A note on transitive topological Markov chains of given entropy and period

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Abstract

We show that, for every positive real number $h$ and every positive integer $p$, there exist oriented graphs $G, G'$ (with countably many vertices) that are strongly connected, of period $p$, of Gurevich entropy $h$, such that $G$ is positive recurrent (thus the topological Markov chain on $G$ admits a measure of maximal entropy) and $G'$ is transient (thus the topological Markov chain on $G'$ admits no measure of maximal entropy).

We also show that any transitive topological Markov chain with infinite entropy carries uncountably many ergodic, invariant probability measures with infinite entropy.

1 Vere-Jones classification of graphs

Definition 1 Let $G$ be an oriented graph and let $u, v$ be two vertices in $G$. We define the following quantities.

- $p^G_{u,v}(n)$ is the number of paths $u_0 \to u_1 \to \cdots \to u_n$ such that $u_0 = u$ and $u_n = v$; $R_{uv}(G)$ is the radius of convergence of the series $\sum p^G_{u,v}(n)z^n$.
- $f^G_{u,v}(n)$ is the number of paths $u_0 \to u_1 \to \cdots \to u_n$ such that $u_0 = u$, $u_n = v$ and $u_i \neq v$ for all $0 < i < n$; $L_{uv}(G)$ is the radius of convergence of the series $\sum f^G_{u,v}(n)z^n$.

Definition 2 Let $G$ be an oriented graph and $V$ its set of vertices. The graph $G$ is strongly connected if for all $u, v \in V$, there exists a path from $u$ to $v$ in $G$. The period of a strongly connected graph $G$ is the greatest common divisor of $(p^G_{u,u}(n))_{u \in V, n \geq 0}$. The graph $G$ is aperiodic if its period is 1.

Proposition 3 (Vere-Jones [8]) Let $G$ be an oriented graph. If $G$ is strongly connected, $R_{uv}(G)$ does not depend on $u$ and $v$; it is denoted by $R(G)$.

If there is no confusion, $R(G)$ and $L_{uv}(G)$ will be written $R$ and $L_{uv}$.

In [8] Vere-Jones gives a classification of strongly connected graphs as transient, null recurrent or positive recurrent. The definitions are given in Table 1 (lines 1 and 2) as well as properties of the series $\sum p^G_{u,v}(n)z^n$ which give an alternative definition.

Proposition 4 (Salama [7]) Let $G$ be a strongly connected oriented graph. If $G$ is transient or null recurrent, then $R = L_{uv}$ for all vertices $u$. Equivalently, if there exists a vertex $u$ such that $R < L_{uv}$, then $G$ is positive recurrent.
Table 1: Properties of the series associated to a transient, null recurrent or positive recurrent graph \( G \) (\( G \) is strongly connected); these properties do not depend on the vertices \( u, v \).

### 2 Topological Markov chains and Gurevich entropy

Let \( G \) be an oriented graph and \( V \) its set of vertices. Then \( \Gamma_G \) is the set of two-sided infinite paths in \( G \), that is,

\[
\Gamma_G = \{ (v_n)_{n \in \mathbb{Z}} \mid \forall n \in \mathbb{Z}, v_n \to v_{n+1} \text{ in } G \} \subset V^\mathbb{Z}.
\]

\( \sigma \) is the shift on \( \Gamma_G \). The **topological Markov chain** on the graph \( G \) is the system \( (\Gamma_G, \sigma) \).

The set \( V \) is endowed with the discrete topology and \( \Gamma_G \) is endowed with the induced topology of \( V^\mathbb{Z} \). The space \( \Gamma_G \) is not compact unless \( G \) is finite.

The topological Markov chain \( (\Gamma_G, \sigma) \) is transitive if and only if the graph \( G \) is strongly connected. It is topologically mixing if and only if the graph \( G \) is strongly connected and of period 1.

If \( G \) is a finite graph, \( \Gamma_G \) is compact and the topological entropy \( h_{\text{top}}(\Gamma_G, \sigma) \) is well defined (see e.g. [2] for the definition of the topological entropy). If \( G \) is a countable graph, the **Gurevich entropy** [3] of the graph \( G \) (or of the topological Markov chain \( \Gamma_G \)) is given by

\[
h(G) = \sup \{ h_{\text{top}}(\Gamma_H, \sigma) \mid H \subset G, H \text{ finite} \}.
\]

This entropy can also be computed in a combinatorial way, as the exponential growth of the number of paths with fixed endpoints.

