# The Representation Theory of $\mathrm{GL}_d(\mathbb{Q}_p)$

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The study of representations of $p$-adic groups was born over fifty years ago when Friedrich Mautner published a paper [Mau] on spherical functions of $p$-adic PGL(2). The Bernstein decomposition expresses the category $\mathfrak{R}(G)$ of smooth admissible complex representations of a reductive $p$-adic group $G$ as the product of certain indecomposable full subcategories, often called the components of $\mathfrak{R}(G)$. For the sake of simplicity, we restrict our attention to $\mathrm{GL}_d(\mathbb{Q}_p)$ in this report. The primary reference is [Roc].

The adjoint functors *Restriction* and *Induction* play a central role in the theory of representations of finite groups. In the theory of representations of $p$-adic groups, the corresponding role is played by parabolic induction and parabolic restriction.

Let $G$ be a reductive $p$-adic group, $L$ a Levi subgroup of $G$. Let $\mathrm{ind}_L^G$ and $\mathrm{res}_L^G$ denote parabolic induction from $L$ to $G$ and parabolic restriction from $G$ to $L$ respectively. Consider the collection of all $(L, \delta)$, where $L$ is any Levi subgroup of $G$, and $\delta$ any irreducible representation of $L$. We define a partial order by $(L, \delta) \leq (L', \delta')$ if $L \subseteq L'$ and $[\mathrm{Hom}_{L'}(\delta', \mathrm{ind}_L^{L'} \delta) = \mathrm{Hom}_L(\mathrm{res}_L^{L'} \delta', \delta)] \neq 0$. The representations which are minimal for this order are called **cuspidal**. A representation is said to be cuspidal if and only if each of its irreducible subquotinets is cuspidal. Every non-cuspidal representation can be obtained as a subrepresentation of some cuspidal representation induced from some reductive subgroup.

The major result in this report is that any representation has a unique splitting as a direct sum of a cuspidal and a non-cuspidal subrepresentation. Methods for the study of the non-cuspidal case are not discussed in this report. The final result demonstrates a further decomposition of the cuspidal subcategory into indecomposable subcategories.

We begin with a preliminary section on $G$-representations, working in the general setting of locally profinite groups. In §2, we assume that our groups have a bi-invariant Haar measure. In §3, we further assume that all irreducible representations satisfy the Schur orthogonality relation (3.1), which along with the very important Proposition 4.2.2, allows us to prove that the cuspidal component of a representation can be split as a direct summand. Proposition 4.2.2 itself is an easy implication of the Cartan decomposition, which can be found, for example, in [Rum].

# 1 Preliminaries

Fix a prime number $p$, and an integer $d \geq 1$. We wish to study representations of the general linear group of degree $d$ over the field of $p$-adic numbers, $\mathrm{GL}_d(\mathbb{Q}_p)$. We use, without proof, various properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Proofs can be found, for example in Chapter 1 of [Kob].

## 1.1 The topology

In this section we describe a fundamental property (locally profiniteness) of the topology of $\mathrm{GL}_d(\mathbb{Q}_p)$ and its subgroups. Using this property alone, we shall in later sections identify certain important characteristics of the kind of objects we wish to study.

**Lemma 1.1.1.** *An open subgroup of a topolological group is also closed.*

*Proof.* Let $H$ be an open subgroup of $G$. The complement of $H$, being the union of all $H$-cosets not containing identity, is open. Therefore $H$ is closed. □

**Definition 1.1.2.** A group is called **locally profinite** if it has a fundamental system neighbourhoods around the identity comprising compact-open subgroups.

*Remark* 1.1.3. A profinite group is one which is both locally profinite and compact. If $G$ is profinite, $K$ a compact normal subgroup in $G$, then $G/K$ is finite. Indeed, profinite groups occur as the inverse limits of discrete finite groups.

We will prove over the next two lemmas that $\mathrm{GL}_d(\mathbb{Q}_p)$ is locally profinite.

**Lemma 1.1.4.** *The sets* $\mathrm{M}_d(\mathbb{Z}_p)$ *and* $\mathrm{GL}_d(\mathbb{Z}_p)$ *are compact-open in* $\mathrm{M}_d(\mathbb{Q}_p)$.

*Proof.* Let $\mathbb{Z}_p^*$ be the set of units of $\mathbb{Z}_p$. We know that $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ are compact-open in $\mathbb{Q}_p$ (a proof can be found in Chapter 1 of [Kob]).
As a topological space, $\mathrm{M}_d(\mathbb{Z}_p)$ is $\mathbb{Z}_p^{d^2}$ sitting inside $\mathrm{M}_d(\mathbb{Q}_p) = \mathbb{Q}_p^{d^2}$, hence is compact-open. Let det denote the determinant map on $\mathrm{GL}_d(\mathbb{Q}_p)$. Since det is continuous, $\det^{-1}\{\mathbb{Z}_p^*\}$ is both closed and open. The set

$$\mathrm{GL}_d(\mathbb{Z}_p) = \det^{-1}\{\mathbb{Z}_p^*\} \cap \mathrm{M}_d(\mathbb{Z}_p)$$

is the intersection of a closed-open set and a compact-open set. Hence it is compact-open. □

**Lemma 1.1.5.** *Let $G$ be any subgroup of* $\mathrm{GL}_d(\mathbb{Q}_p)$. *The topological group $G$ is locally profinite.*

*Proof.* The family of subgroups given by

$$K_0 = \mathrm{GL}_d(\mathbb{Z}_p) \tag{1.1}$$
$$K_i = 1 + p^i \mathrm{M}_d(\mathbb{Z}_p) \qquad i \geq 1 \tag{1.2}$$

form a fundamental system of neighbourhoods around the identity for $\mathrm{GL}_d(\mathbb{Q}_p)$. Hence $\{\mathrm{GL}_d(\mathbb{Q}_p)\}_{i\geq 0}$ is locally profinite. For a subgroup $G$, the $K_i \bigcap G$ form a fundamental system of neighbourhoods around the identity of compact-open subgroups of $G$. □

## 1.2 Representations

In this section, we list out some constructions which can be carried out for any group in general.

**Definition 1.2.1.** A **representation** of $G$ (or a $G$-representation) is a pair $(V, \pi)$ where $V$ is a complex vector space and $\pi$ a group action of $G$ on $V$, such that for each $g \in G$, the action $\pi(g) : V \to V$ is a $\mathbb{C}$-linear.

We will often denote the representation by just $V$ or $\pi$, and $\pi(g)(v)$ by $gv$.

**Example 1.2.2.** The 1-dimensional vector space $\mathbb{C}$ with each $g \in G$ acting as identity is called the **trivial** representation.

Given a representation, there are various canonical representations naturally associated to it. We list out a few of the more important ones.

**Definition 1.2.3.** For $H$ a subgroup of $G$, and $(V, \pi)$ a $G$-representation, we define $(V|_H, \pi|H)$ to be the $H$-representation obtained from $V$ by restricting $\pi$ to $H$.

**Definition 1.2.4.** A **subrepresentation** $V_1$ of $(V, \pi)$ is a $G$-stable vector subspace $V_1 \subseteq V$, that is, a subspace $V_1$ such that one has $\pi(g)V_1 \subseteq V_1$ for all $g \in G$.

**Example 1.2.5.** Let $V$ be a $G$-representation. The elements of the set $V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$ are called the fixed points of $V$. The subspace $V^G$ is a subrepresentation of $V$.

Let $(V, \pi)$ and $(W, \tau)$ be two $G$-representations. The vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is given the structure of a $(G \times G)$-representation under the action given by:

$$((g_1, g_2), f) \mapsto \tau(g_2) \circ f \circ \pi(g_1^{-1}). \tag{1.3}$$

Similarly, the vector space $V \otimes W$ becomes a $(G \times G)$-representation under the unique $\mathbb{C}$-linear action satisfying:

$$((g_1, g_2), v \otimes w) \mapsto \tau(g_1)(v) \otimes \pi(g_1)(w). \tag{1.4}$$

Embedding $G$ diagonally into $G \times G$, we view $\mathrm{Hom}_{\mathbb{C}}(V, W)$ and $V \otimes W$ as $G$-representations. Given $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ we denote by ${}^g f$ the action of $g$ on $f$, to avoid confusion with $\tau(g) \circ f$. In particular, taking $W$ to be the trivial representation endows $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with the structure of a $G$-representation.

**Definition 1.2.6.** The fixed points of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ under the action of $G$ are called **intertwiners**, or $G$**-maps**.

A function $f : V \to W$ is a $G$-map if and only if $f(gv) = g(f(v))$ for all $g \in G$. For example, the canonical linear map $V \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \to \mathrm{End}_{\mathbb{C}} V$ is a $G$-map. In the theory of group representations, $G$-maps play a role similar to the role played by linear transformations in linear algebra.

**Definition 1.2.7.** A $G$-map is said to be an **isomorphism** if it is bijective. Two $G$-representation $V, W$ are said to be isomorphic if there exists an isomorphism from $V$ to $W$.

**Definition 1.2.8.** Given non-zero $G$-representations $V$ and $W$, we say that $W$ is a **subquotient** of $V$ if there exist submodules $V_2 \subsetneq V_1 \subseteq V$ such that $W \cong V_1/V_2$.

*Remark* 1.2.9. A subrepresentation of the image (under a $G$-map) of a representation is a subquotient of the representation.

*Remark* 1.2.10. The relation $< W$ is a subquotient of $V >$ is a partial order on the set of isomorphism classes of $G$-representations.

## 1.3 Smoothness

From now on, $G$ will always be locally profinite. Throughout this report, we are interested in smooth representations. Smoothness corresponds to continuity, if the codomain is given the discrete topology. Since the spaces we are concerned with already have topologies, we do not stress this point of view. Instead, we formally define smooth functions and smooth representations. This section ends with a definition of the category of smooth representations of $G$.

