Interval maps of given topological entropy and Sharkovskii's type

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Abstract

It is known that the topological entropy of a continuous interval map f is positive if and only if the type of f for Sharkovskii's order is $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$; and in this case the topological entropy of f is greater than or equal to $\frac{\log \lambda_p}{2^d}$, where λ_p is the unique positive root of $X^p - 2X^{p-2} - 1$. For every odd $p \geq 3$, every $d \geq 0$ and every $\lambda \geq \lambda_p$, we build a piecewise monotone continuous interval map that is of type $2^d p$ for Sharkovskii's order and whose topological entropy is $\frac{\log \lambda}{2^d}$. This shows that, for a given type, every possible finite entropy above the minimum can be reached provided the type allows the map to have positive entropy. Moreover, if $d = 0$ the map we build is topologically mixing.

1 Introduction

In this paper, an interval map is a continuous map $f\colon I \to I$ where I is a compact nondegenerate interval. A point $x \in I$ is periodic of period n if $f^n(x) = x$ and n is the least positive integer with this property, i.e. $f^k(x) \neq x$ for all $k \in [\![1, n-1]\!]$.

Let us recall Sharkovskii's theorem and the definitions of Sharkovskii's order and type [6] (see e.g. [5, Section 3.3]).

Definition 1.1 *Sharkovskii's order* is the total ordering on $\mathbb{N}$ defined by:

$$3 \lhd 5 \lhd 7 \lhd 9 \lhd \cdots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd \cdots \lhd 2^2\cdot 3 \lhd 2^2 \cdot 5 \lhd \cdots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

(first, all odd integers $n > 1$, then 2 times the odd integers $n > 1$, then successively $2^2\times$, $2^3\times$, ..., $2^k\times$ ... the odd integers $n > 1$, and finally all the powers of 2 by decreasing order).

$a \rhd b$ means $b \lhd a$. The notation $\unlhd, \unrhd$ will denote the order with possible equality.

Theorem 1.2 (Sharkovskii) *If an interval map f has a periodic point of period n then, for all integers $m \rhd n$, f has periodic points of period m.*

Definition 1.3 Let $n \in \mathbb{N} \cup \{2^\infty\}$. An interval map f is *of type n (for Sharkovskii's order)* if the periods of the periodic points of f form exactly the set $\{m \in \mathbb{N} \mid m \unrhd n\}$, where the notation $\{m \in \mathbb{N} \mid m \unrhd 2^\infty\}$ stands for $\{2^k \mid k \geq 0\}$.

It is well known that an interval map f has positive topological entropy if and only if its type is $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$ (see e.g. [5, Theorem 4.58]). The entropy of such a map is bounded from below (see theorem 4.57 in [5]).

Theorem 1.4 (Štefan, Block-Guckenheimer-Misiurewicz-Young) *Let f be an interval map of type $2^d p$ for some odd integer $p \geq 3$ and some $d \geq 0$. Let λ_p be the unique positive root of $X^p - 2X^{p-2} - 1$. Then $\lambda_p > \sqrt{2}$ and $h_{top}(f) \geq \frac{\log \lambda_p}{2^d}$.*

This bound is sharp: for every p, d, there exists a interval map of type $2^d p$ and topological entropy $\frac{\log \lambda_p}{2^d}$. These examples were first introduced by Štefan, although the entropy of these maps was computed later [7, 2].

Moreover, it is known that the type of a topological mixing interval map is p for some odd integer $p \geq 3$ (see e.g. [5, Proposition 3.36]). The Štefan maps of type p are topologically mixing [5, Example 3.21].

We want to show that the topological entropy of a piecewise monotone map can be equal to any real number, the lower bound of Theorem 1.4 being the only restriction. First, for every odd integer $p \geq 3$ and every real number $\lambda \geq \lambda_p$, we are going to build a piecewise monotone map $f_{p,\lambda}\colon [0,1] \to [0,1]$ such that its type is p for Sharkovskii's order, its topological entropy is $\log \lambda$, and the map is topologically mixing. Then we will show that for every odd integer $p \geq 3$, every integer $d \geq 0$ and every real number $\lambda \geq \lambda_p$, there exists a piecewise monotone interval map f such that its type is $2^d p$ for Sharkovskii's order and its topological entropy is $\frac{\log \lambda}{2^d}$.

1.1 Notations

We say that an interval is *degenerate* if it is either empty or reduced to one point, and *nondegenerate* otherwise. When we consider an interval map $f\colon I \to I$, every interval is implicitly a subinterval of I.

Let J be a nonempty interval. Then $\partial J := \{\inf J, \sup J\}$ are the endpoints of J (they may be equal if J is reduced to one point) and $|J|$ denotes the length of J (i.e. $|J| := \sup J - \inf J$). Let $\mathrm{mid}(J)$ denote the middle point of J, that is, $\mathrm{mid}(J) := \frac{\inf J + \sup J}{2}$.