**Proposition 5 (Gurevich [4])** Let \( G \) be a strongly connected oriented graph. Then for all vertices \( u, v \)

\[
h(G) = \lim_{n \to +\infty} \frac{1}{n} \log p_{uv}^G(n) = -\log R(G).
\]

Moreover, the variational principle is still valid for topological Markov chains.

**Theorem 6 (Gurevich [3])** Let \( G \) be an oriented graph. Then

\[
h(G) = \sup \{ h_\mu(\Gamma_G) \mid \mu \text{-invariant probability measure} \}.
\]

In this variational principle, the supremum is not necessarily reached. The next theorem gives a necessary and sufficient condition for the existence of a measure of maximal entropy (that is, a measure \( \mu \) such that \( h(G) = h_\mu(\Gamma_G) \)) when the graph is strongly connected.
Theorem 7 (Gurevich [4]) Let $G$ be a strongly connected oriented graph of finite positive entropy. Then the topological Markov chain on $G$ admits a measure of maximal entropy if and only if the graph $G$ is positive recurrent. Moreover, such a measure is unique if it exists.

3 Construction of graphs of given entropy and given period that are either positive recurrent or transient

Lemma 8 Let $\beta \in (1, +\infty)$. There exist a sequence of non negative integers $(a(n))_{n \geq 1}$ and positive constants $c, M$ such that

- $a(1) = 1$,
- $\sum_{n \geq 1} a(n) \frac{1}{\beta^n} = 1$,
- $\forall n \geq 2, c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + M$,
- $\forall n \geq 1, 0 \leq a(n) \leq M$ if $n$ is not a square.

These properties imply that the radius of convergence of $\sum_{n \geq 1} a(n)z^n$ is $L = \frac{1}{\beta}$ and that $\sum_{n \geq 1} na(n)L^n < +\infty$.

Proof. First we look for a constant $c > 0$ such that

\[
\frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1. \tag{1}
\]

We have

\[
\sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = \sum_{n \geq 2} \beta^{-n} = \frac{1}{\beta(\beta-1)}.
\]

Thus

\[
(1) \iff \frac{1}{\beta} + \frac{c}{\beta(\beta-1)} = 1 \iff c = (\beta-1)^2.
\]

Since $\beta > 1$, the constant $c$ is positive. We define the sequence $(b(n))_{n \geq 1}$ by:

- $b(1) := 1$,
- $b(n^2) := \lfloor c\beta^{n^2-n} \rfloor$ for all $n \geq 2$,
- $b(n) := 0$ for all $n \geq 2$ such that $n$ is not a square.

Then

\[
\sum_{n \geq 1} b(n) \frac{1}{\beta^n} \leq \frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1.
\]

We set $\delta := 1 - \sum_{n \geq 1} b(n) \frac{1}{\beta^n} \in [0, 1[$ and $k := \lfloor \beta^2 \delta \rfloor$. Then $k \leq \beta^2 \delta < k + 1 < k + \beta$, which implies that $0 \leq \delta - \frac{k}{\beta^2} < \frac{1}{\beta^2}$. We write the $\beta$-expansion of $\delta - \frac{k}{\beta^2}$ (see e.g. [1, p 51] for the definition): there exist integers $d(n) \in \{0, \ldots, \lfloor \beta \rfloor \}$ such that $\delta - \frac{k}{\beta^2} = \sum_{n \geq 1} d(n) \frac{1}{\beta^n}$. Moreover, $d(1) = 0$ because $\delta - \frac{k}{\beta^2} < \frac{1}{\beta}$. Thus we can write

\[
\delta = \sum_{n \geq 2} d'(n) \frac{1}{\beta^n}
\]

where $d'(2) = d(2) + k$ and $d'(n) = d(n)$ for all $n \geq 3$.