**Definition 1.3.1.** A function $f : G \to \mathbb{C}$ is called **smooth** if there exists a compact-open subgroup $K$ of $G$, such that $f(kg) = f(g)$ for all $k \in K$, $g \in G$.

**Definition 1.3.2.** A representation $(V, \pi)$ is called **smooth** if the stabilizer of every point is open.

*Remark* 1.3.3. From Lemma 1.1.1, it follows that a representation $(V, \pi)$ is smooth if and only if the stablizer of every point is both open and closed.

*Remark* 1.3.4. Suppose $V$ is given the discrete topology. Then $(V, \pi)$ is smooth if and only if the map $\mathrm{ev}_v : G \to V$ which takes $g$ to $gv$ is continuous for each $v \in V$.

**Definition 1.3.5.** For a compact-open subgroup $K$, we define the vector subspace of $K$-fixed points:

$$V^K := \{v \in V \mid kv = v \text{ for all } k \in K\}.$$

**Definition 1.3.6.** The **smooth component** $V^\infty$ of $V$ is defined as

$$V^\infty := \bigcup_{K <_{c.o} G} V^K.$$

If $K$ is compact-open in $G$, so is $gKg^{-1}$. Also, we have $g(V^K) = V^{gKg^{-1}}$. Hence the vector subspace $V^\infty$ is a subrepresentation of $G$. Given any $G$-representation, we will focus only on its smooth component. Let us now consider smooth $G$-maps between smooth $G$-representations.

**Definition 1.3.7.** Let $V$ and $W$ be smooth $G$-representations. A $G$-map $f : V \to W$ is said to be **smooth** if there exists a compact-open subgroup $K$ such that $f(v) = f(kv) = kf(v)$ for all $k \in K, v \in V$.

If we view $\mathrm{Hom}_{\mathbb{C}}(V, W)$ as a representation of $G \times G$ using the action (1.3), then a $G$-map $f$ is smooth if and only if $f \in \mathrm{Hom}_{\mathbb{C}}(V, W)^\infty$.

*Remark* 1.3.8. Not every element of $\mathrm{Hom}_{\mathbb{C}}(V, W)^\infty$ is a $G$-map.

**Definition 1.3.9.** The **contragredient** of $V$ is defined as $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})^\infty$.

**Lemma 1.3.10.** *For $G$-representations $V_1, V_2$ the contragredient $(V_1 \oplus V_2)^* = V_1^* \oplus V_2^*$.*

*Proof.* From the universal property for direct sums of vector spaces, it follows that $\mathrm{Hom}_{\mathbb{C}}(V_1 \oplus V_2, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_1, \mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_2, \mathbb{C})$. Consider some linear functional $\alpha = \alpha_1 \oplus \alpha_2$. If $K$ fixes $\alpha$, it fixes $\alpha_1 = \alpha|V_1$ and $\alpha_2 = \alpha|V_2$. Conversely, if $K^1, K^2$ fix $\alpha_1, \alpha_2$ respectively, then $K^1 \cap K^2$ fixes $\alpha$. Hence $\mathrm{Hom}_{\mathbb{C}}(V_1 \oplus V_2, \mathbb{C})^\infty = \mathrm{Hom}_{\mathbb{C}}(V_1, \mathbb{C})^\infty \oplus \mathrm{Hom}_{\mathbb{C}}(V_2, \mathbb{C})^\infty$. $\square$

**Lemma 1.3.11.** *The composition of smooth $G$-maps is smooth.*

*Proof.* Consider smooth $G$-maps $f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$, with $f_2$ fixed by some compact-open subgroup $K$. Then $f_1 \circ f_2$ is also fixed by $K$. $\square$

This lemma allows the following definition.

**Definition 1.3.12.** Define $\mathfrak{R}(G)$ (the **category of smooth representations** of $G$) to be the category whose objects are the smooth representations of $G$ and morphisms are the smooth $G$-maps.

We will always work in the category $\mathfrak{R}(G)$. All the considered representations will be smooth, and the same is true for $G$-maps.

## 1.4 The Hecke algebra

From now on, the (locally profinite) group $G$ has a Haar measure $\mu$ which is both left and right invariant. In section §4, we will specialise $G$ to various subgroups of $\mathrm{GL}_d(\mathbb{Q}_p)$, and prove the existence of the required Haar measure for each of those groups. We remind the reader that the collection of all compact-open subgroups forms a fundamental system of neighbourhoods around identity.

When $G$ is finite, the group ring $\mathbb{C}[G]$ plays a very important role in the study of $G$-representations. The corresponding construction used in the study of smooth representations over a locally profinite group is the Hecke algebra of $G$.

For a discrete group $G$, the group ring $\mathbb{C}[G]$ can be viewed as the set of $\mathbb{C}$-valued functions on $G$ with compact support. This suggests the possibility of considering smooth functions with compact support for constructing the Hecke algebra. This section and the next is devoted to demonstrating that this is indeed the 'correct' construction. For now, let $\mathcal{H}(G)$ denote the **space of locally constant, compactly supported $\mathbb{C}$-valued functions on** $G$. We begin by showing that nothing is lost in considering locally constant functions instead of the smooth functions suggested above.

**Lemma 1.4.1.** *A locally constant, compactly supported $\{0,1\}$-valued function from $G$ to $\mathbb{C}$ is smooth.*

*Proof.* Let $f$ be a locally constant, compactly supported $\{0,1\}$-valued function. Let $\Gamma = f^{-1}(\{1\})$ a compact-open subset (careful: $\Gamma$ need not be a subgroup). For a subgroup $K$ to fix $f$, it is sufficient that $K\Gamma = \Gamma$, that is for all $\gamma \in \Gamma$ and $k \in K$, we have $k\gamma \in \Gamma$. Since $\Gamma$ is open, we can write $\Gamma$ as a union of basic open sets, i.e $\Gamma = \bigcup O_i$. We have each $O_i = K^i g_i$, for some compact-open subgroup $K^i$, and some $g_i \in G$. Because $\Gamma$ is compact, the open cover $\{O_i\}$ has a finite subcover. Let $O_1, \dots O_n$ cover $\Gamma$. The compact-open subgroup $K = \bigcap_{i \leq n} K^i$ then fixes $\Gamma$. It follows that $f$ is smooth. $\square$

**Lemma 1.4.2.** *A compactly supported $\mathbb{C}$-valued function on $G$ is locally constant if and only if it is smooth.*

*Proof.* Let $f$ be smooth. There exists a compact-open subgroup $K$ fixing $f$. Then for any $g \in G$, $f$ is constant over $Kg$.
Conversely, suppose $f$ is locally constant. The pre-image under $f$ of a point is both open and closed. Let $X$ be the set of non-zero values taken by $f$. Since $f$ has compact support, $f^{-1}(X)$ is compact. Therefore $f^{-1}(\{x\})$ is compact (hence compact-open) for all $x \in X$. Since $\{f^{-1}(\{x\})\}_{x \in X}$ is a disjoint cover for the compact set $f^{-1}(X)$, we infer that the indexing set of the cover, that is $X$, is finite.
Denote by $\chi_x$ the function which is 1 on $f^{-1}(\{x\})$, and 0 elsewhere. We can now write $f$ as a finite sum, $f = \sum_{x \in X} x\chi_x$. Note that the $\chi_x$ are locally constant, compactly supported $\{0, 1\}$-valued functions. By the previous lemma, it follows that $f$ is the sum of smooth functions. Hence $f$ is smooth. $\square$

**Corollary 1.4.3.** *Let $f$ be a compactly supported smooth $\mathbb{C}$-valued function on $G$. There exists a compact-open subgroup $K$ such that $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k_1, k_2 \in K$.*

*Proof.* Consider the function $f'$ defined by $f'(g) := f(g^{-1})$. Since $f$ is locally constant and compactly supported, so is $f'$. By the previous lemma, $f'$ is smooth. Hence there exists a compact-open subgroup $K^1$ such that $f'(kg) = f'(g)$ for all $k \in K^1$ and $g \in G$. This is equivalent to $f(gk) = f(g)$ for all $k \in K^1$ and $g \in G$. Since $f$ is smooth, there exists $K^2$ such that $f(kg) = (g)$ for all $k \in K^2$ and $g \in G$. Then $K = K^1 \cap K^2$ satisfies $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k_1, k_2 \in K$. $\square$

**Definition 1.4.4.** Given $f_1, f_2 \in \mathcal{H}(G)$, we define the **convolution** $*$ with respect to $\mu$

$$(f_1 * f_2)(g) := \int_{g_1} f_1(g_1) f_2(g_1^{-1} g)$$

**Proposition 1.4.5.** *The pair $(\mathcal{H}(G), *)$ is an associative algebra. It is called the* **Hecke algebra** *of $G$. (We omit the $*$ when no confusion can arise.)*

*Proof.* It is clear that $*$ distributes over addition. Now, let $f = f_1 * f_2$.
Let $m : G \times G \to G$ be the continuous map $(g_1, g_2) \mapsto g_1 g_2$. For $f(g)$ to be non zero, it is necessary that $g$ belong to the image of $m$ restriced to $\text{supp}(f_1) \times \text{supp}(f_2)$. Since $\text{supp}(f_1) \times \text{supp}(f_2)$ is compact, so is $m(\text{supp}(f_1) \times \text{supp}(f_2))$. Hence $f$ has compact support. Also, if $f_1$ is fixed by $K$, so is $f$. Hence $f_1 * f_2$ belongs to $\mathcal{H}(G)$. The associativity of $*$ follows from the identity

$$((f_1 * f_2) * f_3)(g) = \int_{g_1 \in G} \int_{g_2 \in G} f_1(g_1) f_2(g_2) f_3(g_2^{-1} g_1^{-1} g) = (f_1 * (f_2 * f_3))(g)$$

$\square$

*Remark* 1.4.6. The algebra $\mathcal{H}(G)$ need not always have a multiplicative identity. However, it does satisfy the weaker property described in Lemma 1.4.9. Such algebras are said to be **idempotented**.