An interval map $f\colon I \to I$ is *piecewise monotone* if there exists a finite partition of I into intervals such that f is monotone on each element of this partition.

An interval map f has a *constant slope* λ if f is piecewise monotone and if on each of its pieces of monotonicity f is linear and the absolute value of the slope coefficient is λ.

2 Štefan maps

We recall the definition of the Štefan maps of odd type $p \geq 3$.

Let $n \geq 1$ and $p := 2n+1$. The Štefan map $f_p\colon [0, 2n] \to [0, 2n]$, represented in Figure 1, is defined as follows: it is linear on $[0, n-1]$, $[n-1, n]$, $[n, 2n-1]$ and $[2n-1, 2n]$, and

$$f_p(0) := 2n,\ f_p(n-1) := n+1,\ f_p(n) := n-1,\ f_p(2n-1) := 0,\ f_p(2n) := n.$$

Note that $n = 1$ is a particular case because $0 = n-1$ and $n = 2n-1$.

Next proposition summarises the properties of f_p, see [5, Example 3.21] for the proof.

Proposition 2.1 *The map f_p is topologically mixing, its type for Sharkovskii's order is p and $h_{top}(f) = \log \lambda_p$. Moreover, the point n is periodic of period p, and $f_p^{2k-1}(n) = n-k$ and $f_p^{2k}(n) = n+k$ for all $k \in [\![1, n]\!]$.*

3 Mixing map of given entropy and odd type

For every odd integer $p \geq 3$ and every real number $\lambda \geq \lambda_p$, we are going to build a piecewise monotone continuous map $f_{p,\lambda}\colon [0,1] \to [0,1]$ such that its type is p for Sharkovskii's order, its topological entropy is $\log \lambda$, and the map is topologically mixing. We will write f instead of $f_{p,\lambda}$ when there is no ambiguity on p, λ.

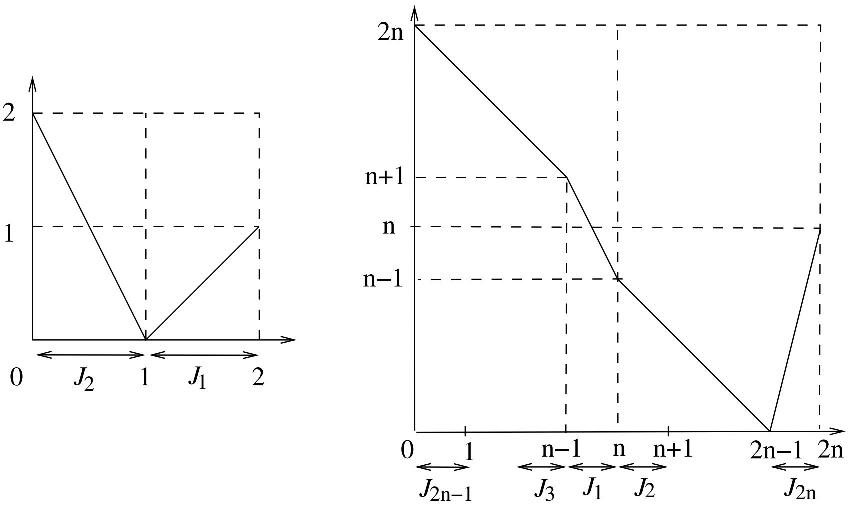

Figure 1: On the left: the map f_3. On the right: the map f_p with $p = 2n + 1 > 3$.

The idea is the following: we start with the Štefan map f_p, we blow up the minimum into an interval and we define the map of this interval in such a way that the added dynamics increases the entropy without changing the type. At the same time, we make the slope constant and equal to λ, so that the entropy is $\log \lambda$ according to the following theorem [1, Corollary 4.3.13], which is due to Misiurewicz-Szlenk [4], Young [8] and Milnor-Thurston [3].

Theorem 3.1 *Let f be a piecewise monotone interval map. Suppose that f has a constant slope $\lambda \geq 1$. Then $h_{top}(f) = \log \lambda$.*

We will also need the next result (see the proof of Lemma 4.56 in [5]).

Lemma 3.2 *Let $p \geq 3$ be an odd integer and $P_p(X) := X^p - 2X^{p-2} - 1$. Then $P_p(X)$ has a unique positive root, denoted by λ_p. Moreover, $P_p(x) < 0$ if $x \in [0, \lambda_p[$ and $P_p(x) > 0$ if $x > \lambda_p$.*

Let $\chi_p(X) := X^{p-1} - X^{p-2} - \sum_{i=0}^{p-2}(-X)^i$. Then $P_p(X) = (X+1)\chi_p(X)$, and thus $\chi_p(x) < 0$ if $x \in [0, \lambda_p[$ and $\chi_p(x) > 0$ if $x > \lambda_p$.

3.1 Definition of the map

We fix an odd integer $p \geq 3$ and a real $\lambda \geq \lambda_p$. Recall that $\lambda_p > \sqrt{2} > 1$ (Theorem 1.4).

