1 Definitions and first properties

Definition 1.1. A power series is a series of functions \( \sum f_n \) where \( f_n : z \mapsto a_n z^n \), \((a_n)\) being a sequence of complex numbers. Depending on the cases, we will consider either the complex variable \( z \), or the real variable \( x \).

Notations 1.2. For \( r \geq 0 \), we will note \( \Delta_r = \{ z \in \mathbb{C} \mid |z| < r \} \), \( K_r = \{ z \in \mathbb{C} \mid |z| \leq r \} \) and \( C_r = \{ z \in \mathbb{C} \mid |z| = r \} \).

Lemma 1.3. [Abel’s lemma] Let \( \sum a_n z^n \) be a power series. We suppose that there exists \( z_0 \in \mathbb{C}^* \) such that the sequence \( (a_n z_0^n) \) is bounded. Then, for all \( r \in ]0, |z_0|[ \), \( \sum a_n z^n \) normally converges on the compact \( K_r \).

Remark 1.4. • Note that it implies the absolute convergence on \( \Delta_{|z_0|} \), i.e. \( \forall z \in \Delta_{|z_0|}, \sum |a_n z^n| \) converges.

• Of course if we suppose \( \sum |a_n| r^n \) convergent, we directly have the normal convergence on \( K_r \) (cf. \( \forall z \in K_r, |a_n z^n| \leq |a_n| r^n \)).

Proof. Let \( z \) be in \( K_r \), we have
\[ |a_n z^n| \leq |a_n| r^n = |a_n z_0^n| \left( \frac{r}{|z_0|} \right)^n = O \left( \left( \frac{r}{|z_0|} \right)^n \right), \]
which gives the result. \( \Box \)

Definition 1.5. We call the radius of convergence of the power series \( \sum a_n z^n \) the number
\[ R = \sup \{ r \geq 0 \mid (a_n r^n) \text{ bounded} \} \in \mathbb{R}^+ = \mathbb{R}^+ \cup \{ +\infty \}. \]
It will sometimes be noted \( RCV(\sum a_n z^n) \).

Theorem 1.6. Let \( R \) be the RCV of a power series \( \sum a_n z^n \).
• For all \( r < R \), \( \sum a_n z^n \) normally converges on the compact \( K_r \).
• For all \( z \) such that \( |z| > R \), \( a_n z^n \xrightarrow{n \to \infty} 0 \).

Remark 1.7. It implies the absolute convergence on \( \Delta_R \).

Proof. The Abel’s lemma gives the first point : \( \forall r \in [0, R[, \exists r' \in ]r, R] \) such that \( (a_n r'^n) \) is bounded, which implies the normal convergence on \( K_r \). For the second point, it’s the contraposition of \( a_n z^n \to 0 \Rightarrow (a_n z^n) \) bounded \( \Rightarrow |z| \leq R \). \( \Box \)
Corollary 1.8. With the same hypothesis, \( R = \sup \{ r \geq 0 \mid \sum a_n r^n \text{ converges} \} = \inf \{ r \geq 0 \mid \sum a_n r^n \text{ diverges} \} \in \mathbb{R}^+ \).

Proof. Let’s note \( R' = \sup \{ r \geq 0 \mid \sum a_n r^n \text{ converges} \} \) and \( R'' = \inf \{ r \geq 0 \mid \sum a_n r^n \text{ diverges} \} \). First, \( R' \leq R'' \) : if not, \( R'' < R' \) and \( \exists r \in ]R'', R'[, r'\) such that \( \sum a_n r'^n \) converges, so we would have \( (a_n r'^n) \) bounded and convergence on \( \Delta_r \) (cf. 1.4), and thus \( R' \geq r \), absurd. By the first point of the theorem, \( R' \geq R \). By the second point, \( R'' \leq R \). So we have \( R \leq R' \leq R'' \leq R \), which gives the result. \( \square \)

Remark 1.9.
- With the same kind of proof, one can show that we also have \( R = \sup \{ r \geq 0 \mid a_n r^n \to 0 \} \).
- To sum up, if we note \( \mathcal{E} \) the domain of convergence of a power series which has a radius of convergence \( R \), we have

\[
\Delta_R \subset \mathcal{E} \subset K_R
\]

and we have absolute convergence on \( \Delta_R \).

Definition 1.10. We call \( \Delta_R = \{ z \in \mathbb{C} \mid |z| < R \} \) the (open) disk of convergence.

Remark 1.11. We can’t say anything \textit{a priori} about the convergence of a power series on the circle \( C_R \), as we will see in the examples.

Examples 1.12.
- \( RCV(\sum z^n) = 1 \) since the constant sequence \( (1) \) is bounded (\( \Rightarrow RCV \geq 1 \) and \( \sum 1 \) diverges (\( \Rightarrow RCV \leq 1 \)). In fact there’s no point in \( C_1 \) where there is oconvergence \( (|z| = 1 \Rightarrow z^n \to 0) \).
- \( RCV(\sum z^n/n) = 1 \) since \( (1/n) \) bounded (\( \Rightarrow RCV \geq 1 \) and \( \sum 1/n \) diverges (\( \Rightarrow RCV \leq 1 \)). Here, the only point of \( C_1 \) where the power series diverges is \( 1 \) : if \( z = e^{i\theta} \neq 1 \), \( \sum z^n/n \) converges if \( \Re(\sum z^n/n) \) and \( \Im(\sum z^n/n) \) converge, ie if \( \sum \cos(n\theta)/n \) and \( \sum \sin(n\theta)/n \) converge. But we’ve already seen that the first one converges if \( e^{i\theta} \neq 1 \), and the same proof shows that it’s the same for the second one.