**Definition 1.4.7.** An element $e \neq 0$ in a ring $A$ is called idempotent if it satisfies $e^2 = e$.

**Definition 1.4.8.** For any compact <u>subset</u> $\Gamma$, we define $\overline{\chi}_\Gamma \in \mathcal{H}(G)$ by

$$\overline{\chi}_\Gamma(g) = \begin{cases} \frac{1}{\mu(\Gamma)} & \text{if } g \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\overline{\chi}_\Gamma$ has integral 1 with respect to $\mu$. If $\Gamma$ is a subgroup, we denote $\overline{\chi}_\Gamma$ by $e_\Gamma$.

For $f \in \mathcal{H}(G)$, we compute

$$(e_K * f)(g) = \int\limits_{k\in K} f(k^{-1}g) = \int\limits_{Kg} f.$$

Substituting $f = e_K$, we see that $e_K$ is idempotent. Further observe that $e_K f = f$ if and only if $f(kg) = f(g)$ for all $g \in G, k \in K$, and similarly $f e_K = f$ if and only if $f(gk) = f(g)$ for all $g \in G, k \in K$.

**Lemma 1.4.9.** *Given $f_1, \cdots f_n \in \mathcal{H}(G)$, there exists an idempotent $e$ such that $ef_ie = f_i$ for all $i$.*

*Proof.* From the corollary 1.4.3, we deduce the existence of a compact-open subgroup $K^i$ such that $e_{K^i} f_i e_{K^i} = f_i$. Taking $K = \bigcap\limits_i K^i$, we have $e_K f_i e_K = f_i$ for all $i$. $\square$

## 1.5 Modules

In analogy with finite groups, we wish to show that every $G$-representation can be viewed as an $\mathcal{H}(G)$-module. However, not all $\mathcal{H}(G)$-modules correspond to representations (recall that we only consider smooth representations). The important property characterising modules which correspond to representations is non-degeneracy.

**Definition 1.5.1.** A module $M$ over $\mathcal{H}(G)$ is called **non-degenerate** if for each $m \in M$, there exists some compact-open subgroup $K$, such that $e_K m = m$.

**Lemma 1.5.2.** *Let $M$ be a non-degenerate module over $\mathcal{H}(G)$. Let $K^1, K^2$ be compact-open subgroups in $G$. For $g \in G$, the equality $e_{K^1} m = e_{K^2} m$ implies*

$$\overline{\chi}_{gK^1} m = \overline{\chi}_{gK^2} m$$

*Proof.* Let $K = K^1 \cap K^2$. We write

$$\begin{aligned} K^1 &= Kt_1 \sqcup \cdots \sqcup Kt_n, \\ gK^1 &= gKt_1 \sqcup \cdots \sqcup gKt_n. \end{aligned}$$

The union is over a finite family as $K^1$ is compact and the $Kt_i$ are open. We have then

$$\begin{aligned} e_{K^1} &= \frac{\overline{\chi}_{Kt_1} + \cdots + \overline{\chi}_{Kt_n}}{n}, \\ \overline{\chi}_{gK^1} &= \frac{\overline{\chi}_{gKt_1} + \cdots + \overline{\chi}_{gKt_n}}{n}, \end{aligned}$$

with obvious notation. Now observe

$$\begin{aligned} (\overline{\chi}_{gK} * \overline{\chi}_{Kt_i})(h) &= \int\limits_{g_1 \in G} \overline{\chi}_{gK}(g_1)\overline{\chi}_{Kt_i}(g_1^{-1}h) \\ &= \int\limits_{g_2 \in Kg^{-1}h} \frac{1}{\mu(gK)}\overline{\chi}_{Kt_i}(g_2). \end{aligned}$$

For $g_2 \in Kg^{-1}h$, $\overline{\chi}_{Kt_i}(g_2)$ is non zero if and only if $Kg^{-1}h = Kt_i$, or equivalently $h \in gKt_i$. We see

$$\begin{aligned} (\overline{\chi}_{gK} * \overline{\chi}_{Kt_i})(h) &= \frac{1}{\mu(gK)} \int\limits_{g_2 \in Kt_i} \frac{1}{\mu(Kt_i)} \\ &= \frac{1}{\mu(K)} \qquad \forall h \in gKt_i, \\ (\overline{\chi}_{gK} * \overline{\chi}_{Kt_i})(h) &= 0 \qquad\qquad \forall h \notin gKt_i. \end{aligned}$$

We get the concise formula

$$\overline{\chi}_{gK}\overline{\chi}_{Kt_i} = \overline{\chi}_{gKt_i},$$

which we use to get

$$\begin{aligned} \overline{\chi}_{gK^1} &= \sum_i \overline{\chi}_{gKt_i} \\ &= \sum_i \overline{\chi}_{gK}\overline{\chi}_{Kt_i} \\ &= \overline{\chi}_{gK}\left(\frac{\sum_i \overline{\chi}_{Kt_i}}{n}\right) \\ \overline{\chi}_{gK^1} &= \overline{\chi}_{gK}e_{K^1}. \end{aligned}$$

Similarly, we have

$$\overline{\chi}_{gK^2} = \overline{\chi}_{gK}e_{K^2}.$$

Finally, we have

$$\begin{aligned} e_{K^1}m &= e_{K^2}m \\ \overline{\chi}_{gK}e_{K^1}m &= \overline{\chi}_{gK}e_{K^2}m \\ \overline{\chi}_{gK^1}m &= \overline{\chi}_{gK^2}m \end{aligned}$$

□

From this point on, we will only consider non-degenerate modules (unless stated otherwise) and simply call them modules. We can now view any $\mathcal{H}(G)$-module $M$ as a $G$-representation, via the action:

$$\lambda gm := (\lambda\overline{\chi}_{gK})m \qquad\qquad \forall\ \lambda \in \mathbb{C},\ g \in G$$

where $K$ is some compact-open subgroup such that $e_K m = m$. Lemma 1.5.2 tells us that the action is well-defined.
Conversely, given a $G$-representation $V$, we can give $V$ the structure of an $\mathcal{H}(G)$-module via:

$$fv := \int\limits_{g\in G} f(g)gv \qquad\qquad \forall\ f \in \mathcal{H}(G), v \in V.$$

**Lemma 1.5.3.** *Let $V$ be a $G$-representation. An element $v \in V$ is fixed by a compact-open subgroup $K$if and only if $e_K v = v$, that is $V^K = e_K V$.*

*Proof.* If some compact-open subgroup $K$ fixes $v$, we have

$$e_K v = \int\limits_{k\in K} \frac{1}{\mu(K)} kv = \int\limits_{k\in K} \frac{1}{\mu(K)} v = v$$

Conversely, if $e_K v = v$, then $kv = \overline{\chi}_{kK} v = e_K v = v$. ☐

We see that $V^K$ is the image of $V$ under an idempotent endomorphism. Hence, we can write $V^K$ as both a subspace and a quotient of $V$. The kernel of $e_K : V \to V$ is a $K$-stable complement to $V^K$. We define

$$V_K := \ker e_K = \{v \in V \mid\ e_K v = 0\} = \{e_K v - v | v \in V\}.$$

This provides us the decomposition

$$V = V^K \oplus V_K. \tag{1.5}$$

Finally, we will describe the module homomorphisms corresponding to $G$-maps. We first identify the $\mathcal{H}(G)$-action on $\mathrm{Hom}_{\mathbb{C}}(M, N)$ corresponding to the $G$-action on $\mathrm{Hom}_{\mathbb{C}}(M, N)$.
Define $\tilde{}: \mathcal{H}(G) \to \mathcal{H}(G)$ by $\tilde{f}(g) = f(g^{-1})$. The $\mathcal{H}(G)$-action on $\mathrm{Hom}_{\mathbb{C}}(M, N)$ is then given by

$$({}^f T)(m) := f(T(\tilde{f}m)),$$

and the module homomorphisms corresponding to smooth maps are exactly those maps which are fixed by some $e_K$. Note the equality $\tilde{e_k} = e_K$.

## 1.6 Irreducibility and Semi-simplicity

In this section, we first identify the 'easiest' kind of representations, and study their direct sums. Such representations are called semi-simple. We then list down here some basic characterisations of semi-simplicity.

**Definition 1.6.1.** A representation $V$ is said to be **irreducible** if it is simple as an $\mathcal{H}(G)$-module, that is

$$(\, 0 \subsetneq V_1 \subseteq V, \quad gV_1 \subseteq V_1 \quad \forall g \in G\,) \quad \Rightarrow V_1 = V$$

*Remark* 1.6.2. The minimal representations mentioned in Remark 1.2.10 are just the irreducible representations.

**Lemma 1.6.3.** *Given $G$-representations $V_1$ and $V_2$, and a non-zero map $f : V_1 \to V_2$, there exists a $G$-representation $W$ which is a subquotient of both $V_1$ and $V_2$.*

*Proof.* The non-zero representation $f(V_1)$ is a quotient of $V_1$ and a subrepresentation of $V_2$. Hence $f(V_1)$ is a subquotient for both $V_1$ and $V_2$. □

**Definition 1.6.4.** A representation $V$ is called **semi-simple** if it can be written as a direct sum of irreducible representations.

