Homological study of some Hilbert schemes

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February 10, 2010

Abstract

This lecture gives a brief historical survey of the Hilbert schemes of points on a surface through cohomological considerations. If the first steps in the theory of Hilbert schemes are based on techniques of pure algebraic geometry, the representation theory is widely used thereafter to understand the cohomology groups : this is the idea of Nakajima, who uses representations of Heisenberg-Clifford algebras.

We can then seek to generalize this method : to an object from algebraic geometry, we try to give a structure of module over a certain type of algebra whose representations can be studied separately. In this sense, Schiffmann and Vasserot studied the K-theory of the Hilbert scheme of points on the plane using elliptic Hall algebras and Hecke algebras.

Contents

1 Hilbert schemes of n points

We begin by defining the schemes which will interest us : the Hilbert schemes of points. The idea is to start from a projective scheme X and use that for a coherent sheaf $\mathcal{F}$ on X, there exists a polynomial, the *Hilbert polynomial* of $\mathcal{F}$, with rational coefficients, which coincides on the positive integers with the *Euler characteristic* of $\mathcal{F}(n)$:

$$\chi(\mathcal{F}(n)) = \sum(-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{F}(n)).$$

For a given polynomial with coefficients in $\mathbb{Q}$, the pull-back then allows us to introduce the following functor defined on locally Noetherian schemes :

$$\mathfrak{hilb}_X^{\phi}(T) = \left\{ \begin{array}{c} Z \text{ closed subscheme of } X \times T \text{ flat on } T \\ \text{with } \forall\, t \in T,\ \phi \text{ Hilbert polynomial of } \mathcal{O}_{Z_t} \end{array} \right\}.$$

Here Z_t denotes the fiber of $Z \hookrightarrow X \times T \xrightarrow{pr_2} T$ over t. We will look at the case of constant Hilbert polynomials, and we can then reformulate the definition. For a positive integer n, we note :

$$\mathfrak{hilb}^n_X(T) = \left\{ \begin{array}{c} Z \text{ closed subscheme of } X \times T \text{ flat and proper on } T \\ \text{with } \forall\ t \in T,\ \dim Z_t = 0 \text{ and } h^0(Z_t, \mathcal{O}_{Z_t}) = n \end{array} \right\}.$$

Practically, an element of $\mathfrak{hilb}^n_X(T)$ will be seen as a family (Z_t) indexed by T of closed subschemes of X. The following theorem is due to Grothendieck :

Theorem 1.1. *With our hypotheses, $\mathfrak{hilb}^n_X$ is representable by a quasi-projective $\mathbb{C}$-scheme, which means that there exists a quasi-projective $\mathbb{C}$-scheme, noted $\operatorname{Hilb}^n X$ or $X^{[n]}$, such that*

$$\operatorname{Mor}_{\mathbb{C}}(\ \bullet\ , X^{[n]}) \simeq \mathfrak{hilb}^n_X.$$

An element $Z \in \mathfrak{hilb}^n_X(\operatorname{Spec}\mathbb{C})$ corresponds to a unique point, still noted Z, of $X^{[n]}$, and morphisms $T \to X^{[n]}$ are provided by families $(Z_t) \in \mathfrak{hilb}^n_X(T)$.

Definition 1.2. The scheme $X^{[n]}$ is called *Hilbert scheme of n points on X*.

The non trivial existence of the *Hilbert-Chow* morphism from the Hilbert scheme of n points to the symmetric power $X^{(n)} := X^n/\mathfrak{S}_n$ of X enables to evaluate the difference there is between considering an element of $X^{[n]}$ and a n-tuple of points of X, eventually counted with multiplicities :

Theorem 1.3. *Let X be smooth and projective oover $\mathbb{C}$. There exists a surjective morphism $\rho : X^{[n]}_{red} \to X^{(n)}$, called* Hilbert-Chow morphism*, and given on the points by*

$$Z \ \mapsto \sum_{P \in supp(Z)} \dim_{\mathbb{C}}(\mathcal{O}_{Z,P})P\ .$$

2 The case of the plane

The theorems of the previous section being also true for quasi-projective schemes, we can work on the Hilbert scheme of n points on the complex plane, which will be noted $(\mathbb{C}^2)^{[n]}$. As a set, we have

$$(\mathbb{C}^2)^{[n]} = \{\ I \text{ ideal in } \mathbb{C}[x,y]\ \mid\ \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\ \}.$$