Exercise 1.13. [Hadamard theorem] Prove that this definition of the radius of convergence is equivalent to the first one :

\[
R = (\limsup |a_n|^{1/n})^{-1}
\]

2 Few methods to find the RCV

Proposition 2.1. Let \( \sum a_n z^n \) be a power series and \( z_0 \in \mathbb{C} \). Then :
- If \( \sum a_n z_0^n \) converges but \( \sum |a_n z_0^n| \) diverges, then \( RCV = |z_0| \).
- Same conclusion if \( \sum a_n z_0^n \) diverges but \( a_n z_0^n \to 0 \).

Proof. For the first point, we have \( RCV \geq |z_0| \) (cf. 1.8), but we can’t have \( RCV > |z_0| \) (cf. 1.6). The second point is a consequence of 1.8 and 1.9. \( \square \)

Proposition 2.2. Let \( \sum a_n z^n \) and \( \sum b_n z^n \) be two power series, and \( R_a, R_b \) their RCV. We have \( a_n = O(b_n) \Rightarrow R_a \geq R_b \).
Proposition 2.8. Let \( a_n \neq 0 \) for \( n \) big enough. Then (with \( 1/0 = +\infty \) and \( 1/\infty = 0 \)):

\[
\exists \lim \frac{a_{n+1}}{a_n} = l \in \mathbb{R}^+ \Rightarrow RCV = \frac{1}{l}.
\]

Proof. We have \( \frac{|a_{n+1}z^{n+1}|}{a_nz^n} \rightarrow l|z| \). By the De D'Alembert rule, \( |z| < 1/l \Rightarrow \sum a_nz^n \) converges, and \( RCV \geq 1/l \) (cf. 1.8). Similarly, if \( |z| > 1/l \), \( \sum a_nz^n \) diverges, and \( RCV \leq 1/l \). \( \square \)

Corollary 2.4. With the same notations, we have \( a_n \sim b_n \Rightarrow R_a = R_b \).

Proof. \( \sim \Rightarrow O \).

Proposition 2.5. Suppose \( a_n \neq 0 \) for \( n \) big enough. Then (with \( 1/0 = +\infty \) and \( 1/\infty = 0 \)):

\[
\exists \lim \frac{a_{n+1}}{a_n} = l \in \mathbb{R}^+ \Rightarrow RCV = \frac{1}{l}.
\]

Proof. We have \( \frac{|a_{n+1}z^{n+1}|}{a_nz^n} \rightarrow l|z| \). By De D'Alembert rule, \( |z| < 1/l \Rightarrow \sum a_nz^n \) converges, and \( RCV \geq 1/l \) (cf. 1.8). Similarly, if \( |z| > 1/l \), \( \sum a_nz^n \) diverges, and \( RCV \leq 1/l \). \( \square \)

Proposition 2.6. Let \( \sum a_nz^n \) a power series and \( R \) its RCV. Then for all \( \alpha \in \mathbb{R} \) the RCV \( R_\alpha \) of the power series \( \sum n^\alpha a_nz^n \) is also \( R \).

Proof. Let \( r < R \) and \( \rho \in [r, R] \). We have

\[
n^\alpha a_n r^n = n^\alpha \left( \frac{r}{\rho} \right)^n a_n \rho^n \Rightarrow (n^\alpha a_n r^n) \text{ bounded} \Rightarrow R_\alpha \geq R.
\]

This is true for all \( \sum a_nz^n \), and for all \( \alpha \), so we also have, with \( \beta = -\alpha \),

\[
R = RCV(\sum n^\beta(n^\alpha a_nz^n)) \geq RCV(\sum n^\alpha a_nz^n) = R_\alpha.
\]

\( \square \)

Examples 2.7.

- By 2.5, \( RCV(\sum z^n/n!) = +\infty \).
- By 2.5, \( RCV(\sum n!z^n) = 0 \).
- By 2.6, \( RCV(\sum z^n/n^2) = 1 \) and we have normal convergence on \( K_1 \).
- We can abusively note \( \sum z^{2n}/5^n \) the power series defined by \( a_{2n+1} = 0 \) and \( a_{2n} = 5^{-n} \) for all \( n \). But we can’t apply directly 2.5. However, it’s clear that we have convergence on \( \Delta \sqrt{5} \) and divergence on its complementary, so \( RCV = \sqrt{5} \).

Proposition 2.8. Let \( R_a \) and \( R_b \) be the RCV of \( \sum a_nz^n \) and \( \sum b_nz^n \). Then \( R_{a+b} = RCV(\sum (a_n+b_n)z^n) \geq m = \min\{R_a, R_b\} \), with equality if \( R_a \neq R_b \). Moreover, on \( \Delta_m \), we have

\[
\sum (a_n+b_n)z^n = \sum a_nz^n + \sum b_nz^n.
\]

Proof. For all \( z \in \Delta_m \), \( \sum a_nz^n \) and \( \sum b_nz^n \) absolutely converges. Hence \( \sum (a_n+b_n)z^n \) also does : \( R_{a+b} \geq m \) and the additivity of limits of sequences gives the additivity formula. If \( R_a < R_b \), for all \( z \in \Delta_{R_a}\backslash K_{R_b} \) we have \( a_nz^n \rightarrow 0 \) and \( b_nz^n \rightarrow 0 \), thus \( (a_n+b_n)z^n \rightarrow 0 \), and \( R_{a+b} \leq R_a = m \). \( \square \)
Example 2.9. Let $\sum a_n z^n = \sum z^n$ and $\sum b_n z^n = \sum ((1/2)^n - 1) z^n$, we have $R_0 = 1 = R_0$ (use 2.5 for the second one). As $a_n + b_n = (1/2)^n$, the domain of convergence of the $\sum (a_n + b_n) z^n$ is clearly $\Delta_2$, so $R_{a+b} = 2 > m$.

