REALIZATIONS FUNCTORS

by

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Abstract. — This article is a written reenactement of my lecture at the Asian-French summer school *Motives and related topics* at the I.H.É.S. in July 2006. The paper focuses on Weil cohomologies, comparison between them and their applications to Zeta functions over finite fields and algebraic cycles.

Résumé. — Cet article est une version rédigée de mon exposé à l'école d'été franco-asiatique *Autour des motifs* à l'I.H.É.S. en juillet 2006. L'essentiel du texte est consacré aux cohomologies de Weil, aux résultats de comparaison les liant et à leurs applications aux fonctions Zêta sur les corps finis et à la théorie des cycles algébriques.

Contents

1. Weil cohomologies

We fix a base field k. We let $\mathcal{V}$ be the category of smooth and projective algebraic varieties over k.

Let F be a field of coefficients. We assume that the characteristic of F is zero. The $\otimes$-category [(1)] of $\mathbf{Z}$-graded finite dimensional F-vector spaces is denoted VecGr_F. The signs in the commutativity constraint of VecGr_F are given by the Koszul rule.

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(1) In this paper, a $\otimes$-category is a « $\otimes$-catégorie linéaire ACU » in the sense of [**15**].

1.1. Definitions. —

Definition 1.1. — A Weil cohomology is a functor $H\colon \mathcal{V}^{\mathrm{opp}} \to \mathrm{VecGr}_F$ with the following properties and additional data:

- H is a $\otimes$-functor: we have a Künneth formula $H(X) \otimes H(Y) \xrightarrow{\sim} H(X \times Y)$ for all $X, Y \in \mathcal{V}$;
- for all $X \in \mathcal{V}$, $H(X)$ lies in nonnegative degrees;
- for all $X, Y \in \mathcal{V}$, we have a canonical isomorphism $H(X \sqcup Y) \xrightarrow{\sim} H(X) \times H(Y)$;
- the F-vector space $H^2(\mathbf{P}^1)$ is one-dimensional, its dual is denoted $F(1)$ (for any F-vector space V and integer n, we define $V(n) = V \otimes_F F(1)^{\otimes n}$);
- for any d-dimensional $X \in \mathcal{V}$, there is a multiplicative trace map $H^{2d}(X)(d) \to F$ inducing perfect Poincaré duality pairings:
$$H^i(X) \otimes H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to F \ ;$$
- there is a cycle class map $\mathrm{cl}\colon CH^\star(X) \to H^{2\star}(X)(\star)$, contravariant in $X \in \mathcal{V}$, compatible with products and normalized with the trace map so that the trace of the cycle classes of 0-cycles be given by the degree;
- $\mathrm{cl}([\infty])$ is the canonical generator of $H^2(\mathbf{P}^1)(1)$.

Proposition 1.2. — *If $H\colon \mathcal{V}^{\mathrm{opp}} \to \mathrm{VecGr}_F$ is a $\otimes$-functor which can be endowed with the structure of a Weil cohomology, then the cycle class is unique.*

The proof of this proposition shall be postponed until the end of this subsection.

Definition 1.3. — A cycle $x \in CH^d(X) \otimes F$ is homologically equivalent to zero (with respect to the Weil cohomology H) if $\mathrm{cl}\, x = 0$ in $H^{2d}(X)(d)$. This is an adequate equivalence relation on cycles. The corresponding category of pure motives is denoted $\mathrm{Mot}_{\mathrm{hom},F}$.

Proposition 1.4. — *Let H be a Weil cohomology. There is an associated realization $\otimes$-functor $r_H\colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$ such that $r_H(h(X)) \simeq H(X)$.*

Proof. — Let X and Y be in $\mathcal{V}$. Let d_X be the dimension X. Let $\alpha \in CH^{d_X}(X \times Y)$. We shall define an action $H(X) \to H(Y)$ of α. As α can be considered as a morphism $h(X) \to h(Y)$ in $\mathrm{Mot}_{\mathrm{rat}}$, it will give the expected functoriality.

The cycle class provides an element $\mathrm{cl}\,\alpha \in H^{2d_X}(X \times Y)(d_X)$. We use the Künneth formula to think of this class as a family of elements in
$$H^{2d_X - p}(X)(d_X) \otimes H^p(Y)$$
for all p. Then, we use the Poincaré duality for X to get a family of elements in
$$H^p(X)^\vee \otimes H^p(Y) \simeq \mathbf{Hom}(H^p(X), H^p(Y)) \ .$$
We thus have defined the action $H(X) \to H(Y)$ of any Chow correspondence α.

The verification of the details are left to the reader. □

Definition 1.5. — If H is Weil cohomology, $M \in \mathrm{Mot}$ and $n \in \mathbf{Z}$, we define $H^n(M)$ to be the degree n part of $r_H(M)$.

Remark 1.6. — The functor r_H factors through the homological equivalence to give a faithful functor $\mathrm{Mot}_{\mathrm{hom},F} \to \mathrm{VecGr}_F$.

We can give a new (equivalent) definition of a Weil cohomology using $\otimes$-functors.

Definition 1.7. — A Weil cohomology is a $\otimes$-functor

$$r\colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$$

such that $r(\mathbf{L})$ lies in degree 2 and for all $X \in \mathcal{V}$, $r(h(X))$ lies in nonnegative degrees [(2)].

Proposition 1.4 shows that a Weil cohomology in the sense of definition 1.1 naturally gives a Weil cohomology in the sense of definition 1.7. The converse is also true (see [**2**, proposition 4.2.5.1]). Then it makes sense to think of $\mathrm{Mot}_{\mathrm{rat}}$ as the universal coefficient category for a Weil cohomology.

Remark 1.8. — If $\mathcal{E}$ is a vector bundle of rank d over $X \in \mathcal{V}$, we denote $\mathbf{P}(\mathcal{E})$ the associated projective bundle. Then, the powers of $c_1(\mathcal{O}(1)) \in CH^1(\mathbf{P}(\mathcal{E}))$ give a canonical isomorphism

$$\bigoplus_{i=0}^{d-1} h(X) \otimes \mathbf{L}^{\otimes i} \xrightarrow{\sim} h(\mathbf{P}(\mathcal{E}))$$

in $\mathrm{Mot}_{\mathrm{rat}}$. This is the motivic projective bundle formula. It follows that for any Weil cohomology H, we also have a projective bundle formula

$$\bigoplus_{i=0}^{d-1} H^{\star-2i}(X)(-i) \xrightarrow{\sim} H^{\star}(\mathbf{P}(\mathcal{E}))\ .$$

Proof of proposition 1.2. — Let $(H, \mathrm{cl}, \mathrm{Tr})$ and $(H, \mathrm{cl}', \mathrm{Tr}')$ be two Weil cohomology structures on the same $\otimes$-functor $H\colon \mathcal{V}^{\mathrm{opp}} \to \mathrm{VecGr}_F$. By definition, the maps cl and cl′ agree on $CH^\star(\mathbf{P}^1)$. It follows from the motivic projective bundle formula that the obvious maps $H^2(\mathbf{P}^n) \to H^2(\mathbf{P}^1)$ are isomorphisms for all $n \geq 1$. As the algebra $CH^\star(\mathbf{P}^n)$ is generated by its homogeneous elements of degree 1, we get that cl and cl′ agree on $CH^\star(\mathbf{P}^n)$ for all n.