**Lemma 1.6.5.** *Let $V$ be a $G$-representation. The following are equivalent:*
(**i**) *$V$ is semi-simple.*
(**ii**) *$V$ is the sum of some irreducible subrepresentations.*
(**iii**) *Given $V_1$ a subrepresentation of $V$, there exists a subrepresentation $V_2$ such that $V = V_1 \oplus V_2$.*

*Proof.* Refer Chapter XVII §2 of [Lang]. □

**Example 1.6.6.** Let $V$ be the 2-dimensional complex vector space spanned by $\{e_1, e_2\}$. Consider the subgroup $B := \{T \in \mathrm{GL}(V) \mid e_1 \text{ is an eigenvector for } T\}$. The $B$-representation $V$ is not semi-simple, since the subrepresentation $\mathbb{C}e_1$ has no $B$-stable complement.

**Lemma 1.6.7.** *Every subquotient of a semi-simple representation is semi-simple.*

*Proof.* Refer Chapter XVII §2.2 of [Lang]. □

**Proposition 1.6.8.** *Suppose $G$ is compact. Then every $G$-representation is semi-simple.*

*Proof.* Recall $V = \bigcup\limits_{K <_{c.o} G} V^K$. Let $K'$ be some compact-open subgroup. Consider the open disjoint cover $\{K'k\}_{k \in K/K'}$ of $K$. Since $K$ is compact, the cover must be finite. Therefore, the index $(K : K')$ is finite. Now $\tilde{K} = \bigcap\limits_{k \in K/K'} kK'k^{-1}$ is a normal subgroup fixing $v$. We deduce that the $K$-subspace spanned by $v$ is a representation of the finite group $K/\tilde{K}$, hence semi-simple Chapter XVIII, §1.2 of [Lang]. Since this is true for every $K'$, it follows from Lemma 1.6.5 that $V$ is semi-simple. □

# 2 Basic properties of representations

## 2.1 Finite representations

**Definition 2.1.1.** We say that a $G$-representation $V$ is **finite** if for each $v \in V$, $\alpha \in V^*$, the function $g \mapsto \alpha(gv)$ is compactly supported.

Finite representations are closely related to the cuspidal representations mentioned at the start of this report. The next two lemmas say that finiteness is a well behaved property.

**Lemma 2.1.2.** *The direct sum $\oplus V_i$ of finitely many $G$-representations $V_1, \cdots, V_n$ is finite if and only if each $V_i$ is finite.*

*Proof.* Let $\alpha = \oplus \alpha_i$, or equivalently $\alpha(g(\oplus v_i)) = \sum_i \alpha_i(gv_i)$ [Corollary 1.3.10]. The map $g \mapsto \alpha(gv)$ is continuous (being the composition $\alpha \circ \mathrm{ev}_v$). Similarly, the maps $\alpha_i(gv_i)$ are continuous. Hence $\mathrm{supp}(g \mapsto \alpha(gv))$ and $\mathrm{supp}(g \mapsto \alpha(gv_i))$ are closed sets.
We have $\mathrm{supp}(g \mapsto \alpha(g(\oplus v_i))) = \bigcup_i \mathrm{supp}(g \mapsto \alpha_i(gv_i))$, which is compact if $\mathrm{supp}(g \mapsto \alpha_i(gv_i))$ is compact for each $i$. Conversely, if $\mathrm{supp}(g \mapsto \alpha(gv))$ is compact, so is $\mathrm{supp}(g \mapsto \alpha_i(gv_i)) = \mathrm{supp}(g \mapsto \alpha(gv)) \bigcap \mathrm{supp}(g \mapsto \alpha_i(gv_i))$, being the intersection of a compact set and a closed set. □

*Remark* 2.1.3. A stronger version of the previous lemma is presented in Corollary 3.1.14.

**Lemma 2.1.4.** *A subquotient of a finite representation is finite.*

*Proof.* First note that a subrepresentation of a finite representation is finite. Indeed, if $W$ is a subrepresentation of $V$, then $\alpha \in W^*$ can always be extended to some $\beta \in V^*$ by defining it to be zero on some vector space complement of $W$. The map $g \mapsto \beta(gv)$ has compact support for each $v \in V$, in particular for each $v \in W$. Hence $W$ is finite.
It is now enough to prove that quotients of finite representations are finite. Let $f : V \twoheadrightarrow V/W$ be the canonical projection. Any linear functional $\alpha$ on $V/W$ can be lifted to a linear functional $\beta = \alpha \circ f$ on $V$. Since $V$ is finite, any map $g \mapsto \beta(gv)$ has compact support. We have the equality $\beta(gv) = \alpha(f(gv)) = \alpha(gf(v))$. It follows from the surjectivity of $f$ that the map $g \mapsto \alpha(g\tilde{v})$ has compact support for each $\tilde{v} \in V/W$. □

## 2.2 Admissibility

**Definition 2.2.1.** We say that a $G$-representation $V$ is **admissible** if for each compact-open subgroup $K$, the subspace $V^K = e_K V$ is finite dimensional.

If a $G$-representation $V$ is admissible, the identity $V = \cup V^K$ allows us to view $V$ as a direct limit of the finite-dimensional representations $V^K$. This often allows us to generalise from statements about finite representations. This technique is illustrated in the proof of Proposition 2.2.3.
The following theorem further justifies the study of admissibilty.

**Theorem 2.2.2.** *A finite irreducible representation is admissible.*

*Proof.* Refer §2.41 of [BZ]. □

**Proposition 2.2.3.** *Let $V$ be an admissible representation. Restricting the canonical $(G \times G)$-map $V \otimes \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \to \mathrm{End}_{\mathbb{C}} V$ to $V \otimes V^*$ induces an isomorphism onto $(\mathrm{End}_{\mathbb{C}} V)^\infty$.*

*Proof.* Let $\theta \in \mathrm{End}_{\mathbb{C}}(V)^{K\times K}$. We have $k_1(\theta(k_2 v)) = \theta(v)$ for all $k_1, k_2 \in K$. Fixing $k_2$ to be the identity, we see that the image of $\theta$ lies in $V^K$. Substituting $k_2 = k_1^{-1}$, we see that $\theta$ is a $K$-map, hence takes $V_K$ to $V_K$. We see that $\theta(V_K) \subseteq V^K \bigcap V_K = 0$. Every element of $\mathrm{End}_{\mathbb{C}}(V)^{K\times K}$ is trivial on $V_K$. Also, $\theta$ being a $K$-map, sends $V^K$ to $V^K$. The map

$$\theta \mapsto \theta|V^K : \mathrm{End}_{\mathbb{C}}(V)^{K\times K} \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(V^K)$$

is an isomorphism.
Since every element of $\mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is fixed by $K$, we have $(V^K)^* = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$. Now suppose $\alpha \in (V^*)^K$. We have $\alpha(\tilde{e_K} v) = \alpha(v)$. We see that $\alpha$ is zero on $V_K$, and hence we can identify $(V^*)^K$ with $\mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})^K = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$.
It follows that $(V^*)^K$ and $(V^K)^*$ are isomorphic as $G$ representations. We can now draw a commutative diagram that defines the bottom arrow as follows

$$\begin{array}{ccc} (V \otimes V^*)^{K\times K} & \xrightarrow{r} & \mathrm{End}_{\mathbb{C}}(V)^{K\times K} \\ {\scriptstyle id}\big\downarrow & & \simeq\big\downarrow \\ V^K \otimes (V^K)^* & \longrightarrow & \mathrm{End}_{\mathbb{C}}(V^K) \end{array} \tag{2.1}$$

in which the left vertical arrow is the identity (after identifying $(V^*)^K$ and $(V^K)^*$) and the bottom horizontal arrow is defined by the diagram. Since $V^K$ is finite dimensional, the bottom horizontal arrow is an isomorphism. It follows that the top horizontal arrow is an isomorphism for all $K$, and so $r$ is an isomorphism. ☐

## 2.3 The fixed points of a compact-open subgroup

**Definition 2.3.1.** For $K$ a compact-open subgroup of $G$, the subspace $\mathcal{H}_K(G) := e_K\mathcal{H}(G)e_K$ is a subalgebra. It is the set of functions which are constant on the double cosets of $K$, that is

$$\mathcal{H}_K(G) = \{f \in \mathcal{H}(G) \mid f(k_1 g k_2) = f \quad \forall k_1, k_2 \in K\}.$$

Since, $e_K$ acts as idenitity on $[V^K = \{v \in V \mid kv = v \text{ for all } k \in K\} = \{v \in V \mid e_K v = v\} = e_K V]$, we see that $V^K$ is an $\mathcal{H}_K(G)$-module. Recall the $K$-stable decomposition (1.5)

$$V = V^K \oplus V_K.$$

Another way to view this decomposition is to note that the $K$-representation $V|_K$ is semi-simple [Proposition 1.6.8], and that the subspace $V^K$ is a $K$-subrepresentation. It provides us with the $K$-map $1_{V^K} : V \twoheadrightarrow V^K$ which has kernel $V_K$.
In the next proposition, we see a sort of converse of the idea of looking at the $K$-fixed points of a representation.