The following fundamental result, due to Fogarty, demonstrates the importance of $(\mathbb{C}^2)^{[n]}$:

Theorem 2.1. *$(\mathbb{C}^2)^{[n]}$ is nonsingular and irreducible of dimension $2n$ and $\rho : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$ is a resolution of singularities.*

Hence, ρ is an isomorphism over the nonsingular locus of $(\mathbb{C}^2)^{(n)}$, consisting in the set of n-tuples of distinct points.

The proof of this theorem imply some objects which are crucial in the sequel, such as :

▷ the *incidence variety*, for a quasi-projective scheme X :

$$X^{[n,n+1]} = \{(Z, Z') \in X^{[n]} \times X^{[n+1]} \mid Z \subset Z'\}\ ;$$

▷ the tangent space of $(\mathbb{C}^2)^{[n]}$ at a closed point Z corresponding to an ideal I :

$$T_Z(\mathbb{C}^2)^{[n]} = \operatorname{Hom}_{\mathbb{C}[x,y]}(I, \mathbb{C}[x,y]/I)\ .$$

Ellingsrud and Strømme obtain a first result about the cohomology of $(\mathbb{C}^2)^{[n]}$:

Theorem 2.2. *For $0 \leq i \leq n$, we have*

$$\begin{cases} \dim H^{2i}(X^{[n]}, \mathbb{Q}) = \#\{\lambda \in \Pi_n,\ l(\lambda) = n - i\}\ , \\ \dim H^{2i+1}(X^{[n]}, \mathbb{Q}) = 0\ , \end{cases}$$

where Π_n denotes the set of partitions of n and $l(\lambda)$ the length λ.

They use a cellular decomposition provided by the action of the torus $T = (\mathbb{C}^*)^2$ on $\mathbb{C}^2$, which prevents any generalization to arbitrary smooth quasi-projective surfaces. The stratification presented hereafter enable to solve this difficulty.

3 Stratification

Once again, the theorem 2.1 can be generalized to the case of a quasi-projective smooth surface S.

Definition 3.1. For all partition $\lambda = (n_1 \geq \ldots \geq n_r) \in \Pi(n)$, one can define the locally closed stratum

$$S^{(n)}_\lambda = \{\ \sum n_i P_i \mid P_i \in S \text{ distinct points }\}$$

of $S^{(n)}$. Denoting $S^{[n]}_\lambda = \rho^{-1}(S^{(n)}_\lambda)$, we obtain a stratification

$$S^{[n]} = \coprod_\lambda S^{[n]}_\lambda$$

of $S^{[n]}$ into locally closed strata.

It's quite simple to show that the strata are nonsingular, and have dimension $2l(\lambda)$, where $l(\lambda)$ denotes the length of λ. The study of the fibers of the Hilbert-Chow morphism on each stratum is closelly related to a new kind of Hilbert schemes :

Definition 3.2. Let $H_n = \mathrm{Hilb}^n \operatorname{Spec} \mathbb{C}[[x, y]]$, parametrizing the ideals of codimension n in $\mathbb{C}[[x, y]]$. The H_n are called the *punctual* Hilbert schemes.

Proposition 3.3. *Let $\lambda = (n_1 \geq \ldots \geq n_r)$ be a partition of n. Then, via the Hilbert-Chow morphism, $S^{[n]}_\lambda$ is a locally trivial fiber bundle (in the analytic topology and in the étale topology) with fiber $H_{n_1} \times \ldots \times H_{n_r}$ over $S^{(n)}_\lambda$.*

The punctual Hilbert schemes have been quite extensively studied, especially by Briançon who proved that H_n is irreducible of dimension $n - 1$. The strata $S^{(n)}_\lambda$ being of codimension $2(n - l(\lambda))$, the Hilbert-Chow morphism is strictly semismall in the following sense :

Definition 3.4. Let $f : X \to Y$ be a projective morphisme of varieties over $\mathbb{C}$, and suppose that $Y = \coprod_\lambda Y_\lambda$ has a stratification into nonsingular locally closed strata. Assume that for all λ, $f : f^{-1}(Y_\lambda) \to Y_\lambda$ is a locally trivial fiber bundle with fiber F_λ in the analytic topology.