We can consider the compositions of cl and cl′ with the Chern character $\mathrm{ch}\colon K_0(X) \to CH^\star(X)_{\mathbf{Q}}$. Once tensored with $\mathbf{Q}$, this ring morphism becomes an isomorphism (see [**6**, example 15.2.16]). Thus, we only have to prove that cl and cl′ agree on the image on the Chern character. We already know that they agree on $\mathrm{ch}([\mathcal{O}(1)]) \in CH^\star(\mathbf{P}^n)_{\mathbf{Q}}$ for all n where $\mathcal{O}(1)$ is the fundamental sheaf on $\mathbf{P}^n$. By functoriality, they agree on $\mathrm{ch}(f^\star[\mathcal{O}(1)])$ for any morphism of the form $f\colon X \to \mathbf{P}^n$ in $\mathcal{V}$. As a result, if $\mathcal{L}$ is a line bundle over $X \in \mathcal{V}$ that is generated by its global sections, then $\mathrm{cl}(\mathrm{ch}([\mathcal{L}])) = \mathrm{cl}'(\mathrm{ch}([\mathcal{L}]))$. As any line bundle over $X \in \mathcal{V}$ is isomorphic to a line bundle of the form $\mathcal{L} \otimes \mathcal{M}^\vee$ where $\mathcal{L}$ and $\mathcal{M}$ are line bundles generated by their global sections, we see that $\mathrm{cl} \circ \mathrm{ch}$ and $\mathrm{cl}' \circ \mathrm{ch}$ agree on classes of line bundles. They shall also agree on $\mathbf{Z}$-linear combinations of classes of line bundles. It follows

[(2)] The Lefschetz motive $\mathbf{L}$ is defined by the canonical decomposition $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbf{L}$. We let $\mathbf{T}$ be the $\otimes$-inverse of $\mathbf{L}$ in the category $\mathrm{Mot}_{\mathrm{rat}}$.

from the projective bundle formula that we can use the splitting principle to prove that $\mathrm{cl} \circ \mathrm{ch} = \mathrm{cl}' \circ \mathrm{ch}$. □

1.2. Traces. —

1.2.1. Definitions. —

***Definition 1.9* (Dold-Puppe [5]).** — Let $\mathcal{T}$ be a $\otimes$-category. Let M be an object of $\mathcal{T}$. We say that M has a strong dual if there exists an object N of $\mathcal{T}$ and maps $\eta\colon \mathbf{1} \to M \otimes N$ and $\varepsilon\colon N \otimes M \to \mathbf{1}$ such that the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\eta \otimes M} & M \otimes N \otimes M \\ & \searrow^{=} & \downarrow{\scriptstyle M \otimes \varepsilon} \\ & & M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{N \otimes \eta} & N \otimes M \otimes N \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon \otimes N} \\ & & N \end{array}$$

If M has a strong dual, the internal Hom. functor $\mathbf{Hom}(M, -)$ exists and (M, η, ε) is unique up to a unique isomorphism. If we define $M^\vee$ as $\mathbf{Hom}(M, \mathbf{1})$, there is a canonical isomorphism $N \simeq M^\vee$ and the canonical morphism is an isomorphism:

$$M^\vee \otimes X \xrightarrow{\sim} \mathbf{Hom}(M, X)$$

for any $X \in \mathcal{T}$.

***Definition 1.10*.** — Let $\mathcal{T}$ be a $\otimes$-category. We say that $\mathcal{T}$ is rigid if all its objects have strong duals.

If $f\colon M \to N$ is a morphism in a rigid $\otimes$-category, we can define the transpose morphism ${}^t f\colon N^\vee \to M^\vee$ of f. It is obtained using the functor $\mathbf{Hom}(-, \mathbf{1})$.

***Definition 1.11*.** — Let $\mathcal{T}$ be a rigid $\otimes$-category. Let $f\colon M \to M$ be an endomorphism in $\mathcal{T}$. We define the trace $\mathrm{Tr}_{\mathcal{T}}\, f \in \mathrm{End}_{\mathcal{T}}(\mathbf{1})$ of f as the following composition:

$$\mathbf{1} \xrightarrow{\eta} M \otimes M^\vee \xrightarrow{f \otimes M^\vee} M \otimes M^\vee \simeq M^\vee \otimes M \xrightarrow{\varepsilon} \mathbf{1} \ .$$

***Proposition 1.12*.** — *Let $\mathcal{T}$ be a rigid $\otimes$-category. Let M be an object of $\mathcal{T}$. Let f and g be elements of $\mathrm{End}_{\mathcal{T}}(M)$. Let $\lambda \in \mathrm{End}_{\mathcal{T}}(\mathbf{1})$. Then, we have some formulas:*

$$\mathrm{Tr}_{\mathcal{T}}(f + g) = \mathrm{Tr}_{\mathcal{T}}\, f + \mathrm{Tr}_{\mathcal{T}}\, g \ ;$$
$$\mathrm{Tr}_{\mathcal{T}}(g \circ f) = \mathrm{Tr}_{\mathcal{T}}(f \circ g) \ ;$$
$$\mathrm{Tr}_{\mathcal{T}}(\lambda \cdot f) = \lambda \cdot \mathrm{Tr}_{\mathcal{T}}\, f \ ;$$
$$\mathrm{Tr}_{\mathcal{T}}({}^t f) = \mathrm{Tr}_{\mathcal{T}}\, f \ .$$

Traces in rigid $\otimes$-categories satisfy an important compatibility with respect to $\otimes$-functors:

***Proposition 1.13* ([15, 5.2.4.1, chapitre I]).** — *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-functor between rigid $\otimes$-categories. Let $f\colon M \to M$ be an endomorphism in $\mathcal{T}$. Then, there is an equality in $\mathrm{End}_{\mathcal{T}'}(\mathbf{1})$:*

$$F(\mathrm{Tr}_{\mathcal{T}}\, f) = \mathrm{Tr}_{\mathcal{T}'}\, F(f).$$

1.2.2. Traces in VecGr_F. — Obviously, the category VecGr_F is rigid and the computation of traces in that category reduces to usual traces of endomorphism of vector spaces:

Proposition 1.14. — *Let $f\colon V \to V$ be an endomorphism in VecGr_F. Then,*

$$\mathrm{Tr}_{\mathrm{VecGr}_F} f = \sum_{n\in\mathbf{Z}} (-1)^n \,\mathrm{Tr}_F(f\colon V^n \to V^n)\ .$$

1.2.3. Traces in $\mathrm{Mot}_{\mathrm{rat}}$. —

Proposition 1.15. — *The category* $\mathrm{Mot}_{\mathrm{rat}}$ *is rigid.*

Proof. — One only needs to prove that for any $X \in \mathcal{V}$, the motive of X has a strong dual. We let M be the motive of X, N be $M \otimes \mathbf{T}^d$. By definition of the category $\mathrm{Mot}_{\mathrm{rat}}$, there are isomorphisms

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}}(\mathbf{1}, M\otimes N) \simeq CH^d(X\times X) \simeq \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}}(N\otimes M, \mathbf{1})\ .$$

We define ε and η to be the morphisms corresponding to the cycle associated to the diagonal Δ_X in $X \times X$. A simple computation shows that it makes $N = h(X) \otimes \mathbf{T}^d$ the strong dual of $M = h(X)$. □

Proposition 1.16. — *Let $X \in \mathcal{V}$. Let $\alpha \in CH^{d_X}(X\times X)$. We consider the morphism $f\colon h(X) \to h(X)$ in $\mathrm{Mot}_{\mathrm{rat}}$ associated to α. Then, we have an equality of integers:*

$$\mathrm{Tr}_{\mathrm{Mot}_{\mathrm{rat}}}(f\colon h(X)\to h(X)) = \deg(\alpha\cdot[\Delta_X])\ ,$$

where Δ_X is the diagonal in $X \times X$.

More generally, if $\beta \in CH^{d_X}(X\times X)$ is a cycle corresponding to a second morphism $g\colon h(X)\to h(X)$ in $\mathrm{Mot}_{\mathrm{rat}}$, we have an equality:

$$\mathrm{Tr}_{\mathrm{Mot}_{\mathrm{rat}}}(g\circ f\colon h(X)\to h(X)) = \deg(\alpha\cdot\tau^\star(\beta))\ ,$$

where $\tau\colon X\times X \to X\times X$ is the intertwining automorphism.

It follows from simple computations in Chow groups of powers of X.