**Proposition 2.3.2.** **(i)** *Let $V$ be a simple $\mathcal{H}(G)$-module. The space $V^K$ is either* 0 *or a simple $\mathcal{H}_K(G)$-module.*
**(ii)** *The correspondence $V \mapsto V^K$ induces a bijection between the following sets:*
- **(a)** *isomorphism classes of irreducible $G$-representations $V$ such that $V^K \neq 0$;*
- **(b)** *isomorphism classes of simple $\mathcal{H}_K(G)$-modules.*

*Proof.* Let $V$ be a simple $\mathcal{H}(G)$-module such that $V^K \neq 0$. Suppose $M$ is some non-zero $\mathcal{H}_K(G)$-submodule of $V^K$. The space $\mathcal{H}(G)M$ is a non-zero $\mathcal{H}(G)$-submodule of $V$, hence is equal to $V$. We therefore have

$$V^K = e_K V = e_K \mathcal{H}(G)M = \mathcal{H}_K(G)M = M,$$

proving (**i**).
We see that $V \mapsto V^K$ gives a map from isomorphism classes of simple $\mathcal{H}(G)$-modules $V$ with $V^K \neq 0$ to isomorphism classes of simple $\mathcal{H}_K(G)$-modules. In the opposite direction let $M$ be a simple $\mathcal{H}_K(G)$-module. Consider the $\mathcal{H}(G)$-module $V = \mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} M$. We have

$$V^K = e_K \mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} M = e_K \otimes M \cong M.$$

However, $V$ need not be simple, and so we will further identify a simple quotient $\tilde{M}$ of $V$ which satisfies $\tilde{M}^K \cong M$.
Let $\{X_i\}_{i \in I}$ be any collection of submodules of $M$ with $X_i^K = 0$ for all $i \in I$. Consider some $x \in \bigcup X_i$ fixed by $K$. Then $x \in X_i$ for some $i$, and hence $x \in X_i^K = 0$, which implies $(\bigcup X_i)^K = 0$. Apply Zorn's Lemma to deduce the existence of an $\mathcal{H}(G)$-submodule $X$ of $V$ which is maximal for the property $e_K X = X^K = X \bigcap (e_K \otimes M) = 0$.
Given any other $\mathcal{H}(G)$-submodule $Y$ with $e_K Y = 0$, we have $e_K(X+Y) = e_K X + e_K Y = 0$. By the maximality of $X$, it follows that $Y \subseteq X$, and $X$ is the unique maximal $\mathcal{H}(G)$-submodule with the property $X^K = 0$. Note that $V_K \subseteq X$.
Consider some $\mathcal{H}(G)$-module $W$ with $X \subsetneq W \subseteq V$. Then $0 \subsetneq W^K \subseteq V^K$. Since $V^K \cong M$ is simple as an $\mathcal{H}_K(G)$-module, we have $W^K = V^K$. Hence, the $\mathcal{H}_K(G)$-module $W$ contains $V^K$. We already have $V_K \subseteq W$, and hence $W = V$ [Recall (1.5)]. Thus $X$ is maximal as a proper $\mathcal{H}(G)$-submodule of $V$ and $\tilde{M} = V/X$ is simple. It further satisfies $\tilde{M}^K \cong M$ as an $\mathcal{H}_K(G)$-module. Indeed it is the unique subquotient $N$ of $\mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} M$ which satisfies $N^K \cong M$. We see that $M \mapsto \tilde{M}$ is the required association.
To demonstrate that the given association is a bijection, it is sufficient to show that $(V^K)\tilde{} = V$. Note that since $V$ is simple, the $\mathcal{H}(G)$-map

$$\mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} V^K \to V : f \otimes v \mapsto fv$$

is surjective, and hence $V$ can be obtained as a quotient of $\mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} V^K$. We know from the prevoius paragraph that $\mathcal{H}(G) \otimes_{\mathcal{H}_K(G)} V^K$ has a unique subquotient $N$ satisfying $N^K \cong V^K$. Hence we have $\tilde{V^K} = N = V$. □

**Lemma 2.3.3.** *For $G = \mathrm{GL}_d(\mathbb{Q}_p)$ and $K$ a compact-open subgroup of $G$, the subalgebra $\mathcal{H}_K(G)$ is of finite type, that is there exists a surjective algebra homomorphism*

$$q : \mathbb{C}{<}X_1, \cdots X_r{>} \longrightarrow \mathcal{H}_K(G), \tag{2.2}$$

*where $\mathbb{C}{<}X_1, \cdots X_r{>}$ is the $\mathbb{C}$-algebra of non-commuting polynomials in $r$ variables.*

*Proof.* Refer Theorem 9 of [Rum]. □

*Remark* 2.3.4. If $K \subset K'$, then $\mathcal{H}_{K'}(G)$ is a subalgebra of $\mathcal{H}_K(G)$, and we have an injection $\mathcal{H}_{K'}(G) \hookrightarrow \mathcal{H}_K(G)$. We point out that the Hecke algebra $\mathcal{H}(G)$ is a direct limit of the $\mathcal{H}_K(G)$ under these injection maps. This is a consequence of Corollary 1.4.3.

# 3 Splitting the category $\mathfrak{R}(G)$

Often when we are trying to understand a category, we come up with a general technique to decompose any object in the category, such that each of the summands is of a distinct 'type'. The concept of splitting a category is an abstraction of this idea. It is described formally in Theorem 3.1.12.

## 3.1 The splitting induced by an irreducible finite representation

This section forms the backbone of this report. It says that (modulo certain conditions) we can isolate the 'maximal isotypic component' corresponding to any finite irreducible representation. The main results are Proposition 3.1.7 and Theorem 3.1.12.

We first state an important property we want $G$ to satisfy.
For every irreducible finite $G$-representation $(V, \pi)$, there exists a positive real number $d(\pi)$, the **formal degree** of $\pi$ such that, for all $v, w \in V$ and all $\alpha, \beta \in V^*$,

$$d(\pi) \int\limits_G \alpha(g^{-1}v)\beta(gw) = \beta(v)\alpha(w) \tag{3.1}$$

This is known as the **Schur orthogonality** relation.

We will always assume that $G$ satisfies the above property. Whenever we specialise $G$ to some specific group, we will refer the reader to a proof that $G$ satisfies this property.

**Lemma 3.1.1.** *Let $W$ be a $G$-representation. Let $f \in \mathcal{H}(G)$, $\alpha \in W^*$, $w \in W$. Then one has*

$$\alpha\left(\int\limits_{g\in G} f(g)gw\right) = \int\limits_{g\in G} f(g)\alpha(gw)$$

*Proof.* The idea of the proof is to show that the integral on the left is really just a finite sum. Since $f$ is smooth, there exists a compact-open subgroup $K$ such that $f(KgK) = f(g)$ for all $g \in G$ and $w \in W^K$. Then for any $h \in G$, we have

$$\int\limits_{g\in hK} f(g)gw = \mu(K)f(h)hw.$$

The following calculation finishes the proof:

$$\begin{aligned}
\alpha\left(\int_{g\in G} f(g)gw\right) &= \alpha\left(\sum_{hK\in K\backslash G}\int_{g\in hK} f(g)gw\right)\\
&= \sum_{hK\in K\backslash G} \alpha\left(\mu(K)f(h)hw\right)\\
&= \sum_{hK\in K\backslash G} \mu(K)\alpha(f(h)hw)\\
&= \sum_{hK\in K\backslash G}\int_{g\in hK} \alpha(f(g)gw)\\
&= \int_{g\in G} f(g)\alpha(gw).
\end{aligned}$$

□

For the rest of this section, fix $(V,\pi)$ to be some irreducible, finite $G$-representation.

**Proposition 3.1.2.** *The map* $\mathcal{H}(G) \xrightarrow{\pi} \mathrm{End}_{\mathbb{C}}(V)^\infty$ *is surjective. Further* $\pi$ *has an algebra section* $\phi$*.*

*Proof.* Let ${}^g\alpha$ denote the action of $g\in G$ on $\alpha\in V^*$. Recall the isomorphism $r: V\otimes V^* \to (\mathrm{End}_{\mathbb{C}}V)^\infty$ from Proposition 2.2.3. We set $T_{v,\alpha}(g) = d(\pi)\alpha(g^{-1}v)$, for $v\in V$ and $\alpha\in V^*$, where $d(\pi)$ is the formal degree mentioned in (3.1). Observe that $T_{gv,{}^g\alpha} = T_{v,\alpha}$. Since $(V,\pi)$ is finite, each $T_{v,\alpha}$ is compactly supported and hence belongs to $\mathcal{H}(G)$. Let $w\in V$. We have

$$\begin{aligned}
\beta(T_{v,\alpha}w) &= \beta\left(\int_{g\in G} d(\pi)\alpha(g^{-1}v)gw\right)\\
&= d(\pi)\int_{g\in G}\alpha(g^{-1}v)\beta(gw) && \text{[Lemma 3.1.1]}\\
&= \alpha(w)\beta(v) && \text{[Schur orthogonality (3.1)].}
\end{aligned}$$

Since this is true for all $\beta\in V^*$, we deduce $T_{v,\alpha}w = \alpha(w)v$.
Now define $\phi : \mathrm{End}_{\mathbb{C}}(V)^\infty \to \mathcal{H}(G)$ to be the unique $\mathbb{C}$-linear map such that

$$\phi(r(v\otimes\alpha)) = T_{v,\alpha}, \qquad \forall\, v\in V, \quad \forall\,\alpha\in V^*.$$