We are going to define points ordered as follows:

$$x_{p-2} < x_{p-4} < \cdots < x_1 < x_0 < x_2 < x_4 < \cdots < x_{p-3} \leq t < x_{p-1},$$
$$\text{with} \quad x_{p-2} = 0,\ x_{p-3} = \frac{1}{\lambda} \quad \text{and} \quad x_{p-1} = 1.$$

The points $x_0, \dots, x_{p-1}$ will form a periodic orbit of period p, that is, $f(x_i) = x_{i+1 \bmod p}$ for all $i \in [\![0, p-1]\!]$.

Remark 3.3 In the following construction, the case $p = 3$ is degenerate. The periodic orbit is reduced to $x_1 < x_0 < x_2$ with $x_1 = 0, x_0 = \frac{1}{\lambda}, x_2 = 1$. We only have to determine the value of t.

The map $f\colon [0,1] \to [0,1]$ is defined as follows (see Figure 2):

- $f(x) := 1 - \lambda x$ for all $x \in [0, \frac{1}{\lambda}] = [x_{p-2}, x_{p-3}]$ (so that $f(0) = 1$ and $f(\frac{1}{\lambda}) = 0$),

- $f(x) := \lambda(x-t)$ for all $x \in [t,1]$ (so that $f(t)=0$ and $f(1)=\lambda(1-t)$),
- definition on $[\frac{1}{\lambda}, t]$: we want to have $f([\frac{1}{\lambda}, t]) \subset [0, x_{p-4}]$ (note that x_{p-4} is the least positive point among $x_0, \ldots, x_{p-1}$), with f of constant slope λ, in such a way that all the critical points except at most one are sent by f on either 0 or x_{p-4}. If $t = \frac{1}{\lambda}$, there is nothing to do. If $t > \frac{1}{\lambda}$, we set $\ell := t - \frac{1}{\lambda}$ (length of the interval), $k := \left\lfloor \frac{\lambda \ell}{2x_{p-4}} \right\rfloor$,

$$J_i := \left[\frac{1}{\lambda} + (i-1)\frac{2x_{p-4}}{\lambda}, \frac{1}{\lambda} + i\frac{2x_{p-4}}{\lambda}\right] \quad \text{for all } i \in [\![1,k]\!], \tag{1}$$

$$K := \left[\frac{1}{\lambda} + k\frac{2x_{p-4}}{\lambda}, t\right]. \tag{2}$$

If $p=3$, we replace x_{p-4} (not defined) by 1 in the above definitions.

It is possible that there is no interval $J_1, \ldots, J_k$ (if $k=0$) or that K is reduced to the point $\{t\}$. On each interval $J_1, \ldots J_k$, f is defined as the tent map of summit x_{p-4}: $f(\min J_i) = 0$, f is increasing of slope λ on $[\min J_i, \mathrm{mid}(J_i)]$ (thus $f(\mathrm{mid}(J_i)) = x_{p-4}$ because of the length of J_i), then f is decreasing of slope $-\lambda$ on $[\mathrm{mid}(J_i), \max J_i]$ and $f(\max J_i) = 0$. On K, f is defined as a tent map with a summit $< x_{p-4}$: $f(\min K) = 0$, f is increasing of slope λ on $[\min K, \mathrm{mid}(K)]$, then f is decreasing of slope $-\lambda$ on $[\mathrm{mid}(K), \max K]$ and $f(\max K) = 0$.

In this way, we get a map f that is continuous on $[0,1]$, piecewise monotone, of constant slope λ. It remains to define t and the points $\{x_i\}_{0 \le i \le p-4}$ (recall that $x_{p-3} = \frac{1}{\lambda}$, $x_{p-2} = 0$ and $x_{p-1} = 1$).

We want these points to satisfy:

$$x_0 \;=\; \lambda(1-t) \tag{3}$$

and

$$\begin{cases} x_1 &= 1 - \lambda x_0 \\ x_2 &= 1 - \lambda x_1 \\ &\vdots \\ x_{p-3} &= 1 - \lambda x_{p-4} \end{cases} \tag{4}$$

and to be ordered as follows:

$$x_{p-2} < x_{p-4} < \cdots < x_1 < x_0 < x_2 < x_4 < \cdots < x_{p-3} \tag{5}$$

$$\frac{1}{\lambda} \le t < x_{p-1}. \tag{6}$$

If $p=3$, the system (4) is empty, and equation (5) is satisfied because it reduces to $0 = x_1 < x_0 = \frac{1}{\lambda}$.

According to the definition of f, the equations (3), (4), (5), (6) imply that $f(x_i) = x_{i+1}$ for all $i \in [\![0, p-2]\!]$ and $f(x_{p-1}) = x_0$.