1.2.4. Lefschetz's trace formula. —

Theorem 1.17. — *Let H be a Weil cohomology. Let $X \in \mathcal{V}$. Let $f\colon h(X)\to h(X)$ be an endomorphism of the motive of X: f corresponds to a cycle class $\alpha \in CH^{d_X}(X\times X)_{\mathbf{Q}}$. Let $[\Delta_X] \in CH^{d_X}(X\times X)$ be the class of the diagonal. Then there is an equality of rational numbers:*

$$\deg(\alpha\cdot[\Delta_X]) = \sum_{n=0}^{2d_X} \mathrm{Tr}_F(f\colon H^n(X)\to H^n(X))\ .$$

Proof. — Thanks to proposition 1.13, if we compute the trace of f in $\mathrm{Mot}_{\mathrm{rat}}$ or in VecGr_F after the application of the realization functor $r_H\colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$, we shall get the same result. Then, the equality follows from the computations of propositions 1.14 and 1.16. □

2. Applications of Weil cohomologies

2.1. Zeta functions. — Let $k = \mathbf{F}_q$ be a finite field with q elements.

In this subsection, we shall prove that the existence of a Weil cohomology over k implies the rationality of the Zeta function of a smooth and projective variety over k and that it satisfies a functional equation, which statements are part of the Weil conjecture.

2.1.1. Definition of Zeta functions. —

Definition 2.1. — Let $f\colon M \to M$ be an endomorphism of an object in a rigid F-linear $\otimes$-category $\mathcal{T}$ (with $\operatorname{End}_{\mathcal{T}}(\mathbf{1}) = F$). We assume that the characteristic of F is zero. We define

$$Z(f,t) = \exp\left(\sum_{n=1}^{\infty} \operatorname{Tr}_{\mathcal{T}}(f^n)\frac{t^n}{n}\right) \in F[[t]] \ .$$

Remark 2.2. — If $r\colon \mathcal{T} \to \mathcal{T}'$ is a F-linear $\otimes$-functor and $f \in \operatorname{End}_{\mathcal{T}}(M)$, then $Z(r(f),t) = r(Z(f),t)$.

2.1.2. Computations in VecGr_F. —

Proposition 2.3. — *Let $f\colon V \to V$ and $g\colon W \to W$ be endomorphisms in* VecGr_F. *We assume that W is 1-dimensional (lying in degree d) and that g is the multiplication by $\lambda \in F$. Then, we have the relation*

$$Z(f \otimes g, t) = Z(f, \lambda t)^{(-1)^d} \ .$$

Proof. — It follows from the equality

$$\operatorname{Tr}_{\operatorname{VecGr}_F}\left[(f\otimes g)^n\right] = (-1)^d \lambda^n \operatorname{Tr}_{\operatorname{VecGr}_F}(f^n) \ ,$$

for all $n \geq 1$. □

Proposition 2.4. — *Let $f\colon V \to V$ be an endomorphism in* VecGr_F. *For any $n \in \mathbf{Z}$, we define $P_n(t) = \det(\operatorname{id} - tf\colon V_n \to V_n) \in F[t]$. Then,*

$$Z(f,t) = \prod_{n\in\mathbf{Z}} P_n(t)^{(-1)^{n+1}} \ .$$

Proof. — First, we observe that if $0 \to V' \to V \to V'' \to 0$ is an exact sequence in VecGr_F such that $f(V') \subset V'$, there are induced endomorphisms $f'\colon V' \to V'$ and $f''\colon V'' \to V''$ and then the additivity of traces implies the following relation

$$Z(f,t) = Z(f',t)\cdot Z(f'',t) \ .$$

We may enlarge the field F so that we can assume that the eigenvalues of f lie in F. Then, by « dévissage », the general formula reduces to the case of a 1-dimensional V (lying in degree zero). For any $\lambda \in F$, we have to consider the endomorphism

$\lambda\colon F \to F$ given by the multiplication by λ. In that case, the formula reduces to the simple equation:

$$Z(\lambda\colon F \to F, t) = \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n}\right) = \frac{1}{1-\lambda t}\ .$$

□

Proposition 2.5. — *Let $f\colon V \to V$ be an automorphism in* VecGr_F. *We consider the dual $V^\vee \to V$ of V and the automorphism ${}^t f^{-1}\colon V^\vee \to V^\vee$. Then, we have an equality*

$$Z({}^t f^{-1}, \frac{1}{t}) = (-t)^{\chi(V)} \cdot \det(f) \cdot Z(f,t)\ ,$$

where $\chi(V) = \sum_{n\in\mathbf{Z}} (-1)^n \dim V_n$ *and* $\det f = \prod_{n\in\mathbf{Z}} \det(f\colon V_n \to V_n)^{(-1)^n}$.

Proof. — As above, we can assume that V is 1-dimensional and lies in degree zero. Then, it reduces to the following trivial equality:

$$\frac{1}{1-\frac{1}{\lambda t}} = \frac{-\lambda t}{1-\lambda t}\ .$$

□

2.1.3. Results in $\mathrm{Mot}_{\mathrm{rat}}$. —

Proposition 2.6. — *The geometric Frobenius morphism $F\colon X \to X$ for all $X \in \mathcal{V}$ induces a $\otimes$-automorphism F of the identity functor of* $\mathrm{Mot}_{\mathrm{rat}}$.

Proof. — For any $X \in \mathcal{V}$, we have the geometric Frobenius morphism $F\colon X \to X$. It induces a morphism $F\colon h(X) \to h(X)$ in $\mathrm{Mot}_{\mathrm{rat}}$. We leave to the reader the (not so easy) exercise to prove that for any morphism $f\colon h(X) \to h(Y)$ in $\mathrm{Mot}_{\mathrm{rat}}$, the following diagram commute:

$$\begin{array}{ccc} h(X) & \xrightarrow{f} & h(Y) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ h(X) & \xrightarrow{f} & h(Y)\ . \end{array}$$

Then, one can formally extend this construction to get a natural Frobenius morphism $F\colon M \to M$ for any *effective* motive $M \in \mathrm{Mot}^{\mathrm{eff}}_{\mathrm{rat}}$. Obviously, the Frobenius on $h(X) \otimes h(Y)$ is the tensor product of the Frobenius morphisms on $h(X)$ and $h(Y)$, so that we have a $\otimes$-endomorphism F of the identity functor of $\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{rat}}$.

One easily sees that $F\colon \mathbf{L} \to \mathbf{L}$ is the multiplication by q, which is an isomorphism as we work with rational coefficients. Then, one can extend the Frobenius morphism from $\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{rat}}$ to $\mathrm{Mot}_{\mathrm{rat}}$ so that for any effective motive M and $r \in \mathbf{N}$, the Frobenius morphism on $M \otimes \mathbf{T}^r$ is $\frac{1}{q^r} F_M \otimes \mathrm{id}_{\mathbf{T}^r}$ where F_M is the Frobenius on M.

As the category Mot is rigid, it follows from [**15**, 5.2.2, chapitre I] that the Frobenius endomorphism of any motive is an isomorphism. More precisely, for any motive M in Mot, the morphism $F_{M^\vee}\colon M^\vee \to M^\vee$ is ${}^t F_M^{-1}$. □

Definition 2.7. — For any motive $M \in \mathrm{Mot}$, we let $F_M \colon M \to M$ be the Frobenius automorphism of M. We define the Zeta function $Z(M,t)$ of M to be $Z(F_M,t)$. If $X \in \mathcal{V}$, the Zeta function $Z(X,t)$ of X is $Z(h(X),t)$.

Proposition 2.8. — *Let $X \in \mathcal{V}$. The Zeta function of X defined above is the usual one:*

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbf{F}_{q^n})}{n} t^n\right) .$$

Proof. — One has to check that for any $n \geq 1$, $\#X(\mathbf{F}_{q^n}) = \mathrm{Tr}_{\mathrm{Mot}}(F^n \colon h(X) \to h(X))$. The set $\#X(\mathbf{F}_{q^n})$ is in bijection with the set of fixed points of F^n acting on $X(\overline{\mathbf{F}_q})$. The differential of F^n is zero, so that the graph G_n of $F^n \colon X \to X$ and the diagonal Δ_X intersects transversaly in $X \times X$. Thus, we have an equality:

$$\deg([G_n] \cdot [\Delta_X]) = \#X(\mathbf{F}_q)$$

since all the intersection multiplicites are one, which finishes the proof thanks to proposition 1.16. □

Theorem 2.9 **(Rationality).** — *Let H be a Weil cohomology with coefficient field F over $\mathbf{F}_q$. For any endomorphism $f \colon M \to M$ in $\mathrm{Mot}_{\mathrm{rat}}$, the Zeta function $Z(f,t)$ belongs to $\mathbf{Q}(t)$. Moreover,*

$$Z(f,t) = \prod_{n \in \mathbf{Z}} P_n(t)^{(-1)^{n+1}} ,$$

where $P_n(t) = \det(\mathrm{id} - tf \colon H^n(M) \to H^n(M)) \in F[t]$. In particular, the Zeta function of a smooth and projective variety X over $\mathbf{F}_q$ belongs to $\mathbf{Q}(t)$.