From $T_{gv,{}^g\alpha} = T_{v,\alpha}$, it follows that $\phi$ is a $G\times G$-homomorphism. Using $T_{v,\alpha}w = \alpha(w)v$, it follows $\pi\phi = \mathrm{id}$ where id denotes the identity map of $\mathrm{End}_{\mathbb{C}}(V)^\infty$. We get the vector space decomposition

$$\mathcal{H}(G) = \ker(\pi)\oplus\mathrm{im}(\phi). \tag{3.2}$$

It remains to show that $\phi$ is an algebra map. Fix $\alpha,\beta\in V^*$ and $v,w\in V$. We claim that declaring

$$\gamma(g^{-1}v) = \frac{(T_{v,\alpha} * T_{w,\beta})(g)}{d(\pi)} \tag{3.3}$$

as $g$ varies over $G$, defines a $\mathbb{C}$-linear functional $\gamma$ on $V$. Indeed, if $h^{-1}v = g^{-1}v$, the calculation

$$\begin{aligned}(T_{v,\alpha} * T_{w,\beta})(g) &= d(\pi)^2 \int\limits_{g_1 g_2 = g} \alpha(g_1^{-1}v)\beta(g_2^{-1}w) \\ &= d(\pi)^2 \int\limits_{(hg^{-1}g_1)g_2 = h} \alpha((hg^{-1}g_1)^{-1}v)\beta(g_2^{-1}w) \\ &= d(\pi)^2 \int\limits_{\tilde{g_1}g_2 = h} \alpha(\tilde{g_1}^{-1}v)\beta(g_2^{-1}w) \\ &= (T_{v,\alpha} * T_{w,\beta})(h)\end{aligned}$$

tells us that $\gamma$ is well defined. Also, since $\{g^{-1}v\}_{g\in G}$ spans the simple module $V$, it follows that $\gamma$ is defined on all of $V$.
We see from (3.3) that $T_{v,\alpha}T_{w,\beta} = T_{v,\gamma}$. It follows that for $a, b \in \mathrm{End}_{\mathbb{C}}(V)^{\infty}$, $\phi(a)\phi(b)$, and hence $\phi(ab) - \phi(a)\phi(b)$ is in the image of $\phi$. Also, $\phi(ab) - \phi(a)\phi(b)$ is in the kernel of $\pi$, and therefore $\phi(ab) - \phi(a)\phi(b) = 0$, that is $\phi$ is an algebra map. $\square$

This result is reminiscent of an analogous result in the representation theory of finite groups: If $\tilde{G}$ is a finite group, $\tilde{V}$ an irreducible $\tilde{G}$-representation, the surjective map $\mathbb{C}[\tilde{G}] \twoheadrightarrow \mathrm{End}_{\mathbb{C}}(\tilde{V})$ has a section $\tilde{\phi}$. Indeed, $\tilde{\phi}(\mathrm{id}_{\tilde{V}})$ is a central primitive idempotent in $\mathbb{C}[\tilde{G}]$.
At this stage, one is tempted to consider $\phi(\mathrm{id}_V)$. Unfortunately, the identity function is not necessarily smooth (unless $G$ is compact). We work around this by using the technique mentioned in §2.2.

**Definition 3.1.3.** For $K$ a compact-open subgroup OF $G$, define $e_K^{\pi} := \phi(1_{V^K})$.

Since $\phi$ is an algebra homomorphism, $e_K^{\pi} \in \mathcal{H}(G)$ is either zero or an idempotent. Recall from (1.4.8) the definition of $e_K$ as the characteristic function of $K$ divided by the measure $\mu(K)$ of $K$.

**Lemma 3.1.4.** *Let $g \in G$ and $K$ a compact-open subgroup of $G$. let $K'$ be an open subgroup of $K$. The following equalities hold:*
**(i)** $e^{\pi}_{gKg^{-1}} = g e^{\pi}_K g^{-1}$,
**(ii)** $e^{\pi}_{K'} e_K = e^{\pi}_K$.

*Proof.* The map $v \mapsto gv$ induces vector space isomorphisms

$$V^K \to V^{gKg^{-1}} \qquad \text{and} \qquad V_K \to V_{gKg^{-1}}.$$

Hence $g1_{V^K} = 1_{V^{gKg^{-1}}}g$ and thus $g1_{V^K}g^{-1} = 1_{V^{gKg^{-1}}}$. Applying $\phi$, (**i**) follows. Consider

$$\frac{1}{\mu(K)} \int\limits_{k\in K} 1_{V^{K'}}k \qquad \in \mathrm{End}_{\mathbb{C}}(V).$$

The containment $V^K \subseteq V^{K'}$ implies that this map is the identity on $V^K$. Further, it is zero on $V_K$, as

$$\frac{1}{\mu(K)} \int\limits_{k\in K} 1_{V^{K'}}k = 1_{V^{K'}} \circ \frac{1}{\mu(K)} \int\limits_{k\in K} k$$

and the kernel of the latter map contains $V_K$. Thus

$$\frac{1}{\mu(K)} \int\limits_{k \in K} 1_{V^{K'}} k = 1_{V^K}.$$

Applying $\phi$ and evaluating at $g \in G$, we obtain

$$\frac{1}{\mu(K)} \int\limits_{k \in K} e^{\pi}_{K'}(gk) = e^{\pi}_K(g),$$

which is exactly (**ii**). $\square$

**Definition 3.1.5.** Let $(W, \tau)$ be a representation of $G$. Define $e^{\pi} : W \to W$ as follows: if $w \in W$ is fixed by some compact-open subgroup $K$, then

$$e^{\pi}(w) := e^{\pi}_K w.$$

We use Lemma 3.1.4 to show that the $e^{\pi}_K$ form a directed system of maps $W^K \to W$. Suppose $w \in W^{K^1}, W^{K^2}$. Let $K$ be the subgroup generated by $K^1$ and $K^2$. It is an open subgroup, therefore also closed. We see that $w$ is fixed under the action of $K$. We have then

$$\begin{aligned} e^{\pi}_{K^2} w = e^{\pi}_{K^2} e_K w = e^{\pi}_K w, \\ e^{\pi}_{K^1} w = e^{\pi}_{K^1} e_K w = e^{\pi}_K w. \end{aligned}$$

Since $V$ is a direct limit of the $V^K$, all the $e^{\pi}_K$ together induce a map $e^{\pi}$ from $V$ to $V$. This is exactly the map $e^{\pi}$.

Also, if $K$ fixes $w$, then $gKg^{-1}$ fixes $gw$, and so

$$e^{\pi} gw = e^{\pi}_{gKg^{-1}} gw = g e^{\pi}_K g^{-1} gw = g e^{\pi}_K w \tag{3.4}$$

and so $e^{\pi}$ is a $G$-map.

**Lemma 3.1.6.** *Given $G$-representations $W_1$, $W_2$ and a $G$-map $f : W_1 \to W_2$, the following diagram commutes.*

$$\begin{array}{ccc} W_1 & \xrightarrow{f} & W_2 \\ {\scriptstyle e^{\pi}}\downarrow & & {\scriptstyle e^{\pi}}\downarrow \\ W_1 & \xrightarrow{f} & W_2 \end{array} \tag{3.5}$$

*Proof.* For $w \in W_1$, let $K$ be a compact-open subgroup fixing both $w$ and $f(w)$. Since, $f$ is a $G$-map, it commutes with the $e^{\pi}_K \in \mathcal{H}(G)$. Now

$$f(e^{\pi}(w)) = f(e^{\pi}_K w) = e^{\pi}_K f(w) = e^{\pi}(f(w))$$

Since this is true for all $w \in W_1$, the maps $e^{\pi}$ commute with $G$-maps. $\square$

Since each $e^{\pi}_K$ is idempotent, so is $e^{\pi}$. We can now define for each representation $W$, a $G$-decomposition

$$W = e^{\pi} W \oplus (1 - e^{\pi}) W = \mathrm{im}(e^{\pi}) \oplus \ker(e^{\pi}), \tag{3.6}$$

which is respected by $G$-maps (Lemma 3.1.6). The $e^{\pi} W$ are going to be exactly the 'maximal isotypic components' alluded to at the beginning of this section.

**Proposition 3.1.7.** *Let $W$ be a $G$-representation.*
(**i**) *The space $e^\pi W$ is a direct sum of copies of $V$.*
(**ii**) *The map $e^\pi : W \to W$ is trivial if and only if $W$ has no subquotient isomorphic to $V$.*
(**iii**) *If all irreducible subquotients of $W$ are isomorphic to $V$, then $W$ is a direct sum of copies of $V$.*

*Proof.* We remind the reader that submodules and quotients of semi-simple modules are semi-simple [Lemma 1.6.7].
(**i**) Let $w \in W$ be fixed by some compact-open subgroup $K$. Then $e^\pi_K w \in e^\pi W$. Consider the surjective $\mathcal{H}(G)$-map

$$he^\pi_K \mapsto he^\pi_K w : \mathcal{H}(G)e^\pi_K \to \mathcal{H}(G)e^\pi_K w.$$

Consider $V \otimes V^*$ as an $\mathcal{H}(G)$-module, under the action

$$v \otimes \alpha \overset{f}{\mapsto} (fv) \otimes \alpha$$

As an $\mathcal{H}(G)$-module, it is a direct sum of copies of the simple module $V$, hence semi-simple. The same is therefore true of the image of $V \otimes V^*$ under

$$V \otimes V^* \cong \mathrm{End}_{\mathbb{C}}(V)^\infty \xrightarrow{\phi} \mathcal{H}(G).$$

Since $\mathcal{H}(G)e^\pi_K$ is contained in this image, $\mathcal{H}(G)e^\pi_K$ is semi-simple and can be written as a sum of copies of $V$ [Lemma 1.6.7]. Therefore the homomorphic image $\mathcal{H}(G)e^\pi_K w$ of $\mathcal{H}(G)e^\pi_K$ is also a sum of copies of $V$.
Varying $w \in W$, we see that $e^\pi W$ is a sum of copies of $V$. By Lemma 1.6.5, it is a direct sum of copies of $V$.

(**ii**) Suppose $e^\pi$ is trivial on $W$. Note that $e^\pi$ acts as identity on $V$. It follows from Lemma 3.1.6 that $e^\pi$ is trivial on each subquotient of $W$. Thus $W$ has no subquotient isomorphic to $V$. Conversely, suppose that $e^\pi$ is non-zero on $W$ so that $e^\pi W \neq 0$. Then, by (**i**), the image of $W$ under $e^\pi$ is a direct sum of copies of $V$, from which $V$ can be obtained as the image under a projection. Hence $V$ it is a subquotient of $W$.