We set $a(1) := b(1)$ and $a(n) := b(n) + d'(n)$ for all $n \geq 2$. Let $M := \beta + k$. We then have:
• $a(1) = 1$,
• $\sum_{n \geq 1} a(n) \frac{1}{\beta^n} = 1$,
• $\forall n \geq 2, c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + \beta \leq c \cdot \beta^{n^2-n} + M$,
• $0 \leq a(2) \leq \beta + k = M$,
• $\forall n \geq 3, 0 \leq a(n) \leq \beta \leq M$ if $n$ is not a square.

The radius of convergence $L$ of $\sum_{n \geq 1} a(n)z^n$ satisfies

$$- \log L = \limsup_{n \to +\infty} \frac{1}{n} \log a(n) = \lim_{n \to +\infty} \frac{1}{n^2} \log a(n^2) = \log \beta \quad \text{because } a(n^2) \sim c\beta^{n^2-n}.$$ 

Thus $L = \frac{1}{\beta}$. Moreover,

$$\sum_{n \geq 1} na(n) \frac{1}{\beta^n} \leq M \sum_{n \geq 1} \frac{1}{\beta^n} + c \sum_{n \geq 1} \beta^{n^2-n} \frac{1}{\beta^n} = M \sum_{n \geq 1} \frac{n}{\beta^n} + c \sum_{n \geq 1} \frac{n^2}{\beta^n} < +\infty.$$  

\[
\begin{proof}
\end{proof}

Lemma 9 ([5], Lemma 2.4) Let $G$ be a strongly connected oriented graph and $u$ a vertex.

i) $R < L_{uu}$ if and only if $\sum_{n \geq 1} f_{uu}^n(n)L_{uu}^n > 1$.

ii) If $G$ is recurrent, then $R$ is the unique positive number $x$ such that $\sum_{n \geq 1} f_{uu}^n(x) = 1$.

Proof. For (i) and (ii), use Table 1 and the fact that $F(x) = \sum_{n \geq 1} f_{uu}^n(x)$ is increasing for $x \in [0, +\infty]$.

\[
\begin{proof}
\end{proof}

Proposition 10 Let $\beta \in (1, +\infty)$. There exist aperiodic strongly connected graphs $G'(\beta) \subset G(\beta)$ such that $h(G'(\beta)) = h(G'(\beta)) = \log \beta$, $G(\beta)$ is positive recurrent and $G'(\beta)$ is transient.

Remark: Salama proved the part of this proposition concerning positive recurrent graphs in [6, Theorem 3.9].

Proof. This is a variant of the proof of [5, Example 2.9].

Let $u$ be a vertex and let $(a(n))_{n \geq 1}$ be the sequence given by Lemma 8 for $\beta$. The graph $G(\beta)$ is composed of $a(n)$ loops of length $n$ based at the vertex $u$ for all $n \geq 1$ (see Figure 1). More precisely, define the set of vertices of $G(\beta)$ as

$$V = \{u\} \cup \bigcup_{n=1}^{+\infty} \{v_k^{n,i} \mid 1 \leq i \leq a(n), 1 \leq k \leq n-1\},$$

where the vertices $v_k^{n,i}$ above are distinct. Let $v_0^{n,i} = v_{n,i}^{n,i} = u$ for all $1 \leq i \leq a(n)$. There is an arrow $v_k^{n,i} \rightarrow v_{k+1}^{n,i}$ for all $0 \leq k \leq n-1, 1 \leq i \leq a(n), n \geq 2$, there is an arrow $u \rightarrow u$, and there is no other arrow in $G(\beta)$. The graph $G(\beta)$ is strongly connected and $f_{uu}^G(\beta)(n) = a(n)$ for all $n \geq 1$.

By Lemma 8, the sequence $(a(n))_{n \geq 1}$ is defined such that $L = \frac{1}{\beta}$ and

$$\sum_{n \geq 1} a(n)L^n = 1,$$

(2)
Figure 1: the graphs $G(\beta)$ and $G'(\beta)$; the bold loop (on the left) is the only arrow that belongs
to $G(\beta)$ and not to $G'(\beta)$, otherwise the two graphs coincide.

where $L = L_{uu}(G(\beta))$ is the radius of convergence of the series $\sum a(n)z^n$. If $G(\beta)$ is transient,
then $R(G(\beta)) = L_{uu}(G(\beta))$ by Proposition 4, but Equation (2) contradicts the definition of
transient (see the first line of Table 1). Thus $G(\beta)$ is recurrent, and $R(G(\beta)) = L$ by Equation (2)
and Lemma 9(ii). Moreover

$$\sum_{n\geq 1} na(n)L^n < +\infty$$

by Lemma 8, and thus the graph $G(\beta)$ is positive recurrent (see Table 1). By Proposition 5,
h$(G(\beta)) = -\log R(G(\beta)) = \log \beta$.