Proof. — Let $f \in \mathrm{End}_{\mathrm{Mot}_{\mathrm{rat}}}(M)$. The Zeta function of f is the same as the one of $r_H(f) \colon r_H(M) \to r_H(M)$ in VecGr_F. Then, proposition 2.4 shows that $Z(f,t)$ is given by the formula above, which implies that $Z(f,t)$ belongs to $F(t)$. As we know that $\mathbf{Q}[[t]] \cap F(t) = \mathbf{Q}(t)$ [(3)], it follows that $Z(f,t)$ belongs to $\mathbf{Q}(t)$. □

Theorem 2.10 **(Functional equation).** — *Let H be a Weil cohomology with coefficient field F over $\mathbf{F}_q$. Let X be a smooth and projective variety over $\mathbf{F}_q$. Assume that X is purely d-dimensional. Then,*

$$Z(X,t) = \varepsilon \cdot q^{\frac{-d\chi(X)}{2}} \cdot t^{-\chi(X)} \cdot Z(X, q^{-d}t^{-1})$$

where $\varepsilon = \pm 1$. If d is odd, then $\varepsilon = 1$. If d is even, then $\varepsilon = (-1)^N$ where N is the multiplicity of $q^{\frac{n}{2}}$ as an eigenvalue of the geometric Frobenius acting on $H^d(X)$.

Proof. — For any motive M, we can compare the Zeta functions of M and $M \otimes \mathbf{T}^d$ (cf. proposition 2.3) and also those of M and $M^\vee$ (cf. proposition 2.5). Then, one can argue using the fact the the dual of the motive $h(X)$ is $h(X) \otimes \mathbf{T}^d$. □

2.2. Numerical equivalence. —

(3) This is a consequence of the following algebraic lemma: if F is a field and $f = \sum_{n \geq 0} a_n t^n \in F[[t]]$, then f belongs to $F(t)$ if and only if there exists an integer m such that for any large enough integer s, the determinant $N_{s,m}$ of the matrix $(a_{s+i+j})_{0 \leq i,j \leq m}$ vanishes.

2.2.1. Definition. —

Definition 2.11. — Let $X \in \mathcal{V}$ and A be $\mathbf{Z}$ or a field of characteristic zero. A cycle of codimension i on X with coefficients in A is numerically equivalent to zero if for any cycle of dimension i on X, we have $\deg(x \cdot y) = 0$ in A. This defines an adequate equivalence relation on cycles. We define $A^i_{\mathrm{num}}(X; A)$ to be the group of equivalence classes of cycles modulo cycles numerically equivalent to zero. If $A = \mathbf{Z}$, this group is denoted $A^i_{\mathrm{num}}(X)$.

Proposition 2.12. — *For any field F of characteristic zero, we have a canonical isomorphism:*

$$A^i_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F \xrightarrow{\sim} A^i_{\mathrm{num}}(X; F) \ .$$

Proof. — Contrary to the proof of the corresponding statement in [**2**, proposition 3.2.7.1], here, we shall not use the existence of a Weil cohomology.

If $F = \mathbf{Q}$, the statement is trivial. Then, we are reduced to proving the the obvious map:

$$A^i_{\mathrm{num}}(X; \mathbf{Q}) \otimes_{\mathbf{Q}} F \to A^i_{\mathrm{num}}(X; F)$$

is an isomorphism. We may assume that X is d-dimensional. We shall use the following lemma:

Lemma 2.13. — *Let V and B be $\mathbf{Q}$-vector spaces with a bilinear pairing $V \times W \to \mathbf{Q}$ such that the obvious map $V \to \mathrm{Hom}_{\mathbf{Q}}(W, \mathbf{Q})$ is injective. Then, for any field F of characteristic zero, the obvious map*

$$V \otimes_{\mathbf{Q}} F \to \mathrm{Hom}_F(W \otimes_{\mathbf{Q}} F, F)$$

is injective.

Proof of the lemma. — We shall prove that for any $\mathbf{Q}$-vector space F, the obvious map

$$V \otimes_{\mathbf{Q}} F \to \mathrm{Hom}_{\mathbf{Q}}(W, F)$$

is injective. This statement is trivial if F is finite-dimensional. Thus, we get that for any finite-dimensional subspace F' of F, the composite map

$$V \otimes_{\mathbf{Q}} F' \to \mathrm{Hom}_{\mathbf{Q}}(W, F') \to \mathrm{Hom}_{\mathbf{Q}}(W, F)$$

is injective. Then, we can take the inductive limit of these maps for any F' in the ordered set of finite-dimensional subspaces of F, which proves that

$$V \otimes_{\mathbf{Q}} F \to \mathrm{Hom}_{\mathbf{Q}}(W, F)$$

is injective. □

We have to prove that the intersection pairing

$$(A^i_{\mathrm{num}}(X; \mathbf{Q}) \otimes_{\mathbf{Q}} F) \times (A^{d-i}_{\mathrm{num}}(X; \mathbf{Q}) \otimes_{\mathbf{Q}} F) \to F$$

is such that the obvious map

$$A^i_{\mathrm{num}}(X; \mathbf{Q}) \otimes_{\mathbf{Q}} F \to \mathrm{Hom}_F(A^{d-i}_{\mathrm{num}}(X; \mathbf{Q}) \otimes_{\mathbf{Q}} F, F)$$

is injective, which follows from the lemma and the fact that

$$A^i_{\mathrm{num}}(X;\mathbf{Q}) \to \mathrm{Hom}_{\mathbf{Q}}(A^{d-i}_{\mathrm{num}}(X),\mathbf{Q})$$

is injective. □

***Conjecture 2.14* (Standard conjecture D).** — *The functor*

$$\mathrm{Mot}_{\mathrm{hom},F} \to \mathrm{Mot}_{\mathrm{num},F}$$

is an equivalence of categories, i.e. *a cycle is numerically equivalent to zero if and only if it is homologically equivalent to zero.*

2.2.2. Finite generation. —

Theorem 2.15. — *Assume that there exists a Weil cohomology with a coefficient field F (of characteristic zero). Then, for any $X \in \mathcal{V}$ and integer i, the abelian group $A^i_{\mathrm{num}}(X)$ is finitely generated and torsion-free.*

Proof. — For any i, there is a surjection of F-vector spaces:

$$A^i_{\mathrm{hom}}(X;F) \to A^i_{\mathrm{num}}(X;F)\ .$$

As $A^i_{\mathrm{hom}}(X;F)$ injects into $H^{2i}(X)(i)$ which is finite-dimensional, the F-vector space $A^i_{\mathrm{num}}(X;F)$ is finite-dimensional. From proposition 2.12, we get that $A^i_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F$ is finite-dimensional. Then, by flat descent, $A^i_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite-dimensional $\mathbf{Q}$-vector space.