Then f is called *strictly semismall* (with respect to the stratification) if for all λ

$$2 \dim F_\lambda = \operatorname{codim} Y_\lambda.$$

In our situation, we also have the irreducibility of the fibers.

This result has been used by Göttsche to compute the Betti numbers of the Hilbert schemes, using the properties of intersection cohomology and perverse sheaves. We write $b_i(Y) = \dim H^i(Y, \mathbb{Q})$ for the i-th Betti number, and $p(Y) = \sum_i b_i(Y) z^i$ for the Poincaré polynomial :

Theorem 3.5. *Let S be a smooth quasi-projective sur face over $\mathbb{C}$. Then*

$$\sum_{n=0}^{\infty} p(S^{[n]}) t^n = \prod_{k>0} \frac{(1+z^{2k-1}t^k)^{b_1(S)}(1+z^{2k+1}t^k)^{b_3(S)}}{(1-z^{2k-2}t^k)^{b_0(S)}(1-z^{2k}t^k)^{b_2(S)}(1-z^{2k+2}t^k)^{b_4(S)}}.$$

4 The Heisenberg algebra

The previous formula suggests an interest in considering all of the cohomology groups of $S^{[n]}$ at the same time. So we denote :

$$\mathbf{V} = \bigoplus_{n \geq 0} H^*(S^{[n]}, \mathbb{Q}).$$

Definition 4.1. Let V be a $\mathbb{Q}$-vector space with a nondegenerate bilinear form $< . , . >$.

- ▷ Let T be the tensor algebra on $V[t, t^{-1}]$. We give t^i the degree i so that we have a graduation $T = \oplus_{i \in \mathbb{Z}} T^i$. The *Heisenberg algebra* $H(V)$ modelled on V is obtained from T by imposing the relations
$$[ut^i, vt^j] = i\delta_{i,-j} < u, v > \mathbf{e}$$
where $\mathbf{e}$ denotes the neutral element of the tensor algebra corresponding to the empty tensor product.
- ▷ The *Fock space* $F(V)$ is the subalgebra of $H(V)$ obtained by replacing $V[t, t^{-1}]$ by $tV[t]$. $F(V)$ becomes a $H(V)$-module, by putting $ut^0.w = 0$ for all $w \in F(V)$ and $ut^{-i}.\mathbf{e} = 0$ for all $i > 0$.

Proposition 4.2. *The Fock space $F(V)$ is an irreducible $H(V)$-module, and if $F(V)^d$ denotes its part of degree d (where t has degree 1), we have*

$$\sum_{d \geq 0} \dim(F(V)^d) t^d = \prod_{k \geq 1} \frac{1}{(1-t^k)^{\dim V}}.$$

The interest for such algebras come from the fact that for a quasi-projective surface S sucht that $V = H^*(S, \mathbb{Q}) = H^0(S, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$, we get the same formula as in 3.5 (with $z = 1$). Nakajima obtained the following result :

Theorem 4.3. *Let S be a quasi-projective smooth surface over $\mathbb{C}$, and $V = H^*(S, \mathbb{Q})$. Assuming $H^1 = H^3 = 0$, we have an isomorphism of $H(V)$-module between $F(V)$ and $\mathbf{V}$.*

Remark 4.4. Here, $< \alpha, \beta >= p_*(\alpha \cap \beta)$ where p denotes the unique morphism $S \to \{pt\}$. If S is such that $H^1 = H^3 = 0$ (which will be the case in the sequel), this form is symetric. If not (non trivial odd cohomology), one should consider *Clifford algebras*.