We are going to show that the system (4) is equivalent to:

$$\forall i \in [\![0, p-4]\!], \quad x_i = \frac{(-1)^i}{\lambda^{p-i-2}} \sum_{j=0}^{p-i-3} (-\lambda)^j. \tag{7}$$

We use a descending induction on i.

- According to the last line of (4), $x_{p-4} = \frac{1}{\lambda}(1 - x_{p-3}) = \frac{1}{\lambda^2}(\lambda - 1)$. This is (7) for $i = p-4$.

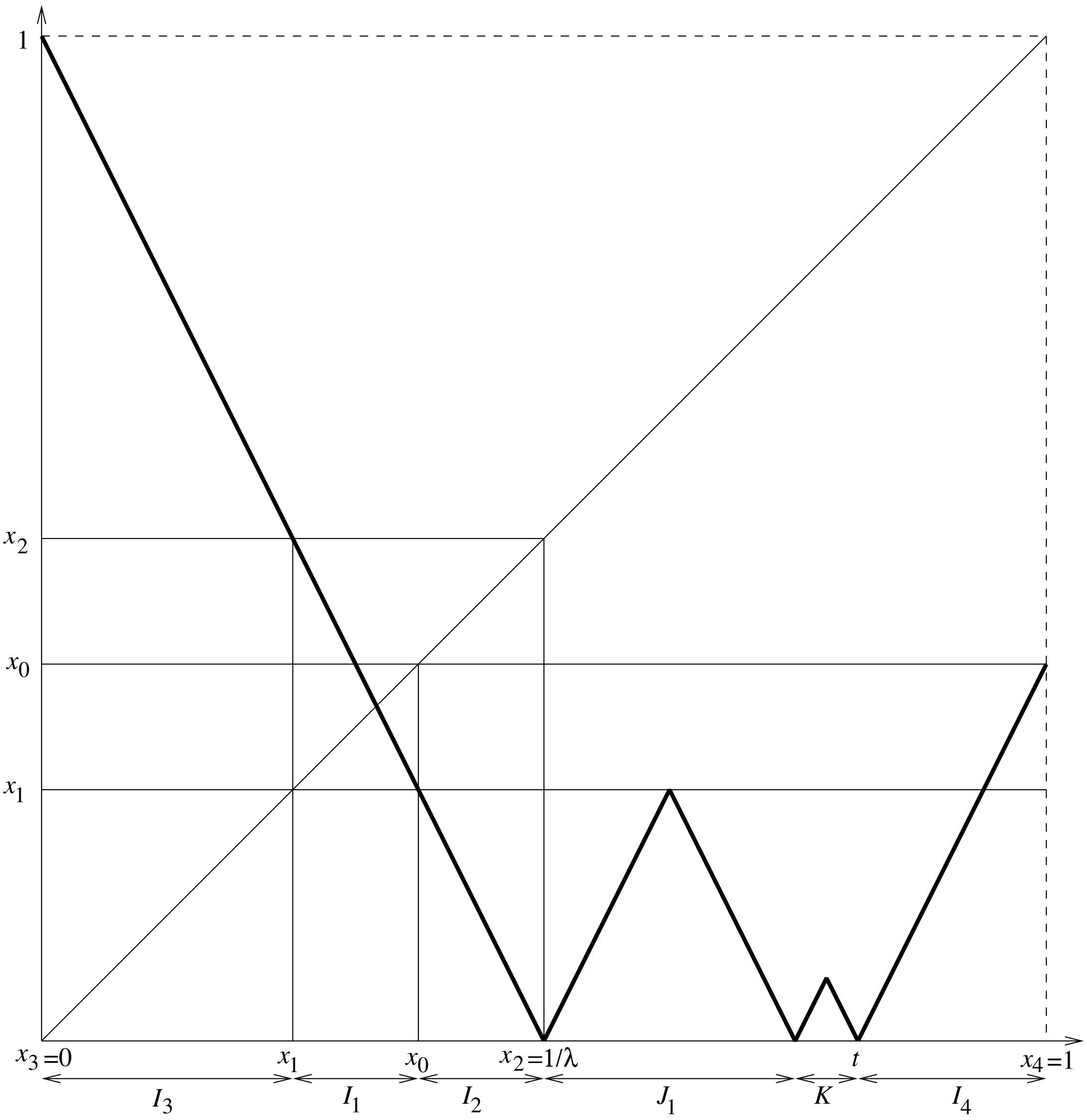

Figure 2: The map f for $p = 5$ and $\lambda = 2$.

- Suppose that (7) holds for i with $i \in [\![1, p-4]\!]$. By (4), $x_i = 1 - \lambda x_{i-1}$, thus

$$\begin{aligned} x_{i-1} &= -\frac{1}{\lambda}(x_i - 1) \\ &= -\frac{(-1)^i}{\lambda^{p-i-1}} \left(\sum_{j=0}^{p-i-3} (-\lambda)^j - (-1)^i \lambda^{p-i-2} \right) \end{aligned}$$

Since p is odd, $-(-1)^i \lambda^{p-i-2} = (-\lambda)^{p-i-2}$. Hence

$$x_{i-1} = \frac{(-1)^{i-1}}{\lambda^{p-i-1}} \sum_{j=0}^{p-i-2} (-\lambda)^j,$$

which gives (7) for $i - 1$. This ends the proof of (7).