The nest result is obvious :

Proposition 2.10. For all $\lambda \in \mathbb{C}^*$, $\sum a_n z^n$ and $\sum \lambda a_n z^n$ have the same RCV $R$.
Moreover, on $\Delta_R$, we have

$$\sum \lambda a_n z^n = \lambda \sum a_n z^n.$$ 

Proposition 2.11. Let $R_a$ and $R_b$ be the RCV of $\sum a_n z^n$ and $\sum b_n z^n$. Then $R_{a+b} = RCV((\sum a_n z^n) * (\sum b_n z^n)) \geq m = \min\{R_a, R_b\}$. Moreover, on $\Delta_m$, we have

$$\sum a_n z^n * (\sum b_n z^n) = (\sum a_n z^n)(\sum b_n z^n).$$

Proof. For all $z \in \Delta_m$, $\sum |a_n z^n|$ and $\sum |b_n z^n|$ absolutely converges. Hence the Cauchy product $(\sum |a_n z^n|) * (\sum |b_n z^n|)$ converges (cf. ch1). But

$$\forall n, \left| \sum_{k=0}^n a_k z^k b_{n-k} z^{n-k} \right| \leq \sum_{k=0}^n |a_k z^k| |b_{n-k} z^{n-k}|,$$

so we get $R_{a+b} \geq m$ and the result given about the Cauchy product in chapter 1 gives the formula.

Examples 2.12.
- We don't have the same result as for the addition if $R_a \neq R_b$ : Let $\sum a_n z^n$ and $\sum b_n z^n$ be defined by $a_0 = 1/2, b_0 = -2$ and $a_n = -1/2^{n+1}, b_n = -3$ for $n \geq 1$. We have $\sum a_n z^n = 1 - \sum_{n \geq 0} z^n / 2^{n+1}$, $\sum b_n z^n = 1 - \sum_{n \geq 0} z^n$, so $R_a = 2 \neq R_b = 1$. We also have

$$\sum a_n z^n = 1 - \frac{1/2}{1 - (z/2)} = \frac{z - 1}{z - 2} \forall z \in \Delta_2,$$

and $\sum b_n z^n = 1 - 3 \frac{1}{1 - z} = \frac{z - 2}{z - 1} \forall z \in \Delta_1$.

Hence by 2.11 $(\sum a_n z^n) * (\sum b_n z^n) = 1$ on $\Delta_1$, so if we note $c_n = \sum_{k=0}^n a_k b_{n-k}$, we have $c_0 = 1$ and $c_n = 0$ for $n \geq 1$. Thus $R_{a+b} = RCV(\sum c_n z^n) = +\infty > m$.
- Let $R, R'$ be the RCV of $\sum a_n z^n$ and $\sum s_n(a) z^n$. We have $\sum s_n(a) z^n = (\sum a_n z^n) * (\sum z^n)$, hence $R' \geq \min\{1, R\}$. We also have

$$\sum a_n z^n = \sum s_n(a) z^n - \sum s_{n-1}(a) z^n = \sum s_n(a) z^n - z \sum_{n \geq 1} s_n(a) z^n,$$

which gives $R \geq R'$. Thus we have

$$\min\{1, R\} \leq R' \leq R$$

which gives $R = R'$ if $1 \geq R$.

3 Properties of the sum

We've already seen :

Theorem 3.1. Let $\sum a_n z^n$ be a power series and $R$ its RCV, $\sum a_n z^n$ normally converges on every $K_r$, $r < R$, which leads to the continuity of the sum function on $\Delta_R$.
Remark 3.2. If $\exists z_0 \in C_R$ such that $\sum a_n z_0^n$ absolutely converges, then we have normal convergence (and continuity) on $K_R$.

**Theorem 3.3.** [Radial continuity] Let’s suppose that $\sum a_n z_0^n$ converges for $z_0 \in C_R$. Then $\sum a_n z^n$ uniformly converges on $[0, z_0]$, ie $t \mapsto \sum a_n z_0^n t^n$ uniformly converges on $[0, 1]$.

**Proof.** We note $s_n(t) = \sum_{k=0}^{n} a_k z_0^k t^k$ for $t \in [0, 1]$ and $r_n = \sum_{k=n+1}^{\infty} a_k z_0^k$. By Abel’s formula we obtain

$$s_n(t) = \sum_{k=0}^{n} (r_{k-1} - r_k)t^k = \sum_{k=0}^{n} (t^{k+1} - t^k)r_k - t^{n+1}r_n + r_{n-1}. $$

For $\epsilon > 0$, $\exists N$ such that $n \geq N \Rightarrow |r_n| \leq \epsilon$, hence, for all $n \geq N$, $p \geq 1$, $t \in [0, 1]$

$$|f_{n+p}(t) - f_n(t)| \leq \sum_{k=n+1}^{n+p} |r_k| (t^k - t^{k+1}) \leq \epsilon(t^{n+1} - t^{n+p+1}) \leq \epsilon,$$

and $|t^{n+1}r_n| \leq \epsilon$, so $(s_n)$ uniformly converges. $\square$

**Remark 3.4.** The Leibniz criterion can also be used in the case of a decreasing real sequence $(a_n)$ which converges to zero. Suppose $R = 1$, then for $x \in [-1, 0]$, $\sum a_n x^n$ satisfies the hypothesis of the Leibniz criterion; so we get $|\sum_{k=n+1}^{\infty} a_n x^n| \leq |a_n x^n| \leq a_n$, which proves the uniform convergence on $[-1, 0]$, and thus the continuity in $-1$.