To simplify, let's consider the case of $S = \mathbb{C}^2$ (so we get $V = \mathbb{Q}$) to construct of the $H(V)$-module structure on $\mathbf{V}$. To represent the action of t^m, one has to create applications $H^*(S^{[n]}) \to H^*(S^{[n+m]})$. Let's generalize the incidence varieties by putting :

$$Z_{n,m} = \{(Z, P, W) \in S^{[n]} \times S \times S^{[n+m]} \mid \rho(W) - \rho(Z) = mP\}.$$

We impose $W \backslash Z$ to be concentrated in a point. Denoting pr and pr' the projections on $S^{[n]}$ to $S^{[n+m]}$, we define for $m > 0$

$$p_m : H^*(S^{[n]}) \to H^*(S^{[n+m]}),\ y \mapsto DP(pr'_*(pr^*(y) \cap Z_{n,m}))$$

where DP denotes the Poincaré duality. Intuitively, if y is represented as the cohomology class Poincaré dual to the fundamental class of a submanifold $Y \subset S^{[n]}$, then $p_m(y)$ will be the class of the closure of $\{Z \sqcup P \mid Z \in Y,\ P \in S^{[m]}_{(m)}\}$.

Still denoting p_m the operator $\mathbf{V} \to \mathbf{V}$ obtained by direct sum, we have the following weak version of the result of Nakajima, which says that all the cohomology of the Hilbert schemes of points can be obtained by just applying the creation operators to $\mathbf{1}$:

Proposition 4.5. *Let $\mathbf{1}$ be the unit of $\mathbb{Q} = H^*(S^{[0]})$. Then the set of*

$$p_{n_1} \circ \ldots \circ p_{n_r}(\mathbf{1}),\ \sum n_j = n$$

is a basis of $\mathbf{V}_n = H^*(S^{[n]})$ for all $n \geq 0$.

Using the intersection pairing, one can similarly define operators $p_{-m} : \mathbf{V}_{n+m} \to \mathbf{V}_n$: this time $p_{-m}(y)$ corresponds to the class of the closure of $\{Z \in S^{[n]} \mid Z \sqcup P \in Y$ for some $P \in S^{[m]}_{(m)}\}$, if y is the class of $Y \subset S^{[n+m]}$. Then we get the main result :

Proposition 4.6. *For all n, m, $[p_n, p_m] = n\delta_{n,-m} id_{\mathbf{V}}$.*

Then we have an $H(V)$-module morphism

$$F(V) \to \mathbf{V},\ t^i \mapsto p_i(\mathbf{1}).$$

As $F(V)$ is irreducible and both have the same Poincaré series, this is an isomorphism.

5 K-theory

The issue of defining similar operators on the Grothendieck groups instead of the cohomology groups is raised by Nakajima. We recall that for a scheme X, $K(X)$ is defined as the free group generated by the isomorphism classes $[\mathcal{F}]$ of coherent sheaves on X, quotiented by the relations $[\mathcal{A}] + [\mathcal{C}] = [\mathcal{B}]$ each time we have a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0.$$

In fact we will use equivariant K-theory in the sequel, which means that, for an algebraic linear group G, we'll be interested in the group $K^G(X)$ of the category of G-equivariant coherent sheaves on X. The operators which will be introduced in the sequel come from *convolution* operators. In the following proposition, we call G-variety a quasi-projective scheme endowed with a G-action :

Proposition 5.1. *Let X_1, X_2 and X_3 be smooth G-varieties admitting* proper *maps to a G-variety Y. There is a natural map*

$$\star : K^G(X_1 \times_Y X_2) \otimes K^G(X_2 \times_Y X_3) \to K^G(X_1 \times_Y X_3).$$

If we note R_G the complexified representation ring of G, the convolution map :

- ▷ endows $K^G(X \times_Y X)$ with a strucutre of an associative R_G-algebra for $X_1 = X_2 = X_3 = X$ (with unit $[\Delta_X]$, $\Delta_X \subset X \times_Y X$ denoting the diagonal) ;
- ▷ endows $K^G(X)$ with the structure of a $K^G(X \times X)$-module for $X_1 = X_2 = X$ and $X_3 = Y = \{pt\}$.