Equation (3) is equivalent to $t = 1 - \frac{1}{\lambda} x_0$. Thus, using (7), we get

$$t = \frac{1}{\lambda^{p-1}} \left(\lambda^{p-1} - \sum_{j=0}^{p-3} (-\lambda)^j \right). \tag{8}$$

Conclusion: with the values of $x_0, \ldots, x_{p-4}$ and t given by (7) and (8), the system of equations (3)-(4) is satisfied (and there is a unique solution). It remains to show that these points are ordered as stated in (5) and (6).

Let i be in $[\![0, p-6]\!]$. By (7), we have

$$\begin{aligned} x_{i+2} - x_i &= \frac{(-1)^i}{\lambda^{p-i-2}} \left(\lambda^2 \sum_{j=0}^{p-i-5} (-\lambda)^j - \sum_{j=0}^{p-i-3} (-\lambda)^j \right) \\ &= \frac{(-1)^i}{\lambda^{p-i-2}} \left(\sum_{j=2}^{p-i-3} (-\lambda)^j - \sum_{j=0}^{p-i-3} (-\lambda)^j \right) \\ &= \frac{(-1)^i}{\lambda^{p-i-2}} (\lambda - 1) \end{aligned}$$

Since $\lambda - 1 > 0$, we have, for all $i \in [\![0, p-6]\!]$,

- $x_i < x_{i+2}$ if i is even,
- $x_{i+2} < x_i$ if i is odd.

By (7), $x_{p-4} = \frac{\lambda - 1}{\lambda^2}$. Since $\lambda > 1$, $x_{p-4} > 0 = x_{p-2}$. Again by (7),

$$\begin{aligned} x_0 - x_1 &= \frac{1}{\lambda^{p-2}} \left(\sum_{j=0}^{p-3} (-\lambda)^j + \lambda \sum_{j=0}^{p-4} (-\lambda)^j \right) \\ &= \frac{1}{\lambda^{p-2}} \left(\sum_{j=0}^{p-3} (-\lambda)^j - \sum_{j=1}^{p-3} (-\lambda)^j \right) \\ &= \frac{1}{\lambda^{p-2}} > 0 \end{aligned}$$

thus $x_0 < x_1$. Moreover,

$$x_{p-3} - x_{p-5} = \frac{1}{\lambda} - \frac{\lambda^2 - \lambda + 1}{\lambda^3} = \frac{\lambda - 1}{\lambda^3} > 0$$

thus $x_{p-5} < x_{p-3}$. This several inequalities imply (5).

By (8), we have

$$t - \frac{1}{\lambda} = \frac{1}{\lambda^{p-1}} \left(\lambda^{p-1} - \lambda^{p-2} - \sum_{j=0}^{p-3} (-\lambda)^j \right) = \frac{1}{\lambda^{p-1}} \cdot \chi_p(\lambda),$$

where χ_p is defined in Lemma 3.2. According to this lemma, $\chi_p(\lambda) \geq 0$ (with equality iff $\lambda = \lambda_p$) because $\lambda \geq \lambda_p$. This implies that $t \geq \frac{1}{\lambda}$ (with equality iff $\lambda = \lambda_p$). Moreover, if $t \geq 1$, then $x_0 = \lambda(1 - t) \leq 0$, which is impossible by (5); thus $t < 1$. Therefore, the inequalities (6) hold.

Finally, we have shown that the map $f_{p,\lambda} = f$ is defined as wanted.

3.2 Entropy

Corollary 3.4 $h_{top}(f_{p,\lambda}) = \log \lambda$.

Proof. This result is given by Theorem 3.1 because, by definition, $f_{p,\lambda}$ is piecewise monotone of constant slope λ with $\lambda > 1$. □

3.3 Type

Lemma 3.5 *Let $g\colon [0,1] \to [0,1]$ be a continuous map. Let $\mathcal{A}$ be a finite family of closed intervals that form a pseudo-partition of $[0,1]$, that is,*

$$\bigcup_{A\in\mathcal{A}} A = [0,1] \quad \text{and} \quad \forall A, B \in \mathcal{A},\ A \neq B \Rightarrow \mathrm{Int}\,(A) \cap \mathrm{Int}\,(B) = \emptyset.$$

We set $\partial\mathcal{A} = \bigcup_{A\in\mathcal{A}} \partial A$. Let $\mathcal{G}$ be the oriented graph whose vertices are the elements of $\mathcal{A}$ and in which there is an arrow $A \dashrightarrow B$ iff $g(A) \cap \mathrm{Int}\,(B) \neq \emptyset$. Let x be a periodic point of period q for g such that $\{g^n(x) \mid n \geq 0\} \cap \partial\mathcal{A} = \emptyset$. Then there exist $A_0, \ldots, A_{q-1} \in \mathcal{A}$ such that $A_0 \dashrightarrow A_1 \dashrightarrow \cdots \dashrightarrow A_{q-1} \dashrightarrow A_0$ is a cycle in the graph $\mathcal{G}$.