We can deduce from the radial continuity a new result about the Cauchy product - compare with the one obtained in ch.1:

**Corollary 3.5.** Let $\sum c_n$ be the Cauchy product of $\sum a_n$ and $\sum b_n$. We suppose that $\sum a_n$, $\sum b_n$ and $\sum c_n$ converge to $A$, $B$ and $C$. Then $C = AB$.

**Proof.** The three power series $f(x) = \sum a_n x^n$, $g(x) = \sum b_n x^n$ and $h(x) = \sum c_n x^n$ have a RCV $\geq 1$, hence absolutely converge for $|x| < 1$ so we can apply the theorem of chapter 1 and get $f(x)g(x) = h(x)$ for these $x$. But by the radial continuity theorem we can apply the double limit theorem for $x \rightarrow 1$ to obtain the result. $\square$

**Definition 3.6.** We call derivative series (resp. primitive series) of a power series $\sum a_n x^n$ the power series defined by $\sum (n+1) a_{n+1} z^n$ (resp. $\sum_{n \geq 1} (a_{n-1}/n) z^n$).

**Remark 3.7.** We know that they have the same RCV than $\sum a_n z^n$, thanks to 2.6 and 2.4: $\sum_{n \geq 1} (a_{n+1}/n) z^n$ converges iff $z \sum_{n \geq 1} (a_{n+1}) z^n = \sum_{n \geq 1} n a_n z^n$ converges; and $\sum_{n \geq 1} (a_{n-1}/n) z^n = z \sum_{n \geq 1} (a_n/(n+1)) z^n$ with $a_n/(n+1) \sim a_n/n$.

**Theorem 3.8.** Let $\sum a_n x^n$ (real variable) be a power series, $f$ its sum, $g$ (resp. $F$) the sum of its derivative (resp. primitive) series and $R$ its RCV. Then, on $] - R, R [$, $f$ is $C^1$ with $f' = g$, and $F$ is the only primitive of $f$ such that $F(0) = 0$.

**Remark 3.9.** This implies

$$\forall x \in ] - R, R [\quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right) dt.$$
Proof. Replacing $f$ by $F$, the first assertion immediately gives the second one. But if we note $f_n(x) = a_n x^n$, we have $f_n \in C^1$ with $f_n'(x) = n a_n x^{n-1}$ for $n \geq 1$ ($f'_0 = 0$). Hence $\sum f_n'$ is the derivatives series of $\sum a_n x^n$ which normally converges on each $[-r, r] \subset [-R, R]$ (cf. 3.7), and we know that it implies : $\sum f_n \in C^1$ on $[-r, r]$ and $f' = (\sum f_n)' = \sum f_n' = g$. We conclude with the fact that $\exists -R, R[= U_0 < r < R] [-r, r]$.

Corollary 3.10. The sum function $f$ of a power series $\sum a_n x^n$ with $RCV = R$ is $C^\infty$ on $]-R, R[$, and $f^{(p)}$ is the sum function of $\sum \frac{(n+p)!}{n!} a_{n+p} x^n$.

The $RCV$ of these power series is also $R$.

Remark 3.11. This implies

$$\forall p, \frac{f^{(p)}(0)}{p!} = a_p$$

Corollary 3.12. If we have $\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} b_n x^n$ on $]-R, R[$ (both power series converging on this interval), then $a_n = b_n$ for all $n$.

Proof. The difference of the sum functions is 0. Hence, all its derivatives at 0 are 0.

4 RPS functions

Definition 4.1. Given a complex number $z_0$ and a function $f : U \to \mathbb{C}$ defined on a neighborhood $U \subset \mathbb{C}$ of $z_0$, we say that $f$ is representable by a power series (=RPS) or analytic at $z_0$ if $\exists r > 0$ and a power series $\sum a_n z^n$ with $RCV \geq r$ such that $\Delta(z_0, r) = \{z \in \mathbb{C} \mid |z_0 - z| < r\} \subset U$ and

$$\forall z \in \Delta(z_0, r), f(z) = \sum a_n (z - z_0)^n.$$  

Remark 4.2.

- For $f : \mathbb{R} \to \mathbb{C}$ and $z_0 = x_0$, replace $U$ by an interval $I \ni x_0$ and $\Delta(z_0, r)$ by $\Delta(x_0, r) \cap \mathbb{R} = [r + x_0, x_0 + r] \ni I(x_0, r)$.

- Most results will be given relatively to $z_0 = 0$, but only for convenience. The generalization is just the consequence of $f$ RPS at $z_0 \Leftrightarrow f(z_0 + \bullet)$ is RPS at 0.

Definition 4.3. $f : U \subset \mathbb{C} \to \mathbb{C}$ is said to be analytic if $f$ is RPS at any point of $U$.

Proposition 4.4. Let $f$ be representable by $\sum a_n z^n$ at 0 on $\Delta(0, r)$. Then $f$ is analytic on $\Delta(0, r)$.