Lemma 2.16. — *Let V be an abelian group such that $V \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite-dimensional $\mathbf{Q}$-vector space. Then,* $\mathrm{Hom}_{\mathbf{Z}}(V,\mathbf{Z})$ *is a finite type and torsion-free group.*

Proof of the lemma. — Let $n = \dim_{\mathbf{Q}} V$. We may assume that $\mathbf{Z}^n \subset V \subset \mathbf{Q}^n$. Then, the obvious map

$$\mathrm{Hom}_{\mathbf{Z}}(V,\mathbf{Z}) \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^n,\mathbf{Z})$$

is injective, which implies that $\mathrm{Hom}_{\mathbf{Z}}(V,\mathbf{Z})$ is finitely generated and torsion-free. □

We already know that $A^{d-i}_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite-dimensional $\mathbf{Q}$-vector space. Thus, the lemma tells us that $\mathrm{Hom}_{\mathbf{Z}}(A^{d-i}_{\mathrm{num}}(X),\mathbf{Z})$ is finitely generated and torsion-free. By construction, this group contains a subgroup isomorphic to $A^i_{\mathrm{num}}(X)$, so that $A^i_{\mathrm{num}}(X)$ is also finitely generated and torsion-free. □

2.2.3. Semi-simplicity. —

***Theorem 2.17* (Jannsen [13]).** — *For any field of coefficients F of characteristic zero, the category* $\mathrm{Mot}_{\mathrm{num},F}$ *is a semisimple abelian category.*

Sketch of proof. — The main step is to prove that for any d-dimensional $X \in \mathcal{V}$, the algebra

$$\mathrm{End}_{\mathrm{Mot}_{\mathrm{num},F}}(h(X)) = A^d_{\mathrm{num}}(X \times X;F)$$

is finite dimensional and semi-simple. We may enlarge the coefficient field F so that there exists a Weil cohomology. Let $\mathcal{R}$ be the Jacobson radical of $\mathrm{End}_{\mathrm{Mot}_{\mathrm{hom},F}}(h(X))$. We shall prove that $\mathcal{R}$ is the kernel of the surjection

$$\mathrm{End}_{\mathrm{Mot}_{\mathrm{hom},F}}(h(X)) \to \mathrm{End}_{\mathrm{Mot}_{\mathrm{num},F}}(h(X))$$

so that $\mathrm{End}_{\mathrm{Mot}_{\mathrm{num},F}}(h(X))$ shall be semisimple.

Let f and g be elements of $\mathrm{End}_{\mathrm{Mot}_{\mathrm{hom},F}}(h(X))$. We know that $\mathrm{Tr}(g \circ f) = \deg(f \cdot \tau^\star(g))$ where τ is the intertwining automorphism on $X \times X$. If f belongs to $\mathcal{R}$, for any g, $g \circ f$ is nilpotent and then $\mathrm{Tr}(g \circ f) = 0$. According to the formula above, it implies that f is numerically equivalent to zero. Conversely, if f is numerically equivalent to zero, then for any g as above, for any $n \geq 0$, we have $\mathrm{Tr}((g \circ (f \circ g)^n) \circ f) = 0$. Then, the endomorphism of $H(X)$ associated to $g \circ f$ is nilpotent, which also means that $g \circ f$ is nilpotent. Thus, f belongs to $\mathcal{R}$. □

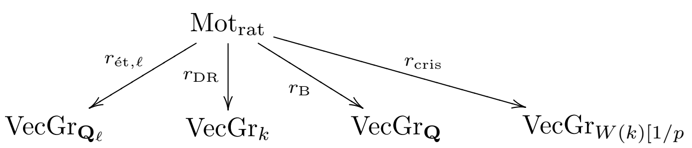
Remark 2.18. — Conservely, Jannsen proved that if $\sim$ is an adequate equivalence relation on cycles such that the associated category of motives $\mathrm{Mot}_{\sim,F}$ is abelian semisimple, then $\sim$ is the numerical equivalence.

3. Examples

3.1. Classical Weil cohomologies. — The classical Weil cohomologies are defined to be the following ones:

- for any prime number ℓ invertible in k and any separable closure $\overline{k}/k$ of k, the étale cohomology $H^\star_{\text{ét}}(X_{\overline{k}}; \mathbf{Q}_\ell)$ is a Weil cohomology with coefficient field $\mathbf{Q}_\ell$;
- if k is of characteristic zero, the algebraic De Rham cohomology $H^\star_{\mathrm{DR}}(X/k)$ is a Weil cohomology with coefficient field k;
- if $\sigma\colon k \to \mathbf{C}$ is a complex embedding of k, the Betti cohomology $H^\star_{\mathrm{B}}(X(\mathbf{C}); \mathbf{Q})$ is a Weil cohomology with coefficient field $\mathbf{Q}$;
- if k is of positive characteristic p, the crystalline cohomology $H^\star_{\mathrm{cris}}(X)$ is a Weil cohomology with coefficient field $K = W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors of k.

Thus, we get different realization functors:



Note that for a given base field k, not all these Weil cohomologies are available.

3.2. Additional structures. —

3.2.1. Étale cohomology. — The étale cohomology groups of schemes are functorial with respect to arbitrary morphisms of schemes. As the Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on the scheme $X_{\overline{k}}$, this Galois group naturally acts on the étale cohomology groups $H^\star_{\text{ét}}(X_{\overline{k}}; \mathbf{Q}_\ell)$. Actually, for any integer q, $H^q_{\text{ét}}(X_{\overline{k}}; \mathbf{Q}_\ell)$ is a continuous representation of the profinite group $\mathrm{Gal}(\overline{k}/k)$.

Moreover, the notation $\mathbf{Q}_\ell(1)$ that we defined for the dual of $H^2_{\text{ét}}(\mathbf{P}^1; \mathbf{Q}_\ell)$ (cf. definition 1.1) is consistent with the usual definition of $\mathbf{Q}_\ell(1)$ as $\mathbf{Z}_\ell(1)[1/\ell]$ where $\mathbf{Z}_\ell(1)$ is $\lim_\nu \mu_{\ell^\nu}(\overline{k})$.

Furthermore, for any $X \in \mathcal{V}$, the image of the cycle class map $CH^n(X) \to H^{2n}(X_{\overline{k}}; \mathbf{Q}_\ell(n))$ is contained in the subspace of fixed points under $\mathrm{Gal}(\overline{k}/k)$ [(4)].

The realisation functor $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_{\mathbf{Q}_\ell}$ can be enriched as a $\otimes$-functor to the category of continuous $\mathbf{Q}_\ell$-representations of $\mathrm{Gal}(\overline{k}/k)$.

3.2.2. Algebraic De Rham cohomology. — The stupid filtration on the De Rham complex of sheaves $\Omega^\star_{X/k}$ on $X \in \mathcal{V}$ leads to a spectral sequence (which is E_1-degenerated):

$$E_1^{pq} = H^p(X, \Omega^q_{X/k}) \Longrightarrow H^{p+q}_{\mathrm{DR}}(X/k) \ .$$

The corresponding filtration on $H^n_{\mathrm{DR}}(X/k)$ is the Hodge filtration F^p. More precisely, $F^p H^n_{\mathrm{DR}(X/k)}$ is the image of the obvious (injective) map:

$$\mathbf{H}^n(X; \cdots \to 0 \to \Omega^p_{X/k} \xrightarrow{d} \Omega^{p+1}_{X/k} \xrightarrow{d} \Omega^{p+2}_{X/k} \xrightarrow{d} \dots) \subset \mathbf{H}^n(X; \Omega^\star_{X/k}) \ .$$

The computation of $H^2_{\mathrm{DR}}(\mathbf{P}^1/k)$ shows that it is isomorphic to k and that it lies in Hodge degree 1. More generally, for any $X \in \mathcal{V}$, the cycle class map $CH^n(X) \to H^{2n}_{\mathrm{DR}}(X/k)$ factors through the subspace $F^n H^{2n}_{\mathrm{DR}}(X/k)$.

Moreover, the Hodge filtration of $r_{\mathrm{DR}}(h(X))$ for any $X \in \mathcal{V}$ extends to a Hodge filtration on $r_{\mathrm{DR}}(M)$ for any motive $M \in \mathrm{Mot}_{\mathrm{rat}}$ and for any morphism $M \to N$ in $\mathrm{Mot}_{\mathrm{rat}}$, the corresponding morphism $r_{\mathrm{DR}}(M) \to r_{\mathrm{DR}}(N)$ is strictly compatible with the Hodge filtration.

3.2.3. Betti cohomology. — From its very definition, the Betti cohomology $H^\star_{\mathrm{B}}(X(\mathbf{C}); \mathbf{Q})$ does not seem to have any additional structure. However, if we extend the coefficient field to $\mathbf{C}$, one can get a differential description of it using the holomorphic Poincaré lemma:

$$H^\star_{\mathrm{B}}(X(\mathbf{C}); \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \mathbf{H}^\star(X(\mathbf{C}); \Omega^\star_{X(\mathbf{C}),\mathrm{hol}}) \ ,$$

where $\Omega^p_{X(\mathbf{C}),\mathrm{hol}}$ denotes the sheaf of holomorphic differential p-forms on $X(\mathbf{C})$. Then, the $\mathbf{C}$-vector space $H^n_{\mathrm{B}}(X(\mathbf{C}); \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ is endowed with an Hodge filtration in the same way as the algebraic De Rham cohomology is.