Proof. For every $n \geq 0$, there exists a unique element $A_n \in \mathcal{G}$ such that $g^n(x) \in \mathrm{Int}\,(A_n)$ because $\{g^n(x) \mid n \geq 0\} \cap \partial\mathcal{A} = \emptyset$. We have $g^n(x) \in A_n$ and $g^{n+1}(x) \in \mathrm{Int}\,(A_{n+1})$, thus $g(A_n) \cap \mathrm{Int}\,(A_{n+1}) \neq \emptyset$; in other words, there is an arrow $A_n \dashrightarrow A_{n+1}$ in $\mathcal{G}$. Finally, $A_q = A_0$ because $g^q(x) = x$. □

Proposition 3.6 *The map $f_{p,\lambda}$ is of type p for Sharkovskii's order.*

Proof. According to the definition of $f = f_{p,\lambda}$, x_0 is a periodic point of period p. It remains to show that f has no periodic point of period q with q odd and $3 \leq q < p$.

We set $I_1 := \langle x_0, x_1\rangle$, $I_i := \langle x_{i-2}, x_i\rangle$ for all $i \in [\![2, p-2]\!]$ and $I_{p-1} := [t, 1]$, where $\langle a, b\rangle$ denotes the convex hull of $\{a, b\}$ (i.e. $\langle a, b\rangle = [a, b]$ or $[b, a]$). The intervals J_i, K have been defined in (1) and (2). The family $\mathcal{A} := \{I_1, \ldots, I_{p-1}, J_1, \ldots, J_k, K\}$ is a pseudo-partition of $[0,1]$. Let $\mathcal{G}$ be the oriented graph associated to $\mathcal{A}$ for the map $g = f$ as defined in Lemma 3.5. If $f(A) \supset B$, the arrow $A \dashrightarrow B$ is replaced by $A \to B$ (full covering). The graph $\mathcal{G}$ is represented in Figure 3; a dotted arrow $A \dashrightarrow B$ means that $f(A) \cap \mathrm{Int}\,(B) \neq \emptyset$ but $f(A) \not\supset B$ (partial covering).

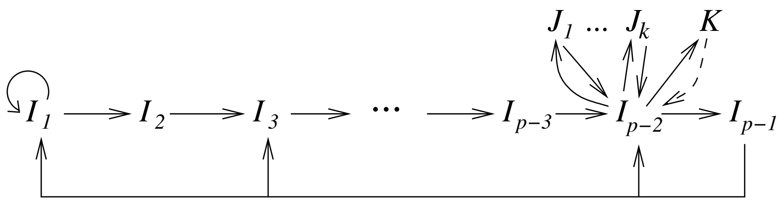

Figure 3: Covering graph $\mathcal{G}$ associated to f.

The subgraph associated to the intervals $I_1, \ldots, I_{p-1}$ is the graph associated to a Štefan cycle of period p (see [5, Lemma 3.16]). The only additional arrows with respect to the Štefan graph are between the intervals $J_1, \ldots, J_k, K$ on the one hand and I_{p-2} on the other hand. There is only one partial covering, which is $K \dashrightarrow I_{p-2}$.

Let q be an odd integer with $3 \leq q < p$. We easily see that, in this graph, there is no primitive cycle of length q (a cycle is primitive if it is not the repetition of a shorter cycle): the cycles not passing through I_1 have an even length, whereas the cycles passing through I_1 have a length either equal to 1, or greater than or equal to $p-1$. Moreover, if x is a periodic point of period q, then $\{f^n(x) \mid n \geq 0\} \cap \partial\mathcal{A} = \emptyset$ (because the periodic points in $\partial\mathcal{A}$ are of period p). According to Lemma 3.5, f has no periodic point of period q. Conclusion: f is of type p for Sharkovskii's order. □

3.4 Mixing

Proposition 3.7 *The map $f_{p,\lambda}$ is topologically mixing.*

Proof. This proof is inspired by [5, Lemmas 2.10, 2.11] and their use in [5, Example 2.13].

We will use several times that the image by $f = f_{p,\lambda}$ of a nondegenerate interval is a nondegenerate interval (and thus all its iterates are nondegenerate).

Let A be a nondegenerate closed interval included in $[0,1]$. We are going to show that there exists an integer $n \geq 0$ such that $f^n(A) = [0,1]$.

We set

$$\mathcal{C}_0 := \bigcup_{i=1}^{k} \partial J_i \cup \{t\}, \quad \mathcal{C}_1 := \{\mathrm{mid}(J_i) \mid i \in [\![1,k]\!]\}, \quad c_K := \mathrm{mid}(K).$$

The set of critical points of f is $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \{c_K\}$.