Proof. Let $z_0 \in \Delta(0, r)$ and $\rho = r - |z_0|$. For $z \in \Delta(z_0, \rho)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n ((z - z_0) + z_0)^n$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a_n z^{n-m} (z-z_0)^m$$

$$= \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} \binom{n}{m} a_n z^{n-m} \right) (z-z_0)^m$$
The last equality is a consequence of the Fubini theorem given in ch1 with \( a_{m,n} = a_n \binom{n}{m} z_0^{n-m}(z - z_0)^m \) (with the convention \( \binom{n}{m} = 0 \) if \( m > n \)). We just have for example to check that \( \sum_n \sum_m |a_{m,n}| \) is finite:

\[
\sum_n \sum_m |a_{m,n}| = \sum_{n=0}^{\infty} |a_n| (|z - z_0| + |z_0|)^n = \sum_{n=0}^{\infty} |a_n|r^n,
\]

with \( 0 \leq r' < \rho + |z_0| = r \) hence \( \sum |a_n|r^n \) converges and we have the result. \( \square \)

**Remark 4.5.** For \( z_0 \in \Delta(0, r) \), it’s important to notice that \( f \) is RPS at \( z_0 \) on the bigger disk centered at \( z_0 \) and contained in \( \Delta(0, r) \), which is \( \Delta(z_0, r - |z_0|) \).

From 3.10, we get a necessary condition for \( f \) to be RPS:

**Proposition 4.6.** If \( f : I \to \mathbb{C} \) is representable by \( \sum a_n x^n \) at 0, then \( \exists r > 0 \) such that \( I(0, r) \subset I \), with \( f \in C^\infty \) on \( I(0, r) \). Moreover we necessarily have \( a_n = f^{(n)}(0)/n! \).

**Example 4.7.** of a function which is not RPS:

\[
f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{if } x \leq 0 \\ \exp(-1/x^2) & \text{if } x > 0 \end{cases}
\]

By induction, one can prove that \( f \) is \( C^\infty \) on \( \mathbb{R} \) with all derivatives \( = 0 \) for all \( x \leq 0 \) and \( f^{(p)}(x) = P_p(1/x) \exp(1/x^2) \) for \( x > 0 \), \( P_p \) being a polynomial. Hence if \( f \)

representable by \( \sum a_n x^n \), \( a_n = f^{(n)}(0)/n! = 0 \Rightarrow f = 0 \) on \( I(0, r) \) for \( r > 0 \), which is false.

**Definition 4.8.** For \( f : I \subset \mathbb{R} \to \mathbb{C} \), the Taylor polynomial of \( f \) at \( a \),

\[
T_n(f, a, \bullet) : x \mapsto \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

the Taylor polynomial of \( f \) at \( a \),

\[
R_n(f, a, \bullet) : x \mapsto f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

the Taylor remainder of \( f \) at \( a \), and

\[
T(f, a, \bullet) : x \mapsto \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

the Taylor series of \( f \) at \( a \).

**Corollary 4.9.** A function \( f : I \to \mathbb{C} \) is RPS at 0 iff \( \exists r > 0 \) such that \( I(0, r) \subset I \) such that \( f \) is \( C^\infty \) on \( I(0, r) \) and

\[
\forall x \in I(0, r), \quad R_n(f, 0, x) \xrightarrow{n \to \infty} 0.
\]

In such a case, \( f \) is representable by its Taylor series at 0.

**Remark 4.10.**

• Of course we have the same result replacing 0 by \( a \) - just use \( f_a = f(\bullet + a) \).
• About the Taylor remainder: one can prove by induction, using integrations by parts, that we have, for $f \in C^{n+1}$:

\[
R_n(f, a, x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt
\]

This implies, for example, that

\[
|R_n(f, a, x)| \leq \int_a^x \frac{(x-t)^n}{n!} |f^{(n+1)}(t)|dt
\]

\[
\leq \max_{(a,x)} |f^{(n+1)}| \int_a^x \frac{(x-t)^n}{n!} dt
\]

\[
= \max_{(a,x)} |f^{(n+1)}| \left[ \frac{(x-a)^{n+1}}{(n+1)!} - \frac{(a-t)^{n+1}}{(n+1)!} \right]
\]

because the sign of $x-t$ is constant on $[(a, x)] (= [a, x]$ if $a \leq x$, $= [x, a]$ if not). Hence we have

\[
|R_n(f, a, x)| \leq \max_{(a,x)} |f^{(n+1)}| \frac{|a-t|^{n+1}}{(n+1)!}
\]

This gives a sufficient condition for $f \in C^{\infty}$ to be RPS at $a$:

\[
\exists r > 0, \exists M \geq 0, \forall x \in [a-r, a+r], \forall n, |f^{(n)}(t)| \leq M.
\]

\[
(|a-t|^{n+1}/(n+1)! \to 0 \text{ since } RCV(\sum z^n/n!) = +\infty).
\]

**Proposition 4.11.** Let $\sum a_n z^n$ a power series with $RCV= R > 0$, sum function $f$. We suppose $a_0 \neq 0$. Then $1/f$ is RPS at $0$.

**Proof.** We can suppose $a_0 = 1$ (consider $f \leftrightarrow f/a_0$). Let’s first prove

**Lemma 4.12.** $RCV(\sum u_n z^n) > 0 \iff \exists q > 0, |u_n| < q^n$.

**Proof.** For $\Rightarrow$, we note $r = RCV(\sum u_n z^n) > 0$. Fix $r' \in [0, r]$: we have $(u_n r^n)$ bounded by some constant $M \geq 1$, and we get $\forall n, |u_n| \leq M(1/r')^n \leq q^n$ with $q = M/r'$. For the other implication we have $u_n = O(q^n)$, hence $RCV(\sum u_n z^n) \geq RCV(\sum q^n z^n) = 1/q > 0$.