We consider $S = \mathbb{A}^2$. The schemes $S^{[n]}$ being nonproper for $n > 0$, we use again the action of the torus $T = (\mathbb{C}^*)^2$. It's not hard to see that the fixed points of $S^{[n]}$ under this action are in bijection with $\Pi(n)$ via

$$\lambda = (\lambda_1, \dots, \lambda_r) \mapsto I_\lambda =< y^{\lambda_1}, xy^{\lambda_2}, \dots, x^{r-1}y^{\lambda_r}, x^r >\subset \mathbb{C}[x, y].$$

In particulier, $(S^{[n]})^T$ is finite. If we note

$$q : T \to \mathbb{C}^*,\ (z_1, z_2) \mapsto z_1^{-1}, \quad t : T \to \mathbb{C}^*,\ (z_1, z_2) \mapsto z_2^{-1},$$

we have $R_T = \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ and we then note $\mathbf{K} = \mathbb{C}(q^{1/2}, t^{1/2})$. A theorem of Thomason gives us the following isomorphism, where ι is the embedding $S^{[n],T} \hookrightarrow S^{[n]}$ and where we denote $M_\mathbf{K}$ for $M \otimes_{R_T} \mathbf{K}$ if M is a R_T-module :

$$\iota_* : K^T(S^{[n],T})_\mathbf{K} = \underset{\lambda \vdash n}{\oplus} \mathbf{K}[I_\lambda] \xrightarrow{\sim} K^T(S^{[n]})_\mathbf{K}.$$

Denoting $I_{\mu,\lambda} = (I_\mu, I_\lambda)$, we also have

$$K^T(S^{[n]} \times S^{[m]})_\mathbf{K} = \underset{\lambda \vdash n,\ \mu \vdash m}{\oplus} \mathbf{K}[I_{\mu,\lambda}].$$

Hence each element of $K^T(S^{[n]})_\mathbf{K}$ or $K^T(S^{[n]} \times S^{[m]})_\mathbf{K}$ is a linear combination of classes of coherent sheaves with proper support. This allows us to define convolution operations :

$$\star : K^T(S^{[n]} \times S^{[m]})_\mathbf{K} \otimes K^T(S^{[m]} \times S^{[k]})_\mathbf{K} \to K^T(S^{[n]} \times S^{[k]})_\mathbf{K},$$

$$\star : K^T(S^{[n]} \times S^{[m]})_\mathbf{K} \otimes K^T(S^{[m]})_\mathbf{K} \to K^T(S^{[n]})_\mathbf{K}.$$

We consider the following associative $\mathbf{K}$-algebra

$$\mathbf{E_K} = \bigoplus_{k \in \mathbb{Z}} \prod_n K^T(S^{[n+k]} \times S^{[n]})_\mathbf{K},$$

where the product ranges over all integers n for which $n \geq \max(0, -k)$. It acts on the $\mathbf{K}$-vector space

$$\mathbf{L_K} = \bigoplus_{n \geq 0} K^T(S^{[n]})_\mathbf{K}.$$

In order to define a relevant subalgebra of $\mathbf{E_K}$, the incidence varieties will once again be usefull :

- ▷ Let's note $Z_n = \{(x, Z) \mid x \in Z\} \subset S \times S^{[n]}$ the universal family and $\boldsymbol{\tau}_n = p_* \mathcal{O}_{Z_n}$ the *tautological bundle of* $S^{[n]}$, where p is the projection $S \times S^{[n]} \to S^{[n]}$.
- ▷ Denoting p_1 and p_2 the projections of $S^{[n]} \times S^{[n+1]}$ to $S^{[n]}$ and $S^{[n+1]}$, we also note $\boldsymbol{\tau}_{n,n+1} = \ker(p_2^* \boldsymbol{\tau}_{n+1} \to p_1^* \boldsymbol{\tau}_n)$ the *tautological bundle of* $S^{[n,n+1]}$, which specializes to $\mathbb{C}[x, y]/J \to \mathbb{C}[x, y]/I$ over a point $(I, J) \in S^{[n,n+1]}$.
- ▷ We can similarly construct $\tau_{n+1,n} \in K^T(S^{[n+1]} \times S^{[n]})_\mathbf{K}$.
- ▷ Finally, the *tautological bundle of* $S^{[n]} \times S^{[n]}$ is $\boldsymbol{\tau}_{n,n} = \pi^* \boldsymbol{\tau}_n$ where π is one of the natural projections of $S^{[n]} \times S^{[n]}$ to $S^{[n]}$.