Step 1: there exists $i_0 \geq 0$ such that $f^{i_0}(A) \cap (\mathcal{C}_0 \cup \mathcal{C}_1) \neq \emptyset$ and there exists $n_0 \geq 0$ such that $0 \in f^{n_0}(A)$.

Let

$$J_i' := [\min J_i, \mathrm{mid}(J_i)] \text{ and } J_i'' := [\mathrm{mid}(J_i), \max J_i] \quad \text{for all } i \in [\![1,k]\!],$$
$$\mathcal{F} := \left\{ \left[0, \frac{1}{\lambda}\right], [t,1], K \right\} \cup \{J_i', J_i'' \mid i \in [\![1,k]\!]\}.$$

If $A \subset B$ for some $B \in \mathcal{F}$ and $B \neq K$, then $|f(A)| = \lambda |A|$. If $A \subset K$, then $|f(A)| \geq \frac{\lambda |A|}{2}$ and $f(A) \subset I_{p-2}$, thus $|f^2(A)| = \lambda |f(A)| \geq \frac{\lambda^2}{2}|A|$. We have $\lambda > 1$ and $\frac{\lambda^2}{2} > 1$ because $\lambda > \sqrt{2}$ (Theorem 1.4). If for all $i \geq 0$, there exists $A_i \in \mathcal{F}$ such that $f^i(A) \subset A_i$, then what precedes implies that $\lim_{i \to +\infty} |f^i(A)| = +\infty$. This is impossible because $f^i(A) \subset [0,1]$. Thus there exist $i_0 \geq 0$ and $c \in \mathcal{C}_0 \cup \mathcal{C}_1$ such that $c \in f^{i_0}(A)$. If $c \in \mathcal{C}_0$, then $f(c) = 0$, and hence $0 \in f^{i_0+1}(A)$. If $c \in \mathcal{C}_1$, then $f(c) = x_{p-4}$ and hence $0 \in f^{i_0+3}(A)$. This ends step 1.

Step 2: there exist $n_1 \geq n_0$ and $j \in [\![1, p-1]\!]$ such that $f^{n_1}(A) \supset I_j$.

Recall that $I_1 = [x_1, x_0]$, $I_i = \langle x_{i-2}, x_i \rangle$ for all $2 \leq i \leq p-2$ and $I_{p-1} = [t,1] = [t, x_{p-1}]$. We set $I_0 := I_1$. By definition, for all $0 \leq i \leq p-1$, there exists $\delta_i > 0$ such that $I_i = \langle x_i, x_i + (-1)^{i+1}\delta_i \rangle$. Moreover, f is linear of slope $-\lambda$ on each of the intervals $I_0, \ldots, I_{p-2}$ and of slope $+\lambda$ on I_{p-1}.

We set $B_{-2} := f^{n_0}(A)$. This is a nondegenerate closed interval containing 0, thus there exists $b > 0$ such that $B_{-2} = [0, b]$ with $0 = x_{p-2}$. We set $B_i := f^{i+2}(B_{-2})$ for all $i \geq -2$, and we define $m \geq -2$ as the least integer such that B_m is not included in a interval of the form I_j (such an integer m exists by step 1).

If $b > x_{p-4}$, then $B_{-2} \supset I_{p-2}$ and $m = -2$. Otherwise, $B_{-2} \subset I_{p-2}$ and $B_{-1} = [1 - \lambda b, 1] = [x_{p-1} - \lambda b, x_{p-1}]$ because $f|_{I_{p-2}}$ is of slope $-\lambda$. If $1 - \lambda b < t$, then $B_{-1} \supset I_{p-1}$ and $m = -1$. Otherwise, $B_{-1} \subset I_{p-1}$ and $B_0 = [x_0 - \lambda^2 b, x_0]$ because $f|_{I_{p-1}}$ is of slope $+\lambda$. We go on in a similar way.

- If $m > 0$, then $B_0 \subset I_0$ and $B_1 = [x_1, x_1 + \lambda^3 b]$.

- If $m > 1$, then $B_1 \subset I_1$ and $B_2 = [x_2 - \lambda^4 b, x_2]$.

$\vdots$

- If $m > p-3$, then $B_{p-3} \subset I_{p-3}$ and $B_{p-2} = \langle x_{p-2}, x_{p-2} + (-1)^{p+1}\lambda^p b\rangle = [0, \lambda^p b]$.

Notice that B_{p-2} is of the same form as B_{-2}. What precedes implies that

$$\forall i \in [\![-2, m]\!],\ B_i = \left\langle x_{i \bmod p}, x_{i \bmod p} + (-1)^{r+1}\lambda^{i+2} b\right\rangle,\ \text{where } i = qp + r,\ r \in [\![0, p-1]\!],$$
$$\forall i \in [\![-2, m-1]\!],\ B_i \subset I_{i \bmod p},$$
$$B_m \supset I_{m \bmod p}.$$

This ends step 2 with $n_1 := n_0 + m + 2$ and $j := m$.