(**iii**) Assume $(1 - e^\pi)W \neq 0$. Then it follows from (**ii**) that $(1 - e^\pi W)$, and hence $W$, has a subquotient different from $V$. This is a contradiction. Along with (3.6), this tells us that we have $W = e^\pi W$. From (**i**), it follows that $W$ is a direct sum of copies of $V$. □

We are now ready to define the subcategories of $\mathfrak{R}(G)$ that will cause the splitting.

**Definition 3.1.8.** Denote by $\mathfrak{R}^\pi(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects are all representations of $W$ of $G$ such that each irreducible subquotient of $W$ is isomorphic to $V$.

**Lemma 3.1.9.** *Any object $\tilde{V} \in \mathfrak{R}^\pi(G)$ is isomorphic to $\oplus_I V$, a direct sum of copies of $V$.*

*Proof.* Since any subquotient of $\tilde{V}$ is isomorphic to $V$, it follows from Proposition 3.1.7(**ii**) that $(1 - e^\pi)V = 0$. From (3.6) we get $V = e^\pi V$. Proposition 3.1.7(**i**) finishes the proof. □

**Definition 3.1.10.** Denote by $\mathfrak{R}_\pi(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects are all representations $W$ of $G$ such that no subquotient of $W$ is isomorphic to $V$.

**Lemma 3.1.11.** *There are no non-zero $G$-maps between objects in $\mathfrak{R}^\pi(G)$ and objects in $\mathfrak{R}_\pi(G)$.*

*Proof.* This is a direct consequence of Lemma 1.6.3 and Lemma 3.1.9. □

**Theorem 3.1.12.** *The category $\mathfrak{R}(G)$ splits as follows*

$$\mathfrak{R}(G) = \mathfrak{R}^\pi(G) \times \mathfrak{R}_\pi(G). \tag{3.7}$$

*This means that for each representation $W$ of $G$, there exist unique subrepresentations $W^\pi$ and $W_\pi$ of $W$ with $W^\pi \in \mathfrak{R}^\pi(G)$ and $W_\pi \in \mathfrak{R}_\pi(G)$ such that $W = W^\pi \oplus W_\pi$ and*

$$\mathrm{Hom}_G(W_1, W_2) = \mathrm{Hom}_G(W_1^\pi, W_2^\pi) \oplus \mathrm{Hom}_G(W_{1\pi}, W_{2\pi}).$$

*Proof.* Observe that defining $W^\pi = e^\pi W$ and $W_\pi = (1 - e^\pi)W$ gives us a satisfactory decomposition [Recall Proposition 3.1.7(**i**), (**ii**)]. It remains to show the uniqueness of the decomposition.
Note that $e^\pi$ acts as identity on every subrepresentation of $W$ isomorphic to $V$. Also, it acts as zero on every other irreducible subrepresentation [Lemma 3.1.6]. Therefore any subrepresentation of $W$ which is isomorphic to $V$ is contained in $e^\pi W$. Hence any candidate for $W^\pi$ is contained in $e^\pi W$. Let if possible $W^\pi$ is a proper subrepresentation of $e^\pi W$. We see that $W_\pi \cong W/W^\pi$ has $e^\pi W/W^\pi$ as a subquotient, and hence has $V$ as a subquotient, a contradiction. Therefore $W^\pi := e^\pi W$ is unique.
Suppose now that there exists $W' \in \mathfrak{R}_\pi(G)$ such that $W = W^\pi \oplus W'$. Consider the composition maps

$$f_1 : W' \overset{\subseteq}{\hookrightarrow} W \xrightarrow{\mathrm{id}} W^\pi \oplus (1 - e^\pi)W \twoheadrightarrow W^\pi$$
$$f_2 : W' \overset{\subseteq}{\hookrightarrow} W \xrightarrow{\mathrm{id}} W^\pi \oplus (1 - e^\pi)W \twoheadrightarrow W_\pi.$$

Note that $f_1 \oplus f_2$ is the inclusion map $W' \hookrightarrow W$. By Lemma 3.1.11, $f_1$ is the zero map. Therefore $f_2$ must be injective, that is $W' \subseteq (1 - e^\pi)W$. Similarly, $(1 - e^\pi)W \subseteq W'$, and hence $W' = (1 - e^\pi)W$. This proves the uniqueness of $W_\pi := (1 - e^\pi)W$. □

**Corollary 3.1.13.** *If $W_1, W_2$ are two $G$-representations, then we have*

$$\mathrm{Hom}_G(W_1, W_2) = \mathrm{Hom}_G(W_1^\pi, W_2^\pi) \oplus \mathrm{Hom}_G(W_{1\pi}, W_{2\pi}).$$

*Proof.* This is merely a restatement of Lemma 3.1.11:

$$\begin{aligned}
&\mathrm{Hom}_G(W_1^\pi \oplus W_{1\pi}, W_2\pi \oplus W_{2\pi}) \\
&= \mathrm{Hom}_G(W_1^\pi, W_2^\pi) \oplus \mathrm{Hom}_G(W_1^\pi, W_{2\pi}) \oplus \mathrm{Hom}_G(W_{1\pi}, W_2^\pi) \oplus \mathrm{Hom}_G(W_{1\pi}, W_{2\pi}) \\
&= \mathrm{Hom}_G(W_1^\pi, W_2^\pi) \oplus \mathrm{Hom}_G(W_{1\pi}, W_{2\pi})
\end{aligned}$$

since $\mathrm{Hom}_G(W_1^\pi, W_{2\pi}) = 0$ and $\mathrm{Hom}_G(W_{1\pi}, W_2^\pi) = 0$ [Lemma 3.1.11]. □

**Corollary 3.1.14.** *Any finite representation is a direct sum of irreducible finite representations.*

*Proof.* Let $W$ be a finite representation of $G$ and write $W^f$ for the sum of all irreducible finite $G$-subrepresentations of $W$. Suppose that $W/W^f \neq 0$. Then $W/W^f$ admits an irreducible subquotient $\pi$. The representation $\pi$ is finite [Lemma 2.1.4]. Then, by Proposition 3.1.7(**ii**), $e^\pi(W/W^f) \neq 0$ and thus

$$e^\pi W^f \subsetneq e^\pi W.$$

However, by part (**i**), $e^\pi W \subseteq W^f$, whence $e^\pi W \subseteq e^\pi W^f$, a contradiction. It follows that $W/W^f = 0$, that is is $W = W^f$. □

## 3.2 Splitting off all finite irreducible representations

We have seen that a single irreducible finite representation $\pi$ of $G$ induces the splitting $\mathfrak{R}(G) = \mathfrak{R}^\pi(G) \times \mathfrak{R}_\pi(G)$. We now consider whether the class of all irreducible finite representations of $G$ induces a similar splitting. Theorem 3.2.5 answers this in the affirmative provided a certain finiteness condition is satisfied.

**Definition 3.2.1.** Let $\mathfrak{R}^f(G)$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are all finite representations of $G$.

**Definition 3.2.2.** Let $\mathfrak{R}^{nf}(G)$ for the full subcategory of $\mathfrak{R}(G)$ whose objects are all representations $W$ of $G$ such that no subquotient of $W$ is finite.

First we note down a lemma similar to 3.1.11.

**Lemma 3.2.3.** *There are no non-zero maps between $\mathfrak{R}^f(G)$ and $\mathfrak{R}^{nf}(G)$.*

*Proof.* This a direct consequence of Lemma 1.6.3 and Lemma 2.1.4. □

The question under consideration is whether $\mathfrak{R}(G) = \mathfrak{R}^f(G) \times \mathfrak{R}^{nf}(G)$ holds? In concrete terms, if $W$ is a representation, and $W^f$ denotes the sum of all finite $G$ subspaces of $W$, then does $W^f$ admit a $G$ complement? Note that any such complement is unique as there are no non-zero $G$-maps between objects in $\mathfrak{R}^f(G)$ and $\mathfrak{R}^{nf}(G)$ [Recall the proof of Corollary 3.1.13.]

Let $\Pi_f$ denote a set of representatives for the distinct isomorphism classes of irreducible finite representations of $G$. Note the following

**Lemma 3.2.4.** *The category $\mathfrak{R}^f(G)$ splits as $\mathfrak{R}^f(G) = \prod_{\pi \in \Pi_f} \mathfrak{R}^\pi(G)$.*

*Proof.* This is the result of applying Lemma 1.6.3 to Corollary 3.1.14 and Corollary 3.1.9. □

**Theorem 3.2.5.** *The category $\mathfrak{R}(G)$ splits as $\mathfrak{R}^f(G) \times \mathfrak{R}^{nf}(G)$ if the following finiteness condition* **(FC)** *holds:*
**For each compact-open subgroup $K$ of $G$, there are only finitely many isomorphism classes of irreducible finite $G$-representations $V$ such that $V^K \neq 0$.**

*Proof.* Suppose **(FC)** holds. For each representation $W$, we define $e_f : W \to W$ by

$$e_f(w) = \sum_{\pi \in \Pi_f} e^\pi w, \qquad w \in W.$$

To see that $e_f$ is well defined, note that for any compact-open subgroup $K$ such that $w \in W^K$ with $e^\pi w \neq 0$, we have $e^\pi w = e^\pi_K w \neq 0$, whence $e^\pi_K \neq 0$ and so $\pi^K \neq 0$. Thus **(FC)** implies that for each $w \in W$, the above sum has only finitely many non-zero terms. Clearly, $e_f$ is either zero or an idempotent $G$-map. It induces the $G$-decomposition

$$W = e_f W \oplus (1 - e_f)W.$$

It follows from Lemma 3.2.4 that $e_f$ acts as identity on finite representations. Hence $e_f W$ belongs to $\mathfrak{R}^f(G)$. Also, Lemma 3.1.6 and **(FC)** imply that $e_f$ commutes with $G$-maps. It follows that $(1 - e_f)W$ belongs to $\mathfrak{R}^{nf}(G)$. Therefore,

$$\mathfrak{R}(G) = \mathfrak{R}^f(G) \times \mathfrak{R}^{nf}(G). \tag{3.8}$$

□

# 4 Cuspidal representations of $\mathrm{GL}_d(\mathbb{Q}_p)$

In this final section, we put $G = \mathrm{GL}_d(\mathbb{Q}_p)$, and $G^o = \{g \in G \mid det(g) \in \mathbb{Z}_p^*\}$. The group $G$ does not satisfy **(FC)**. However, the previous theory is still applicable if we consider the restriction of $G$-representations to $G^o$, which does satisfy **(FC)**.