If $1/f$ is RPS $\sum b_n z^n$ on $\Delta(0, R') \subset \Delta(0, R)$, we get (cf. 2.11) on $\Delta(0, R')$

\[
(\sum a_n z^n) \ast (\sum b_n z^n) = (\sum a_n z^n)(\sum b_n z^n) = 1
\]

which implies (cf. 3.12)

\[
b_0 = 1 \text{ and } \forall n \geq 1, b_n = -a_1 b_{n-1} - \cdots - a_n b_0.
\]

Let $q > 0$ such that $|a_n| \leq q^n$ and let’s prove by induction that $|b_n| \leq q'^n$ with $q' = 2q$. This is true for $n = 0$ and if $|b_{n-1}| \leq q'^{n-1}$, we have

\[
|b_n| \leq \sum_{k=1}^n |a_k| |b_{n-k}| \leq \sum_{k=1}^n q'^k q'^{n-k} = \sum_{k=1}^n \frac{1}{2^k} q'^{n} \leq q'^n.
\]
Hence by the lemma we have $RCV(\sum b_n z^n) = R_0 > 0$ and the formula 1 proves that the sum function of $\sum b_n z^n$ is equal to $1/f$ on $\Delta(0, \min\{R, R_0\})$. □

Remark 4.13. About the composition of two RPS functions : Suppose $f(z) = \sum a_n z^n$ on $\Delta(0, R)$ and $g(z) = \sum b_n z^n$ on $\Delta(0, R')$ with $b_0 = 0 = g(0)$ : then $\exists \rho < R'$ such that $z \in \Delta(0, \rho) \Rightarrow g(z) \in \Delta(0, R)$ by continuity of $g$, and for $z \in \Delta(0, \rho)$, we have $f(g(z)) = \sum_n a_n g(z)^n$. But, by Cauchy product, $g^n$ is RPS on $\Delta(0, \rho)$, and we can note $g(z)^n = \sum_p b_{n,p} z^p$ for some complex numbers $b_{n,p}$.

Hence,

$$f(g(z)) = \sum_n \sum_p a_n b_{n,p} z^p = \sum_p (\sum_n a_n b_{n,p}) z^p$$

if we can apply the Fubini theorem to the double series $(a_n b_{n,p})$.

5 Classical examples

Definition 5.1. We note $\exp(z) = e^z$, $\cos z$ and $\sin z$ the sum functions of the following power series :

$$\sum \frac{z^n}{n!}, \sum \frac{(-1)^n}{(2n)!} z^{2n} \text{ and } \sum \frac{(-1)^n}{(2n + 1)!} z^{2n+1}.$$

Remark 5.2.
- The three power series have $RCV=\infty$ : we already know that for the first one. But if we note these series respectively $\sum a_n z^n, \sum b_n z^n$ and $\sum c_n z^n$ ($a_n = 1/n!$) we remark that $|b_n| \leq a_n$ and $|c_n| \leq a_n$.
- Following this definition, we clearly have, for $z \in \mathbb{C}$,

$$\cos(-z) = \cos z \text{ and } \sin(-z) = -\sin z.$$

Proposition 5.3. We have the following facts :

1. The derivative series of $\exp$, $\sin$ and $\cos$ are respectively $\exp$, $\cos$ and $-\sin$.
2. For all $z, z' \in \mathbb{C}$, $e^{z+z'} = e^z e^{z'}$.
3. For all $z \in \mathbb{C}$, $\cos z + i \sin z = e^{iz}$.
4. For all $z \in \mathbb{C}$, $e^z = \lim_{n \to \infty} (1 + \frac{z}{n})^n$.

Proof. The first point is a consequence of 3.8. For 2, we use the Cauchy product (and $RCV(\sum z^n/n!) = \infty$, so we have absolute convergence everywhere) to get

$$e^z e^{z'} = \sum_{n \geq 0} \left( \sum_{k=0}^n \frac{(-1)^k}{k!} \frac{(-1)^{n-k}}{(n-k)!} z^{n-k} \right) = \sum_{n \geq 0} \frac{(z+z')^n}{n!} = e^{z+z'}.$$

With the notations of 5.2, $b_n + ic_n = i^n a_n$ so we get 3. Let’s prove 4 : we note $E = \mathbb{N} \subset \mathbb{R}$ and for all $k \in \mathbb{N}$ (with the convention $\binom{n}{k} = 0$ if $k > n$),

$$\alpha_k \begin{cases}
E &\to \mathbb{C} \\
n &\mapsto \binom{n}{k} \frac{1}{n^k} z^k,
\end{cases}$$

so we get for all $n \in E$,

$$A(n) = \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{+\infty} \alpha_k(n).$$
Let’s try to apply the double-limit theorem for \( n \to \infty \): we first have for all \( k \geq 0 \)
\[
\forall n \geq k, \quad \alpha_k(n) = \frac{z^k}{k!} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{n} \right) \xrightarrow{n \to \infty} \frac{z^k}{k!}.
\]
But we also have
\[
\forall n \geq k, \quad |\alpha_k(n)| = \left| \frac{z^k}{k!} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{n} \right) \right| \leq \frac{|z|^k}{k!},
\]
and this inequality is also true for \( n < k \): we have the normal convergence (cf. \( \sum |z|^n/n! \) converges). The double-limit theorem gives the result.

\[ \square \]

**Remark 5.4.** As a consequence of 5.2 and 3 we have
\[
\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.
\]

**Lemma 5.5.** We note \( E = \{ x \in \mathbb{R}^+ \mid \cos(x) = 0 \} \). Then \( \exists x = \inf E \in \mathbb{R}_+^* \).