Definition 3.1. — A pure $\mathbf{Q}$-Hodge structure of weight $n \in \mathbf{Z}$ is a finite dimensional $\mathbf{Q}$-vector space V endowed with a decomposition of the $\mathbf{C}$-vector space $V_{\mathbf{C}} = V \otimes_{\mathbf{Q}} \mathbf{C}$ as

$$V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q} \ ,$$

such that for all (p,q), $V^{p,q}$ and $V^{q,p}$ are exchanged by the complex conjugation on $V_{\mathbf{C}}$. The Hodge filtration on $V_{\mathbf{C}}$ is defined by $F^p V_{\mathbf{C}} = \sum_{p' \geq p} V^{p', n-p'}$.

If V is a pure $\mathbf{Q}$-Hodge structure of weight n, then one can recover the $V^{p,q}$ from the Hodge filtration. More precisely, one can define a second filtration $\overline{F^p}$ which is the complex conjugate of the Hodge filtration F^p on $V_{\mathbf{C}}$. Then, $V^{p,n-p} = F^p V_{\mathbf{C}} \cap \overline{F^{n-p}} V_{\mathbf{C}})$.

(4) The Tate conjecture asserts that for a base field k of finite type over its prime field, the $\mathbf{Q}_\ell$-vector space spanned by the image of $CH^n(X)$ is precisely the subspace of fixed points under $\mathrm{Gal}(\overline{k}/k)$.

One of the main analytic results of classical Hodge theory is the following theorem:

Theorem 3.2. — *Let X be a compact* $\mathbf{C}$*-analytic variety. If X has a Kähler metric, then the Hodge filtration on $H^n(X;\mathbf{C})$ endows $H^n(X;\mathbf{Q})$ with a pure* $\mathbf{Q}$*-Hodge structure of weight n.*

As any projective and smooth variety over $\mathbf{C}$ has a Kähler metric, it follows that the Betti cohomology spaces of an algebraic variety X are endowed with $\mathbf{Q}$-Hodge structures. If we let $\mathrm{HS}_{\mathbf{Q}}$ denote the category of $\mathbf{Q}$-Hodge structures, one may extend this definition to get a realization functor $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{HS}_{\mathbf{Q}}$.

3.2.4. Crystalline cohomology. — The crystalline cohomology vector spaces $H^\star_{\mathrm{cris}}(X) = H^\star_{\mathrm{cris}}(X/W(k))[1/p]$ are endowed with a semilinear Frobenius endomorphism $\phi\colon H^\star_{\mathrm{cris}}(X) \to H^\star_{\mathrm{cris}}(X)$.

3.3. Comparison theorems. — We have just seen that there are many different Weil cohomologies, with different coefficients fields. However, if k is of characteristic zero, there are comparison theorems between the corresponding realization functors, provided the coefficient field is extended properly:

Theorem 3.3. — *If $\overline{k}$ is embedded in* $\mathbf{C}$ *and ℓ is a prime number, there is a canonical isomorphism of $\otimes$-functors* $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_{\mathbf{Q}_\ell}$*:*

$$r_{\acute{e}t,\ell} \xrightarrow{\sim} r_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \ .$$

If k is embedded in $\mathbf{C}$*, there is a canonical isomorphism of $\otimes$-functors* $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_{\mathbf{C}}$*:*

$$r_{\mathrm{DR}} \otimes_k \mathbf{C} \xrightarrow{\sim} r_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{C} \ .$$

If p is a prime number and $\overline{k}$ is an algebraic extension of $\mathbf{Q}_p$*, there exists a isomorphism of $\otimes$-functors* $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_{B_{\mathrm{DR}}}$*:*

$$r_{\mathrm{DR}} \otimes_k B_{\mathrm{DR}} \simeq r_{\acute{e}t,p} \otimes_{\mathbf{Q}_p} B_{\mathrm{DR}} \ .$$

These theorems are easy extensions of comparison theorems between cohomology groups of projective and smooth varieties for different Weil cohomologies. The comparison between étale cohomology and Betti cohomology is due to M. Artin (SGA 1 XI 4.4). The comparison of the algebraic De Rham cohomology with the Betti cohomology follows from results by Serre [**16**], and has been extended to smooth varieties that are not assumed to be projective by Grothendieck in [**7**] (see also [**8**]). The last statement is one of the comparison theorems from p-adic Hodge theory. The period ring B_{DR} is one of the rings that are needed to state the comparison theorems in that setting. It has been developped by many authors: Tate, Fontaine and Messing, Breuil, etc. In this form, this is due to Faltings and Tsuji.

There are also comparison theorems in mixed characteristic:

Theorem 3.4 **(Berthelot-Ogus)**. — *Let A be a complete discrete valuation ring, with fraction field k and perfect residue field k_0. We assume that k is of characteristic* *zero and k_0 of characteristic $p > 0$. Let $\mathcal{X}$ be a projective and smooth scheme over A. Then, there is an isomorphism:*

$$H^{\star}_{\mathrm{DR}}(\mathcal{X}_k/k) \simeq H^{\star}_{\mathrm{cris}}(\mathcal{X}_{k_0}) \otimes_{W(k_0)[1/p]} k\ .$$

4. Absolute Hodge cycles, motivated cycles

4.1. Absolute Hodge cycles. — We assume that the base field k is of finite transcendance degree over $\mathbf{Q}$. We fix an algebraic closure $\overline{k}$ of k.

The $\mathbf{Q}$-vector space of absolute Hodge cycles on a smooth and projective variety defined in [**3**] can be described as follows:

Definition 4.1. — Let $X \in \mathcal{V}$. An absolute Hodge cycle of codimension n on X is a family of elements:

- $x_{\mathrm{DR}} \in H^{2n}_{\mathrm{DR}}(X/k)(n)$;
- $x_\sigma \in H^{2n}_{\mathrm{B}}(X(\mathbf{C})_\sigma; \mathbf{Q})$ for any embedding $\sigma\colon k \to \mathbf{C}$;
- $x_\ell \in H^{2n}_{\text{ét}}(X_{\overline{k}}; \mathbf{Q}_\ell)(n)$ for any prime number ℓ,

such that:

- x_{DR} belongs to $F^0(H^{2n}_{\mathrm{DR}}(X/k)(n))$;
- for any prime number ℓ, x_ℓ is fixed under the action of $\mathrm{Gal}(\overline{k}/k)$;
- these elements constitute a compatible family under the comparison isomorphisms of theorem 3.3.

We let $C^n_{\mathrm{AH}}(X)$ be the (finite-dimensional) $\mathbf{Q}$-vector space of absolute Hodge cycles of codimension n on X.

Remark 4.2. — Obviously, if x is an algebraic cycle of codimension n on X, then the family of the images of x under the cycle class maps is an absolute Hodge cycle. We shall say that such absolute Hodge cycles are algebraic.

Remark 4.3. — If H is a classical Weil cohomology, then $C^n_{\mathrm{AH}}(X)$ identifies to a subgroup of $H^{2n}(X)(n)$. Then, it makes sense to say that an element in $H^{2n}(X)(n)$ "is" an absolute Hodge cycle.

Definition 4.4. — One may define a $\otimes$-category whose objects are the $h(X)$ for $X \in \mathcal{V}$ in such a way that the set of morphisms $h(X) \to h(Y)$ is $C^{d_X}_{\mathrm{AH}}(X \times Y)$ (which identifies to a subset of $\mathrm{Hom}(H(X), H(Y))$ for all classical Weil cohomologies H). Then, one may take the pseudo-abelian envelope of that category and invert the "Lefschetz motive" to get a $\otimes$-category $\mathrm{Mot}_{\mathrm{AH}}$ and get $\otimes$-functors $\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{Mot}_{\mathrm{AH}}$ and "realizations functors" $r_H\colon \mathrm{Mot}_{\mathrm{AH}} \to \mathrm{VecGr}_F$ for any classical Weil cohomology H with coefficient field F.

