta^{n^2 - n} \leq a(n^2) \leq c \cdot \beta^{n^2 - n} + M$,
- $\forall n \geq 1,\ 0 \leq a(n) \leq M$ *if n is not a square.*

These properties imply that the radius of convergence of $\sum_{n \geq 1} a(n) z^n$ is $L = \frac{1}{\beta}$ and that $\sum_{n \geq 1} n a(n) L^n < +\infty$.

Proof. First we look for a constant $c > 0$ such that

$$\frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2 - n} \frac{1}{\beta^{n^2}} = 1. \tag{1}$$

We have

$$\sum_{n \geq 2} \beta^{n^2 - n} \frac{1}{\beta^{n^2}} = \sum_{n \geq 2} \beta^{-n} = \frac{1}{\beta(\beta - 1)}.$$

Thus

$$(1) \iff \frac{1}{\beta} + \frac{c}{\beta(\beta - 1)} = 1 \iff c = (\beta - 1)^2.$$

Since $\beta > 1$, the constant $c := (\beta - 1)^2$ is positive. We define the sequence $(b(n))_{n \geq 1}$ by:

- $b(1) := 1$,
- $b(n^2) := \lfloor c \beta^{n^2 - n} \rfloor$ for all $n \geq 2$,

- $b(n) := 0$ for all $n \geq 2$ such that n is not a square.

Then

$$\sum_{n\geq 1} b(n)\frac{1}{\beta^n} \leq \frac{1}{\beta} + c\sum_{n\geq 2}\beta^{n^2-n}\frac{1}{\beta^{n^2}} = 1.$$

We set $\delta := 1 - \sum_{n\geq 1} b(n)\frac{1}{\beta^n} \in [0,1)$ and $k := \lfloor \beta^2\delta \rfloor$. Then $k \leq \beta^2\delta < k+1 < k+\beta$, which implies that $0 \leq \delta - \frac{k}{\beta^2} < \frac{1}{\beta}$. We write the β-expansion of $\delta - \frac{k}{\beta^2}$ (see e.g. [1, p 51] for the definition): there exist integers $d(n) \in \{0,\ldots,\lfloor\beta\rfloor\}$ such that $\delta - \frac{k}{\beta^2} = \sum_{n\geq 1} d(n)\frac{1}{\beta^n}$. Moreover, $d(1) = 0$ because $\delta - \frac{k}{\beta^2} < \frac{1}{\beta}$. Thus we can write

$$\delta = \sum_{n\geq 2} d'(n)\frac{1}{\beta^n}$$

where $d'(2) := d(2) + k$ and $d'(n) := d(n)$ for all $n \geq 3$.

We set $a(1) := b(1)$ and $a(n) := b(n) + d'(n)$ for all $n \geq 2$. Let $M := \beta + k$. We then have:

- $a(1) = 1$,
- $\sum_{n\geq 1} a(n)\frac{1}{\beta^n} = 1$,
- $\forall n \geq 2,\ c\cdot\beta^{n^2-n} \leq a(n^2) \leq c\cdot\beta^{n^2-n} + \beta \leq c\cdot\beta^{n^2-n} + M$,
- $0 \leq a(2) \leq \beta + k = M$,
- $\forall n \geq 3,\ 0 \leq a(n) \leq \beta \leq M$ if n is not a square.

The radius of convergence L of $\sum_{n\geq 1} a(n)z^n$ satisfies

$$-\log L = \limsup_{n\to+\infty}\frac{1}{n}\log a(n) = \lim_{n\to+\infty}\frac{1}{n^2}\log a(n^2) = \log\beta \quad \text{because } a(n^2) \sim c\beta^{n^2-n}.$$

Thus $L = \frac{1}{\beta}$. Moreover,

$$\sum_{n\geq 1} na(n)\frac{1}{\beta^n} \leq M\sum_{n\geq 1} n\frac{1}{\beta^n} + c\sum_{n\geq 1} n^2\beta^{n^2-n}\frac{1}{\beta^{n^2}} = M\sum_{n\geq 1}\frac{n}{\beta^n} + c\sum_{n\geq 1}\frac{n^2}{\beta^n} < +\infty.$$

□

Lemma 9 ([5], Lemma 2.4) *Let G be a strongly connected oriented graph and u a vertex.*

i) *$R < L_{uu}$ if and only if $\sum_{n\geq 1} f^G_{uu}(n)L^n_{uu} > 1$.*

ii) *If G is recurrent, then R is the unique positive number x such that $\sum_{n\geq 1} f^G_{uu}(n)x^n = 1$.*

Proof. For (i) and (ii), use Table 1 and the fact that $F(x) = \sum_{n\geq 1} f^G_{uu}(n)x^n$ is increasing for $x \in [0,+\infty[$. □

Proposition 10 *Let $\beta \in (1,+\infty)$. There exist aperiodic strongly connected graphs $G'(\beta) \subset G(\beta)$ such that $h(G(\beta)) = h(G'(\beta)) = \log\beta$, $G(\beta)$ is positive recurrent and $G'(\beta)$ is transient.*

Remark: Salama proved the part of this proposition concerning positive recurrent graphs in [6, Theorem 3.9].