**Proof.** We just have to prove that \( E \neq \emptyset \). If not, \( \cos x > 0 \) for all \( x \geq 0 \) (cf. \( \cos 0 = 1 \) and \( \cos \) is continuous). This would imply the strict convexity of \(-\cos\) on \( \mathbb{R}_+ \), which cannot happen since for all \( x \in \mathbb{R}_+ \), \( -\cos x < 0 \) (the only negative convex functions on \( \mathbb{R}_+ \) are the constant functions).

\[ \square \]

**Definition 5.6.** The constant \( 2\alpha \) will be noted \( \pi \).

**Corollary 5.7.** We have the following facts:
1. For all \( x \in \mathbb{R} \), \( \cos^2 x + \sin^2 x = 1 \).
2. \( e^{i\pi/2} = i \), which implies \( \forall x \in \mathbb{R}, \cos(x + \frac{\pi}{2}) = -\sin x \) and \( \sin(x + \frac{\pi}{2}) = \cos x \).
3. \( e^{i\pi} = -1 \), which implies \( \forall x \in \mathbb{R}, \cos(\pi - x) = -\cos x \) and \( \sin(\pi - x) = \sin x \).
4. \( e^{i2\pi} = 1 \), which implies the \( 2\pi \)-periodicity of the functions of the real variable \( x \mapsto \sin x, \cos x \).

**Proof.** Using the continuity and the algebraic properties of \( \tau : z \mapsto \bar{z} \), we have for all \( z \in \mathbb{C} \),
\[
\exp z = \tau \left( \lim_{n \to \infty} \sum_{k=0}^{n} \frac{z^k}{k!} \right) = \lim_{n \to \infty} \left( \tau \left( \sum_{k=0}^{n} \frac{z^k}{k!} \right) \right) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} \frac{z^k}{k!} \right) = \exp \bar{z}.
\]
Hence for \( z = ix \in i\mathbb{R} \), by 5.3.2, we have \((e^{ix})^{-1} = e^{-ix} = e^{-i\pi} \), which gives \( e^{i\pi} = 1 \) and then 1. But \( \cos(\pi/2) = 0 \), so 5.3.3 implies \( e^{i\pi/2} = i \). Then \( e^{i\pi} = (e^{i\pi/2})^2 = 1 \) and \( e^{i2\pi} = (e^{i\pi/2})^4 = 1 \). Just take the real and imaginary parts of \( e^{ix}e^{i\lambda} = e^{(x+\lambda)i} \) for \( \lambda \in \{1/2, 1, 2\} \) to obtain the complementary assertions in 2, 3 and 4.

\[ \square \]

**Remark 5.8.**
- More generally for \( a, b \in \mathbb{R} \), the classical trigonometric formulas
  \[
  \begin{align*}
  \cos(a + b) &= \cos a \cos b - \sin a \sin b \\
  \sin(a + b) &= \cos a \sin b + \sin a \cos b
  \end{align*}
  \]
  are a consequence of \( e^{ia}e^{ib} = e^{i(a+b)} \).
• The hyperbolic sine and cosine are defined as follow for \( z \in \mathbb{C} \) :
\[
\begin{align*}
\sinh z &= -i \sin(iz) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{e^z - e^{-z}}{2}, \\
\cosh z &= \cos(iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \frac{e^z + e^{-z}}{2},
\end{align*}
\]
generalizing the definition known for \( x \in \mathbb{R} \).

Example 5.9. There’s a classical way to calculate the sum of power series of the form \( \sum P(n)z^n/n! \) for a given polynomial \( P \in \mathbb{C}[X] \). First the RCV is \(+\infty\) by De D’Alembert rule. Then the idea is to decompose \( P \) on the base \( \{1, X, X(X-1), \ldots, X(X-1)\ldots(X-d+1)\} \) if \( \deg P = d \). Practically, with \( \prod_{i=0}^{+\infty}(X-i) = 1 \),
\[
\begin{align*}
\deg P = d \quad \Rightarrow \quad \exists! (a_0, \ldots, a_d) \in \mathbb{C}^{d+1} \mid P &= \sum_{k=0}^{d} a_k \prod_{i=0}^{k-1}(X-i) \\
&= \sum_{n \geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n \geq 0} \frac{n \ldots (n-k+1) z^n}{n!} \\
&= \sum_{n \geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n \geq k} \frac{n \ldots (n-k+1) z^n}{n!} \\
&= \sum_{n \geq 0} \frac{P(n)}{n!} z^n = \sum_{k=0}^{d} a_k \sum_{n \geq k} \frac{z^n}{(n-k)!} = \sum_{k=0}^{d} a_k z^k e^z.
\end{align*}
\]

Theorem 5.10. The function \( x \in \mathbb{R} \mapsto -\ln(1-x) \) is representable by the power series \( \sum_{n \geq 1} x^n/n \) on \([-1, 1]\).

Proof. More precisely, we have : the primitive series of \( \sum z^n \) (which has RCV= 1) is \( \sum_{n \geq 1} z^n/n \). Hence we have the result since \( \ln \) is defined on \( \mathbb{R}^+ \) as the primitive \( F \) of \( x \mapsto 1/x \) such that \( F(1) = 0 \).