The following result is then obtained by Schiffmann and Vasserot, which gives an answer the question of Nakajima (with the convention $\tau_{i,j}^{-1} = \tau_{i,j}^{*}$) :

Theorem 5.2. *Let $\mathbf{H_K}$ be the subalgebra of $\mathbf{E_K}$ spanned by*

$$\begin{array}{lll} \prod_n [\boldsymbol{\tau}_{n,n+1}^{\otimes l}], & \prod_n [\boldsymbol{\tau}_{n+1,n}^{\otimes l}], & l \in \mathbb{Z}, \\ \prod_n [\wedge^l \boldsymbol{\tau}_{n,n}], & \prod_n [\wedge^l \boldsymbol{\tau}_{n,n}^{*}], & l \in \mathbb{Z}_{>0}. \end{array}$$

Then :

1. *$\mathbf{H_K}$ is isomorphic to a certain one-dimensional central extension of the spheriacl Double Affine Hecke Algebra $\mathbf{S\ddot{H}}_\infty$ of type $\mathcal{G}l_\infty$.*
2. *As an $\boldsymbol{\mathcal{E}}$-module, $\mathbf{L_K}$ is isomorphic to $\mathbf{K}[x_1, x_2, \ldots]^{\mathfrak{S}_\infty}$.*

The definition of $\mathbf{H_K}$ comes from the fact that the *nested Hilbert schemes* $S^{[n,n+k]} = \{(Z, Z') \in S^{[n]} \times S^{[k]} \mid Z \subset Z'\}$ are smooth only if $k = 0, 1$.

It would be too long to define $\mathbf{S\ddot{H}}_\infty$ here. However, this algebra is isomorphic to $\mathbf{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty}$ as a $\mathbf{K}$-vector space. Let's give some informations about the action of $\boldsymbol{\mathcal{E}}$ on $\mathbf{L_K}$:

- ▷ First, the complexification of the Fock space $F(\mathbb{Q})$ can be seen as $\mathbb{C}[p_1, p_2, \ldots]$, t^m acting by

$$\begin{cases} m\dfrac{\partial}{\partial p_m} & (m > 0) \\ p_{-m} & (m < 0) \end{cases} ;$$

- ▷ We have a $\mathbb{Z}^2$-graduation on $\boldsymbol{\mathcal{E}}$, one of the axes acting, in a similar way as the Heisenberg algebra, by

$$\begin{cases} -m\dfrac{q^{m/2}}{1-t^m}\dfrac{\partial}{\partial p_m} & (m > 0) \\ \dfrac{t^{-m/2}}{1-q^{-m}}\, p_{-m} & (m < 0) \end{cases} ,$$

  with $p_m = (\sum_{i=0}^n x_i^m)_n$ (cf. $\mathbf{K}[x_1, x_2, \ldots]^{\mathfrak{S}_\infty}$ is defined as the projective limit of the $\mathbf{K}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$). The action of the other axis involve *Macdonald polynomials*.

Here, we don't know how such a result can be generalized to other surfaces. A first step would be to consider *toric* surfaces, starting with $\mathbb{P}^2$.