The graph $G'(\beta)$ is obtained from $G(\beta)$ by deleting a loop $u \rightarrow \cdots \rightarrow u$ of length $n_0$ for
some $n_0 \geq 2$ such that $a(n_0) \geq 1$ (such an integer $n_0$ exists because $L < +\infty$). Obviously one
has $L_{uu}(G'(\beta)) = L$ and

$$\sum_{n\geq 1} f_{uu}^{G'(\beta)}(n)L^n = 1 - L^{n_0} < 1.$$ 

Since $R(G'(\beta)) \leq L_{uu}(G'(\beta))$, this implies that $G'(\beta)$ is transient. Moreover $R(G'(\beta)) = L_{uu}(G'(\beta))$ by Proposition 4, so $R(G'(\beta)) = R(G(\beta))$, and hence $h(G'(\beta)) = h(G(\beta))$ by
Proposition 5. Finally, both $G(\beta)$ and $G'(\beta)$ are of period 1 because of the loop $u \rightarrow u$. □

Corollary 11 Let $p$ be a positive integer and $h \in (0, +\infty)$. There exist strongly connected
graphs $G, G'$ of period $p$ such that $h(G) = h(G') = h$, $G$ is positive recurrent and $G'$ is transient.

Proof. For $G$, we start from the graph $G(\beta)$ given by Proposition 10 with $\beta = e^{hp}$. Let $V$
denote the set of vertices of $G(\beta)$. The set of vertices of $G$ is $V \times \{1, \ldots, p\}$, and the arrows in $G$
are:

- $(v, i) \rightarrow (v, i + 1)$ if $v \in V$, $1 \leq i \leq p - 1$,
- $(v, p) \rightarrow (w, 1)$ if $v, w \in V$ and $v \rightarrow w$ is an arrow in $G(\beta)$.

According to the properties of $G(\beta)$, $G$ is strongly connected, of period $p$ and positive recurrent.
Moreover, $h(G) = \frac{1}{p}h(G(\beta)) = \frac{1}{p}\log \beta = h$.

For $G'$, we do the same starting with $G'(\beta)$. □

According to Theorem 7, the graphs of Corollary 11 satisfy that the topological Markov
chain on $G$ admits a measure of maximal entropy the topological Markov chain on $G'$ admits
no measure of maximal entropy ; both are transitive, of entropy $h$ and carried by a graph of
period $p$. 

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4 Topological Markov chains with infinite entropy

**Proposition 12** Any transitive topological Markov chain with infinite entropy carries uncountably many ergodic, invariant probability measures with infinite entropy.

**Proof.** Let $(\Gamma_G, \sigma)$ be a topological Markov chain, where $G$ is a strongly connected graph such that $h(G) = +\infty$. Let $v$ be a vertex of $G$ and choose a sequence $(n_k)_{k \geq 1}$ such that $f_{vv}^G(n_k) \geq 2^{k^2 n_k}$ (such a sequence exists because $h(G) = +\infty$).

Set $a_k := f_{vv}^G(n_k)$ and define $f = \sum_{k=1}^{\infty} a_k z^{n_k}$.

The loop shift $\sigma_f$ embeds into the topological Markov chain $(\Gamma_G, \sigma)$.

For all $k \geq 1$, define $q_k = (2^k a_k n_k)^{-1}$ and $p_0 = \sum_{k=1}^{\infty} q_k$.

Consider $\sigma_f$ as a vertex shift. Let $\mu$ be the Markov measure assigning the base vertex $v$ probability $p_0$, with transition probability $q_k/p_0$ from $v$ to each vertex following $v$ and lying in a first return loop of length $n_k$. Then $\mu$ is an ergodic, atomless invariant probability measure of $\sigma_f$ and $h_{\mu}(\sigma_f) = +\infty$.

By use of other subsequences, we easily get uncountably many distinct ergodic measures with infinite entropy. \qed

**References**