Proof. This is a variant of the proof of [5, Example 2.9].

Let u be a vertex and let $(a(n))_{n\geq 1}$ be the sequence given by Lemma 8 for β. The graph $G(\beta)$ is composed of $a(n)$ loops of length n based at the vertex u for all $n \geq 1$ (see Figure 1). More precisely, define the set of vertices of $G(\beta)$ as

$$V := \{u\} \cup \bigcup_{n=1}^{+\infty} \{v_k^{n,i} \mid i \in [\![1, a(n)]\!], k \in [\![1, n-1]\!]\},$$

where the vertices $v_k^{n,i}$ above are distinct. Let $v_0^{n,i} = v_n^{n,i} = u$ for all $i \in [\![1, a(n)]\!]$. There is an arrow $v_k^{n,i} \to v_{k+1}^{n,i}$ for all $k \in [\![0, n-1]\!], i \in [\![1, a(n)]\!], n \geq 2$; there is an arrow $u \to u$; and there is no other arrow in $G(\beta)$. The graph $G(\beta)$ is strongly connected and $f_{uu}^{G(\beta)}(n) = a(n)$ for all $n \geq 1$.

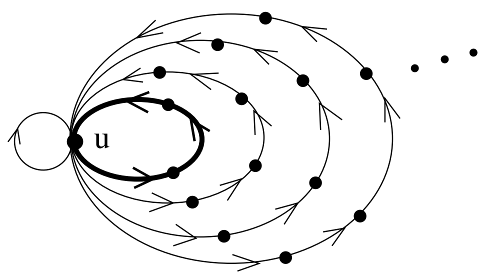

Figure 1: the graphs $G(\beta)$ and $G'(\beta)$; the bold loop belongs to $G(\beta)$ and not to $G'(\beta)$, otherwise the two graphs coincide.

By Lemma 8, the sequence $(a(n))_{n\geq 1}$ is defined such that $L = \frac{1}{\beta}$ and

$$\sum_{n\geq 1} a(n)L^n = 1, \tag{2}$$

where $L = L_{uu}(G(\beta))$ is the radius of convergence of the series $\sum a(n)z^n$. If $G(\beta)$ is transient, then $R(G(\beta)) = L_{uu}(G(\beta))$ by Proposition 4. But Equation (2) contradicts the definition of transient (see the first line of Table 1). Thus $G(\beta)$ is recurrent, and $R(G(\beta)) = L$ by Equation (2) and Lemma 9(ii). Moreover

$$\sum_{n\geq 1} na(n)L^n < +\infty$$

by Lemma 8, and thus the graph $G(\beta)$ is positive recurrent (see Table 1). By Proposition 5, $h(G(\beta)) = -\log R(G(\beta)) = \log \beta$.

The graph $G'(\beta)$ is obtained from $G(\beta)$ by deleting a loop starting at u of length n_0 for some $n_0 \geq 2$ such that $a(n_0) \geq 1$ (such an integer n_0 exists because $L < +\infty$). Obviously one has $L_{uu}(G'(\beta)) = L$ and

$$\sum_{n\geq 1} f_{uu}^{G'(\beta)}(n)L^n = 1 - L^{n_0} < 1.$$

Since $R(G'(\beta)) \leq L_{uu}(G'(\beta))$, this implies that $G'(\beta)$ is transient. Moreover $R(G'(\beta)) = L_{uu}(G'(\beta))$ by Proposition 4, so $R(G'(\beta)) = R(G(\beta))$, and hence $h(G'(\beta)) = h(G(\beta))$ by Proposition 5. Finally, both $G(\beta)$ and $G'(\beta)$ are of period 1 because of the arrow $u \to u$. $\square$

Corollary 11 *Let p be a positive integer and $h \in (0, +\infty)$. There exist strongly connected graphs G, G' of period p such that $h(G) = h(G') = h$, G is positive recurrent and G' is transient.*

Proof. For G, we start from the graph $G(\beta)$ given by Proposition 10 with $\beta = e^{hp}$. Let V denote the set of vertices of $G(\beta)$. The set of vertices of G is $V \times [\![1, p]\!]$, and the arrows in G are:

- $(v, i) \to (v, i+1)$ if $v \in V$, $i \in [\![1, p-1]\!]$,
- $(v, p) \to (w, 1)$ if $v, w \in V$ and $v \to w$ is an arrow in $G(\beta)$.

According to the properties of $G(\beta)$, G is strongly connected, of period p and positive recurrent. Moreover, $h(G) = \frac{1}{p} h(G(\beta)) = \frac{1}{p} \log \beta = h$.

For G', we do the same starting with $G'(\beta)$. □

According to Theorem 7, the graphs of Corollary 11 satisfy that the topological Markov chain on G admits a measure of maximal entropy whereas the topological Markov chain on G' admits no measure of maximal entropy; both are transitive, of Gurevich entropy h and supported by a graph of period p.

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