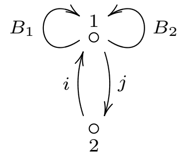
6 Quiver varieties

For $I \in (\mathbb{C}^2)^{[n]}$, the multiplication by x and y give two endomorphisms B_1 and B_2 of $V = \mathbb{C}[x,y]/I \simeq \mathbb{C}^n$ such that if we note $v = 1 \mod I \in V$, we have

$$\begin{array}{rl} (i) & [B_1, B_2] = 0 \\ (ii) & v \text{ is cyclic for } (B_1, B_2) \ . \end{array}$$

The second condition means that if a subspace $W \subset V$ contains v, $(B_1(W) \subset W,\ B_2(W) \subset W) \Rightarrow W = V$. Nakajima has proved that we have an isomorphism between $(\mathbb{C}^2)^{[n]}$ and $\{(B_1, B_2, v) \in \mathrm{End}(\mathbb{C}^n)^2 \times \mathbb{C}^n \mid (i), (ii)\}/G$, where $G = \mathcal{G}l_n(\mathbb{C})$. Hence, the study of the Hilbert schemes of points on the plane is linked to a particular quiver variety of Nakajima, the one associated to the loop $A_0 = \circ\circlearrowright$, which we will define here.

We first define $\bar{E}_{(n,1)} = \mathrm{End}(\mathbb{C}^n)^2 \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ to be the representation space of the double quiver associated to A_0, noted $\bar{A}_0$:



The canonical symplectic form ω is given by

$$\omega((B_1, B_2, i, j), (B_1', B_2', i', j')) = \mathrm{Tr}(B_1 B_2' - B_2 B_1' + ij' - i'j)$$

and the moment map relative to the action of G on $\bar{E}_{(n,1)}$:

$$\mu \left| \begin{array}{ccc} \bar{E}_{(n,1)} & \to & \mathfrak{g} \\ (B_1, B_2, i, j) & \mapsto & [B_1, B_2] + ij \end{array} \right. .$$

There are two ways to define the symplectic quotient of $\bar{E}_{(n,1)}$ by G. One is the classical "algebraico-geometrical" quotient, which will be noted $\mu^{-1}(0)/G$. The second one comes from the geometric invariant theory : for $\chi = \det^{-1}$, we will note $\mu^{-1}(0)/_\chi G$ the algebraic quotient by G of the set of *stable points*, (B_1, B_2, i, j) being stable if for all $z \in \mathbb{C}^*$, the orbit

$$G.(B_1, B_2, i, j, z) = \{(g.(B_1, B_2, i, j), \chi(g)^{-1} z) \mid g \in G\}$$

is a closed subset of $\bar{E}_{(n,1)} \times \mathbb{C}$. Then :

Proposition 6.1. *We have the following isomorphisms :*

$$(\mathbb{C}^2)^{(n)} \simeq \mu^{-1}(0)/G \quad \textit{and} \quad (\mathbb{C}^2)^{[n]} \simeq \mu^{-1}(0)/_\chi G.$$

The Hilbert-Chow morphism is just the natural morphism $\mu^{-1}(0)/_\chi G \twoheadrightarrow \mu^{-1}(0)/G$.

This is a consequence of the following points :
- ▷ $(\mathbb{C}^2)^{[n]} \simeq \{(B_1, B_2, v) \in \mathrm{End}(\mathbb{C}^n)^2 \times \mathbb{C}^n \mid (i), (ii)\}/G$;
- ▷ $((B_1, B_2, i, j)$ stable and $[B_1, B_2] + ij = 0) \Leftrightarrow j = 0$;
- ▷ $((B_1, B_2, i, 0)$ stable and $[B_1, B_2] = 0) \Leftrightarrow$ condition (ii), where the homomorphism i is identified with any vector v such that $< v >= \mathrm{Im}(i)$.

Using crystal structures, Nakajima has established a link between the cohomology groups of Hilbert schemes of points and the lagrangian subvarieties of quiver varieties. These results involve nontrivial finite groups $\Gamma \subset \mathcal{S}l_2(\mathbb{C})$ and the associated Hilbert schemes of fixed points $((\mathbb{C}^2)^{[n]})^\Gamma$. But for such $\Gamma \neq \{id\}$, the associated quiver variety do *not* contain any loop.

However, the study of the lagrangian variety $\Lambda^n = \rho^{-1}(D^{(n)})$, for D a fixed line in $\mathbb{C}^2$, should lead to a better comprehension of the cohomology groups of $(\mathbb{C}^2)^{[n]}$.