Definition 5.11. We define the complex logarithm as the sum of the power series \(-\sum_{n \geq 1} (1-z)^n/n \), defined on \( \Delta(1, 1) \), and we note it \( \ln z \).

Proposition 5.12. We have

• for all \( z \in \Delta(1, 1) \), \( \exp(\ln z) = z \);
• for all \( z \in \Delta(0, \ln 2) \), \( \ln(\exp z) = z \).

Proof. Following 4.13, we write, for \( z \in \Delta_1 \),
\[
\ln^n(1-z) = (-1)^n(\sum_{k \geq 1} z^k/k)^n = (-1)^n \sum_{k \geq 0} a_{k,n} z^k,
\]
and we set \( b_{k,n} = (-1)^n a_{k,n} z^k/n! \). We have \( |b_{k,n}| = a_{k,n} |z|^k/n! \) because \( a_{k,n} \geq 0 \) (cf. \( \alpha_n \geq 0, \beta_n \geq 0 \Rightarrow \sum_{k \geq 0} a_{k,n} \alpha_k \beta_{n-k} \geq 0 \)), hence the series \( \sum_{n \geq 0} b_{k,n} \) converges to \((-1)^n \ln^n(1-|z|)/n! \). Since the series \( \sum(-\ln(1-|z|))/n \) converges, we can apply the Fubini’s theorem, which gives (cf. 4.13) :
\[
\exp(\ln(1-z)) = \sum_{k \geq 0} \left( \sum_{n \geq 0} \frac{a_{k,n}}{n!} \right) z^k = \sum_{k \geq 0} c_k z^k.
\]
The point is that we know that this quantity is 1 - \( x \) if \( z = x \in [-1, 1] \). Thus, by 3.12, we have \( c_0 = 1, c_1 = -1 \) and \( c_k = 0 \) if \( k > 1 \). Finally we get the result
\( \exp(\ln(1-z)) = 1 - z. \)

For the other assumption, we first remark that the left member is well defined:

\[ z \in \Delta(0, \ln 2) \Rightarrow |e^z - 1| = |\sum_{n \geq 1} k^n/n!| \leq \sum_{n \geq 1} |z|^n/n! = e^{|z|} - 1 \in [0, 1]. \]

Then we write

\[ \ln(\exp z) = \ln(1 - (1 - e^z)) = \sum_{n \geq 1} \sum_{k \geq 0} b_{k,n}. \]

with this time \( b_{k,n} = (-1)^n a_{k,n} z^k/n, \) where

\[ (-1)^n \sum_{k \geq 0} a_{k,n} z^k = (1 - e^z)^n = (-1)^n (\sum_{p \geq 1} z^p/p!)^n. \]

Again, by induction (and using the definition of the coefficients of the Cauchy product), one can show that \( a_{k,n} \geq 0. \) This implies \( |b_{k,n}| = a_{k,n} |z|^k/n \) and thus

\[ \sum_{k \geq 0} |b_{k,n}| = \sum_{k \geq 0} a_{k,n} |z|^k/n = (-1)^n (1 - e^{|z|})^n/n = (e^{|z|} - 1)^n/n \]

with \( e^{|z|} - 1 \in [0, 1]. \) Hence \( \sum_{n \geq 1} (e^{|z|} - 1)^n/n \) converges and we can, here again, apply the Fubini’s theorem. The end of the proof is the same as in the first case, using the known results when \( z = x \in ]-\infty, \ln 2[. \)

\[ \square \]

**Proposition 5.13.** For all \( x \in ]-1, 1[, \)

1. \( \arctan(x) = \sum_{n \geq 0} (-1)^n/(2n + 1) x^{2n+1} \)
2. \( \arctanh(x) = \sum_{n \geq 0} x^{2n+1}/2n + 1 = \frac{1}{2} \ln \frac{1 + x}{1 - x} \)
3. \( \forall \alpha \in \mathbb{N}, (1 + x)^\alpha = \sum_{n \geq 0} (\alpha)_n x^n \) with \( (\alpha)_n = \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} \) and \( (\alpha)_0 = 1. \)
4. \( \frac{1}{\sqrt{1 - x^2}} = \sum_{n \geq 0} (2n)! \cdot 2^{2n} (n!)^2 x^{2n} \)
5. \( \arcsin(x) = \sum_{n \geq 0} (2n)!/2^{2n} (n!)^2 2n + 1 x^{2n+1} \)
6. \( \frac{1}{\sqrt{1 + x^2}} = \sum_{n \geq 0} (-1)^n (2n)!/2^{2n} (n!)^2 x^{2n} \)
7. \( \arcsinh(x) = \sum_{n \geq 0} (-1)^n (2n)!/2^{2n} (n!)^2 2n + 1 x^{2n+1} \)

**Proof.** 1 and 2 follow from 3.8; 5 and 7 follow from 3.8 and 4 and 6, which follow from 3. So, let’s prove 3 : the only power series which can represent \( x \mapsto (1 + x)^\alpha \) is the one given, which is the Taylor series of \( \phi. \) The power series \( \sum (\alpha)_n x^n \) has RCV= 1 by the ratio test and if we note \( S \) its sum function we have

\[ S'(x) = \sum_{n \geq 0} \left( \frac{\alpha}{n + 1} \right) (n + 1) x^n = \sum_{n \geq 0} (\alpha)_n (\alpha - n) x^n = \alpha S(x) - x S'(x). \]

Hence, since \( S(0) = 1, \) \( S(x) = (1 + x)^\alpha \) for all \( x \in ]-1, 1[. \)

\[ \square \]