If the commutativity constraint of $\mathrm{Mot}_{\mathrm{AH}}$ is modified properly, the category $\mathrm{Mot}_{\mathrm{AH}}$ becomes a Tannakian category.

Definition 4.5. — Let $X \in \mathcal{V}$. Let $x \in H^{2n}_{\mathrm{DR}}(X/k)(n)$. Let $\sigma\colon k \to \mathbf{C}$ be an embedding. We say that x is a Hodge cycle relative to σ if

– the image of x in $H^{2n}_{\mathrm{B}}(X(\mathbf{C})_\sigma;\mathbf{C})(n)$ *via* the comparison isomorphism is in the rational subspace $H^{2n}_{\mathrm{B}}(X(\mathbf{C})_\sigma;\mathbf{Q})(n)$;
– the element x lies in Hodge bidegree $(0,0)$.

Remark 4.6. — The Hodge conjecture asserts that any Hodge cycle is an algebraic cycle. As any absolute Hodge cycle is a Hodge cycle, the Hodge conjecture implies that the functor $\mathrm{Mot}_{\mathrm{hom},\mathbf{Q}} \to \mathrm{Mot}_{\mathrm{AH}}$ is an equivalence.

The main result of [**3**] can be considered as a weaker form of the Hodge conjecture:

***Theorem 4.7* (Deligne)**. — *Any Hodge cycle over an abelian variety is an absolute Hodge cycle.*

4.2. Motivated cycles. — Let k be a field of characteristic zero and H be a classical Weil cohomology.

Let $X \in \mathcal{V}$. Let D be an ample divisor on X. For any $0 \leq i \leq 2d_X$, we have a commutative square:

$$\begin{array}{ccc} A^i_{\mathrm{hom},\mathbf{Q}}(X) & \xrightarrow{\cdot[D]^{d_X-2i}} & A^{d_X-i}_{\mathrm{hom},\mathbf{Q}}(X)] \\ \downarrow & & \downarrow \\ H^{2i}(X)(i) & \xrightarrow[\sim]{\cdot(\mathrm{cl}[D])^{d_X-2i}} & H^{2d_X-2i}(X)(d-i) \end{array}$$

where $A^\star_{\mathrm{hom},\mathbf{Q}}(X)$ denote groups of cycles on X modulo the homological equivalence and the vertical (injective) maps are given by the cycle class map. The bottom map is an isomorphism: this is the hard Lefschetz theorem. Then, the upper map is injective.

***Conjecture 4.8* (Standard conjecture B)**. — *Let $X \in \mathcal{V}$. Let D be an ample divisor on X. For any i, the (injective) map $A^i_{\mathrm{hom},\mathbf{Q}}(X) \to A^{d_X-i}_{\mathrm{hom},\mathbf{Q}}(X)$ induced by the multiplication by $[D]^{d_X-2i}$ is a bijection.*

The idea of the definition of motivated cycles in [**2**] is to enlarge the groups $A^\star_{\mathrm{hom},\mathbf{Q}}(X)$ so as to force the analog of standard conjecture B to be true.

Definition 4.9. — We let Cohom be the $\otimes$-category of "motives" defined as $\mathrm{Mot}_{\mathrm{AH}}$ (cf. definition 4.4) but such that the set of morphisms $h(X) \to h(Y)$ is the whole group $H^{2d_X}(X \times Y)(d_X)$.

Definition 4.10. — There exists a smallest $\mathbf{Q}$-linear pseudoabelian sub-$\otimes$-category $\mathrm{Mot}_{\mathrm{mot}}$ of Cohom containing $\mathrm{Mot}_{\mathrm{hom},\mathbf{Q}}$ and such that for any $X \in \mathcal{V}$ and D an ample divisor on X, the injective map $A^i_{\mathrm{mot}}(X) \to A^{d_X-i}_{\mathrm{mot}}(X)$ induced by the multiplication by $(\mathrm{cl}[D])^{2d_X-2i}$ is a bijection, where $A^n_{\mathrm{mot}}(X) = \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{mot}}}(\mathbf{L}^{\otimes n}, h(X))$. The elements of $A^n_{\mathrm{mot}}(X)$ are called "motivated cycles".

By construction, the faithful functor $\mathrm{Mot}_{\mathrm{hom},\mathbf{Q}} \to \mathrm{Mot}_{\mathrm{mot}}$ is an equivalence of categories if and only if standard conjecture B is true.

If H' is another classical Weil cohomology, the categories of motivated motives $\mathrm{Mot}_{\mathrm{mot}}$ defined using H and H' are naturally equivalent. If k is of finite transcendance degree over $\mathbf{Q}$, there exists an obvious faithful functor $\mathrm{Mot}_{\mathrm{mot}} \to \mathrm{Mot}_{\mathrm{AH}}$ (*i.e.* a motivated cycle is an absolute Hodge cycle).

The following proposition is an analog of the fact that standard conjecture B implies standard conjecture C:

Proposition 4.11. — *For any $X \in \mathcal{V}$, the Künneth projectors in* $\mathrm{End}_{\mathrm{Cohom}}(h(X))$ *are motivated cycles.*

It follows that one can alter the commutativity constraint on $\mathrm{Mot}_{\mathrm{mot}}$ to get a Tannakian category.

The arguments of [**3**] have been improved in [**1**] to get a stronger version of theorem 4.7:

***Theorem 4.12* (André)**. — *Any Hodge cycle over an abelian variety is a motivated cycle.*

5. Mixed realizations

5.1. The abelian category of mixed realizations. — Let k be a field embeddable in $\mathbf{C}$ and $\overline{k}$ be an algebraic closure of k. Delinge [**4**] and Jannsen [**12**] have defined an abelian category of mixed realizations in the same spirit as absolute Hodge cycles were defined in definition 4.1. The definition can be sketched as follows:

Definition 5.1. — The abelian category MR_k of mixed realizations is the category whose objects are families of objects:

- H_{DR} is a k-vector space equipped with a Hodge filtration and a weight filtration;
- H_σ is a mixed $\mathbf{Q}$-Hodge structure (for any embedding $\sigma\colon k \to \mathbf{C}$);
- H_ℓ is a $\mathbf{Q}_\ell$-vector space with a continuous action of $\mathrm{Gal}(\overline{k}/k)$

with comparison isomorphisms (like in theorem 3.3) compatible with the additional data. The morphisms in MR_k are the families of linear maps that are compatible with the additional data and the comparison isomorphisms.

This category MR_k is a Tannakian category. For any smooth (and quasi-projective) variety U over k, the De Rham, Betti and étale cohomology spaces of U fit together as an object $H(U)$ of MR_k. Jannsen defined a category of mixed motives as the sub-Tannakian category of MR_k generated by objects $H(U)$ for any smooth variety U. If we let Sm_k be the category of smooth (and quasi-projective) k-schemes, there is a well-defined functor ${\mathrm{Sm}_k}^{\mathrm{opp}} \to \mathrm{MR}_k$.

The main problem with that definition is that morphisms have no geometric interpretation whereas they have such an origin in $\mathrm{Mot}_{\mathrm{rat}}$.

5.2. Triangulated categories of motives. — Another approach of mixed motives have been developped since the 1990s. The idea is to construct a triangulated category of mixed motives, and then one may expect to construct an abelian category of mixed motives as the heart of some t-structure of the triangulated category. There are several constructions of these triangulated categories. However, we do not know any unconditional construction of a good candidate for the abelian category of mixed motives.

For any perfect field k, Voevodsky defined in [**18**] a triangulated category of motives $\mathrm{DM}_{\mathrm{gm}}(k)$. By construction, there is a *covariant* functor $\mathrm{Sm}_k \to \mathrm{DM}_{\mathrm{gm}}(k)$ that maps a smooth variety U to its motive $M(U)$ (in the previous sections, the natural functor from $\mathcal{V}$ ro $\mathrm{Mot}_{\mathrm{rat}}$ was contravariant). Levine defined another category $\mathcal{DM}(k)$ with a contravariant motive functor from the category Sm_k.