Step 3: there exists $n_2 \geq n_1$ such that $f^{n_2}(A) = [0, 1]$.

Let $n_1 \geq 0$ and let $j \in [\![1, p-1]\!]$ be such that $f^{n_1}(A) \supset I_j$ (step 2). In the covering graph of Figure 3, we see that there exists an integer $q \geq 0$ such that, for every vertex C of the graph, there exists a path of length q, with only arrows of type $\to$, starting from I_j and ending at C. This implies that $f^q(I_j) = [0, 1]$, that is, $f^{n_1+q}(A) = [0, 1]$.

We have shown that, for every nondegenerate closed interval $A \subset [0, 1]$, there exists n such that $f^n(A) = [0, 1]$. We conclude that f is topologically mixing. □

4 General case

4.1 Square root of a map

We first recall the definition of the so-called *square root* of an interval map. If $f\colon [0, b] \to [0, b]$ is an interval map, the square root of f is the continuous map $g\colon [0, 3b] \to [0, 3b]$ defined by

- $\forall x \in [0, b]$, $g(x) := f(x) + 2b$,
- $\forall x \in [2b, 3b]$, $g(x) := x - 2b$,
- g is linear on $[b, 2b]$.

The graphs of g and g^2 are represented in Figure 4.

The square root map has the following properties, see e.g. [5, Examples 3.22 and 4.62].

Proposition 4.1 *Let f be an interval map of type n, and let g be the square root of f. Then g is of type $2n$ and $h_{top}(g) = \frac{h_{top}(f)}{2}$. If f is piecewise monotone, then g is piecewise monotone too.*

4.2 Piecewise monotone map of given entropy and type

Theorem 4.2 *Let $p \geq 3$ be an odd integer, let d be a non negative integer and λ a real number such that $\lambda \geq \lambda_p$. Then there exists a piecewise monotone map f whose type is $2^d p$ for Sharkovskii's order and such that $h_{top}(f) = \frac{\log \lambda}{2^d}$. If $d = 0$, the map f can be built in such a way that it is topologically mixing.*

Proof. If $d = 0$, we take $f = f_{p,\lambda}$ defined in Section 3.

If $d > 0$, we start with the map $f_{p,\lambda}$, then we build the square root of $f_{p,\lambda}$, then the square root of the square root, etc. According to Proposition 4.1, after d steps we get a piecewise monotone interval map f of type $2^d p$ and such that $h_{top}(f) = \frac{h_{top}(f_{p,\lambda})}{2^d} = \frac{\log \lambda}{2^d}$. □

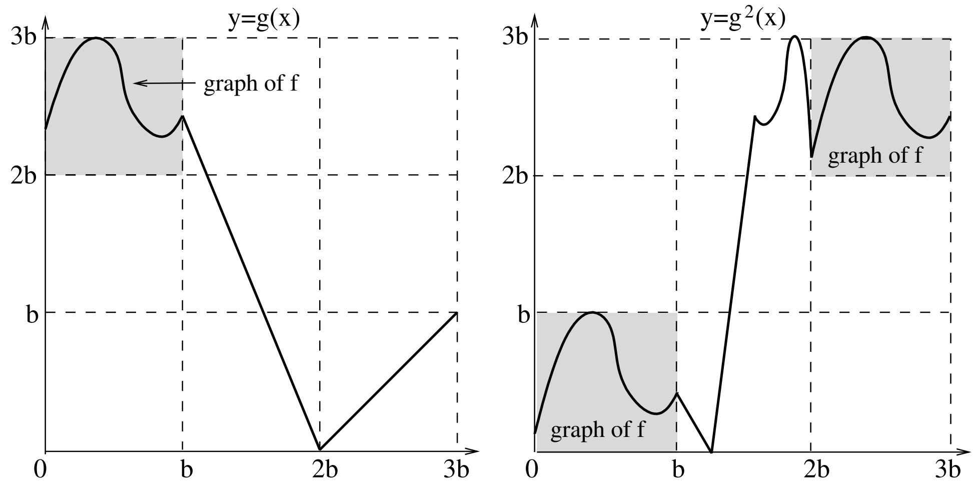

Figure 4: The left side represents the map g, which is the square root of f. The right side represents the map g^2.

Corollary 4.3 *For every positive real number h, there exists a piecewise monotone interval map f such that $h_{top}(f) = h$.*

Proof. Let $d \geq 0$ be an integer such that $\frac{\log \lambda_3}{2^d} \leq h$ and set $\lambda := \exp(2^d h)$. Then $\lambda \geq \lambda_3$ and, according to Theorem 4.2, there exists a piecewise monotone interval map f of type $2^d 3$ such that $h_{top}(f) = \frac{\log \lambda}{2^d} = h$. □

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