These two constructions have been compared by Levine for k of characteristic zero and by Ivorra for any perfect field provided we take $\mathbf{Q}$ as a coefficient ring:

Theorem 5.2 **(Levine** [**14**, Part I, Chapter VI, 2.5.5]**, Ivorra** [**11**]**)**
For any field of characteristic zero k, there is an equivalence of triangulated categories $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \simeq \mathcal{DM}(k)$.
For any perfect field k, there is an equivalence of triangulated categories $\mathrm{DM}_{\mathrm{gm}}(k;\mathbf{Q})^{\mathrm{opp}} \simeq \mathcal{DM}(k;\mathbf{Q})$.

Category of pure motives and triangulated motives are also related by the following embedding theorem:

Theorem 5.3 **(Voevodsky)**. — *Let k be a perfect field. There is an fully faithful functor* $\mathrm{Mot}_{\mathrm{rat}}(k)^{\mathrm{opp}} \to \mathrm{DM}_{\mathrm{gm}}(k)$.

5.3. Triangulated realization functors. — Various kinds of triangulated realization functors naturally have arisen. If $\mathcal{D}$ is triangulated category, we may say that a covariant realization functor is a triangulated functor $\mathrm{DM}_{\mathrm{gm}}(k) \to \mathcal{D}$. The idea is that when we compose this functor with the natural functor from the category of smooth schemes to $\mathrm{DM}_{\mathrm{gm}}(k)$, we get a functor $\mathrm{Sm}_k \to \mathcal{D}$ rather than a functor from the opposite category ${\mathrm{Sm}_k}^{\mathrm{opp}}$. We may also say that this realization functor is of homological type. Conversely, if we consider a triangulated functor $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \to \mathcal{D}$ or $\mathcal{DM}(k) \to \mathcal{D}$, we shall say that it is a contravariant realization functor, or a realization functor of cohomological type.

5.3.1. Covariant realization functors. —

Theorem 5.4 **(Huber** [**9**]**)**. — *Let k be a field embeddable in $\mathbf{C}$. There exists a well defined category $\mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k}$ of mixed realizations, equipped with various triangulated functors:*

– *an étale component functor* $\mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k} \to \mathrm{D}^{\mathrm{b}}(k_{\acute{e}t};\mathbf{Q}_\ell)$*;*
– *an algebraic De Rham component functor* $\mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k} \to \mathrm{D}^{\mathrm{b}}(\mathrm{VecGr}_k)$*;*
– *a Betti component functor* $\mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k} \to \mathrm{D}^{\mathrm{b}}(\mathrm{MHS}_{\mathbf{Q}})$ *for any embedding* $\sigma\colon k \to \mathbf{C}$.

There is a triangulated realization functor $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k}$ *of cohomological type, whose components are compatible with the definition of the aforementioned cohomology theories for smooth varieties.*

Theorem 5.5 **(Ivorra** [**10**]**)**. — *Let k be a field and ℓ be a prime number invertible in k. There exists a well defined étale realization functor* $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \to \mathrm{D}^{\mathrm{b}}(k_{\acute{e}t};\mathbf{Z}_\ell)$

of cohomological type. If k is embeddable in $\mathbf{C}$*, this functor, once tensored with* $\mathbf{Q}$*, is naturally isomorphic to the étale component of the realization functor of theorem 5.4.*

Remark 5.6. — The theory of relative cycles of [**17**] enabled Ivorra to define a triangulated category $\mathrm{DM}_{\mathrm{gm}}(S)$ for any noetherian base scheme S (which has also been studied independently by Cisinski and Déglise). Moreover, his construction of the triangulated étale realization functor is possible with that degree of generality.

Remark 5.7. — The hard part in the constructions of triangulated functors is to extent the functoriality of cohomology theory from morphism of schemes to finite correspondences (*i.e.* to construct transfers). Huber used Galois-theoretic arguments to do this process: this method is highly versatile but requires the coefficient ring to be tensored with $\mathbf{Q}$. In his work, Ivorra used another method: he proved that the Godement resolution of the constant étale sheaf $\mathbf{Z}/\ell^{\nu}\mathbf{Z}$ naturally has transfers.

Remark 5.8. — Levine also constructed realizations functors of cohomological type $\mathcal{DM}(k) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{MR}_k}$. However, it is not clear whether or not these constructions are compatible with respect to the equivalence of categories $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \simeq \mathcal{DM}(k)$ from theorem 5.2.

Remark 5.9. — One can use these triangulated realization functors to construct regulators. For any $X \in \mathrm{Sm}_k$, and integers (p,q), the motivic cohomology group $H^p(X;\mathbf{Z}(q))$ is defined as $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X),\mathbf{Z}(q)[p])$ (5). Thus, one can apply triangulated functors to get interesting maps from $H^p(X;\mathbf{Z}(q))$ to other groups. For instance, the étale realization functors brings a regulator map

$$H^p(X;\mathbf{Z}(q)) \to H^p_{\text{ét,cont}}(X;\mathbf{Z}_\ell(q)) ,$$

where $H^p_{\text{ét,cont}}(X;\mathbf{Z}_\ell(q))$ denotes a continuous étale cohomology group of X.

5.3.2. Covariant realization functors. — One of the features of Voevodsky's construction of triangulated categories of motives lies in its sheaf-theoretic aspects. One may define $\mathrm{DM}^{\mathrm{eff}}(k)$ as the subtriangulated category of $\mathbf{A}^1$-local objects in the derived category of Nisnevich sheaves with transfers over Sm_k (a complex of Nisnevich sheaves K over Sm_k is $\mathbf{A}^1$-local if for any $X \in \mathrm{Sm}_k$ and $q \in \mathbf{Z}$, the obvious map $H^q(X,K) \to H^q(\mathbf{A}^1 \times X, K)$ is an isomorphism). The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ naturally embeds in $\mathrm{DM}^{\mathrm{eff}}(k)$ as a triangulated subcategory (cf. [**18**, theorem 3.2.6]).

Theorem 5.10 **(Suslin-Voevodsky** [**18**, proposition 3.3.3]**)**
Let k be a perfect field which is of finite cohomological dimension. Let $n \geq 1$ *be an integer invertible in k. One can define an étale version* $\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbf{Z}/n\mathbf{Z})$ *of* $\mathrm{DM}^{\mathrm{eff}}(k;\mathbf{Z}/n\mathbf{Z})$ *and there is an equivalence of categories*

$$\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k;\mathbf{Z}/n\mathbf{Z}) \simeq \mathrm{D}^{\mathrm{b}}(k_{\acute{e}t};\mathbf{Z}/n\mathbf{Z}) .$$

(5) The motive $\mathbf{Z}(1)$ is defined in such a way that $M(\mathbf{P}^1) = \mathbf{Z} \oplus \mathbf{Z}(1)[2]$, and $\mathbf{Z}(q)$ is defined for any q as its qth $\otimes$-power.

Thus, the associated sheaf functor (for the étale topology) $\mathrm{DM}^{\mathrm{eff}}(k;\mathbf{Z}/n\mathbf{Z}) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(k;\mathbf{Z}/n\mathbf{Z})$ induces a triangulated realization functor of homological type:

$$\mathrm{DM}^{\mathrm{eff}}(k;\mathbf{Z}/n\mathbf{Z}) \to \mathrm{D}^{\mathrm{b}}(k_{\text{ét}};\mathbf{Z}/n\mathbf{Z}) .$$

One may extent this to a non-effective version $\mathrm{DM}(k)$ of $\mathrm{DM}^{\mathrm{eff}}(k)$ defined using $\mathbf{Z}(1)$-spectra.

Another construction has been proposed by Cisinski and Déglise. Let k be a perfect field and F be a field of coefficients of characteristic zero. We let $\mathrm{VecGr}^{\infty}_F$ be the $\otimes$-category of F-vector spaces. Let E be a presheaf of complexes of F-vector spaces on the category of schemes of finite type over k. Under some circumstances (homotopy invariance, Künneth formula, Nisnevich descent, proper descent...), they prove that there is a $\otimes$-functor $r\colon \mathrm{DM}(k) \to \mathrm{D}(\mathrm{VecGr}^{\infty}_F)$ such that for any $X \in \mathrm{Sm}_k$, there is a canonical isomorphism $r(M(X)) \simeq \mathbf{Hom}_F(E(X), F)$. They claimed that their method applies to the algebraic De Rham cohomology and to rigid cohomology.

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