A proof of the existence of left-right invariant Haar measure for $G$ and $G^o$ can be found in Theorem 5.1 of [Glö]. A proof of (3.1), the Schur orthogonality relation, can be found in §2.41 of [BZ].

## 4.1 The restriction to $G^o$

In this section, we show that if $V$ is a $G$-representation, then $V|_{G^o}$ is sum of finitely many irreducible $G^o$-representations. We also give a necessary and sufficient condition for two $G$-representations $V$ and $W$ to be isomorphic when viewed as $G^o$-representations.

**Lemma 4.1.1.** (**i**) *The subgroup $G^o$ is normal in $G$.*
(**ii**) *The quotient $G/G^o$ is isomorphic to $\mathbb{Z}$.*
(**iii**) *Let $Z$ be the center of $G$. Then $ZG^o$ has finite index in $G$.*

*Proof.* Let $\nu$ be the $p$-adic valuation on $\mathbb{Z}_p$. The subgoup $G^o$ is the kernel of the surjective map $\nu \circ \det : G \twoheadrightarrow \mathbb{Z}$. This proves (**i**) and (**ii**). Observe that $Z$ is just the set of all scalar matrices (diagonal matrices with identical diagonal entries) in $G$. The image of $Z$ under $\nu \circ \det : G \twoheadrightarrow \mathbb{Z}$ is then $d\mathbb{Z}$. We see that $G/ZG^o \cong \mathbb{Z}/d\mathbb{Z}$, and hence $ZG^o$ has finite index in $G$. □

**Definition 4.1.2.** The group $X_{nr}(G)$ of **unramified characters** of $G$ is the set of group homomorphisms from $G/G^o$ to $\mathbb{C}^*$.

$$X_{nr}(G) = \mathrm{Hom}(G/G^o, \mathbb{C}^*) \tag{4.1}$$

Let $(V, \pi)$ be a representation of $G$. For each $\nu \in X_{nr}(G)$, we write $\pi\nu$ for the $G$-representation given by $(\pi\nu)(g) := \pi(g)\nu(g)$. Such representations are called **unramified twists** of $\pi$.

**Proposition 4.1.3.** *Let $(V, \pi)$ and $(V', \pi')$ be irreducible $G$-representations. Then the following are equivalent:*
**(i)** $\pi \cong \pi'\eta$ *for some* $\eta \in X_{nr}(G)$*.*
**(ii)** $\mathrm{Hom}_{G^o}(V|_{G^o}, V'|G^o) \neq 0$.
**(iii)** $\pi|_{G^o} \cong \pi'|_{G^o}$.

*Proof.* Clearly, **(i)** $\Rightarrow$**(iii)** $\Rightarrow$**(ii)**. We prove that **(ii)** implies **(i)**.
Set $\mathcal{I} = \mathrm{Hom}_{G^o}(V|_{G^o}, V'|G^o)$. Since $ZG^o$ has finite index in $G$, the restriction to $ZG^o$ of any irreducible smooth $G$-representation $\tau$ is a finite sum of simples. (See, for example, 2.9 of [BZ]). As $Z$ acts by scalars on any such $\tau$, the restriction $\tau|G^o$ is also a finite sum of simples. In particular $\mathcal{I}$) is a finite dimensional $\mathbb{C}$-vector space. Consider the $G$-action on $\mathcal{I}$ given by:

$$^gT := \pi'(g) \circ T \circ \pi(g)^{-1}.$$

Clearly, $G^O$ acts trivially and thus the $G$-action yields a $G/G^O$-representation. Being a finite dimensional representation of an abelian group, it admits a one-dimensional irreducible subrepresentation. Hence there is a $T \in \mathcal{I}$ such that $^gT = \nu(g)T$ for all $g \in G$. It follows that $\pi \cong \pi'\nu^{-1}$. □

## 4.2 Verifying the Finiteness Condition

Here we will show that $G^o$ satisfies **(FC)**, and therefore $\mathfrak{R}(G^o)$ splits as in Theorem 3.1.12. In the next section, we will adapt this splitting to $\mathfrak{R}(G)$.
Our proof of Theorem 4.2.2 uses the next very fundamental theorem (which we do not prove here).

**Theorem 4.2.1** (Uniform admissibility)**.** *Let $K$ be a compact-open subgroup of $G^o$. There is a constant $N$ such that* $\dim W^K \leq N$*, for all irreducible representations $W$ of $G^o$.*

*Proof.* Refer §4 of [BZ]. □

**Proposition 4.2.2.** *The group $G^o$ satisfies* **(FC)***.*

*Proof.* From Proposition 2.3.2, it is enough to show that for any compact-open subgroup $K$, there are only finitely many isomorphism classes of $\mathcal{H}_K(G^o)$-modules.
Theorem 4.2.1 and Proposition 2.3.2 tell us that the dimension of $\mathcal{H}_K(G^o)$-modules are uniformly bounded above. Hence to prove **(FC)**, it suffices to show that for a given positive integer $n$, there are only finitely many isomorphism classes of simple finite representations $V$ of $G^o$ with $\dim V^K = n$.
Recall the surjection $q : \mathbb{C}{<}X_1, \cdots X_r{>} \longrightarrow \mathcal{H}_K(G)$ from Lemma 2.3.3. We set $I = \ker q$ and use $q$ to view any $\mathcal{H}_K(G)$-module as a $\mathbb{C}{<}X_1, \cdots X_r{>}$-module on which $I$ acts trivially. Let $\mathfrak{R}_n$ denote the set of $\mathbb{C}{<}X_1, \cdots X_r{>}$-module structures $\rho$ on $\mathbb{C}^n$ on which $I$ acts trivially. Any such $\rho$ is determined by the map $\rho : \mathbb{C}{<}X_1, \cdots X_r{>} \longrightarrow \mathrm{M}_n(\mathbb{C})$, or equivalently, by the $r$-tuple

$$(\rho(X_1), \cdots, \rho(X_r)) \in (\mathrm{M}_n(\mathbb{C}))^r \cong \mathbb{C}^{n^2 r}$$

The set $\mathfrak{R}_n$ can be identified as the set of points of $\mathbb{C}^{n^2 r}$ which correspond to some $\rho$.
Note that any polynomial equation in $(\mathrm{M}_n(\mathbb{C}))^r$ corresponds to $n^2$ equations in

$\mathbb{C}^{n^2 r}$. Now $\rho$ factors via $q$. This implies that $(X_1, \cdots X_r) \in (\mathrm{M}_n(\mathbb{C}))^r$ if and only if for each $p \in I$, $p(\rho(X_1), \cdots \rho(X_r)) = 0$. The set $\{p \circ \rho\}_{\rho \in I}$ corresponds to some set of polynomials over $\mathbb{C}^{n^2 r}$. We see that the image of $\rho$ in $\mathbb{C}^{n^2 r}$ is the zero-set of some set of polynomials in $n^2 r$ variables over $\mathbb{C}$. In this way, $\mathfrak{R}_n$ acquires the structure of an affine variety.
Let $(V, \pi)$ be a simple finite module over $\mathcal{H}(G^o)$ such that $\dim V^K = n$. Let $\mathfrak{R}_\pi$ denote the subset of $\mathfrak{R}_n$ consisting of all module structures isomorphic to $V^K$. Consider $f \in \mathbb{C}<X_1, \cdots X_r>$ such that $q(f) = e^\pi$. Since $e^\pi_K$ acts as the identity on each element of $\mathfrak{R}_\pi$ and as zero on each element of $\mathfrak{R}_n/\mathfrak{R}_\pi$, we see that $(x_1, \cdots x_r) \in \mathrm{M}_n(\mathbb{C})^r$ corresponds to an element of $\mathfrak{R}_\pi$ if and only if

$$f(x_1, \cdots x_r) = \mathrm{id},$$

where id denotes the identity element of $\mathrm{M}_n(\mathbb{C})$. It follows that $\mathfrak{R}_\pi$ is closed in $\mathfrak{R}_n$. In the same way, $(x_1, \cdots x_r) \in \mathrm{M}_n(\mathbb{C})^r$ corresponds to an element of $\mathfrak{R}_n/\mathfrak{R}_\pi$ if and only if

$$f(x_1, \cdots x_r) = 0.$$

Hence $\mathfrak{R}_n/\mathfrak{R}_\pi$ is also closed. Therefore $\mathfrak{R}_n$ is open and closed and so must be a union of connected components. As $\mathfrak{R}_n$ has only finitely many components [Har][Corollary 1.6], we see that there can only be finitely many isomorphism classes of irreducible finite $G^o$-representations $V$ with $\dim V^K = n$. This completes the proof. □

*Remark* 4.2.3. A group $G$ satisfies **(FC)** if and only if it satifies uniform admissibility. The previous proposition proves a one way implication. A complete proof can be found in [Roc].

## 4.3 Cuspidal components of $\mathrm{GL}_d(\mathbb{Q}_p)$

The irreducible representations of $G$ and $G^o$ are closely related and the proper splitting of $\mathfrak{R}(G)$ is obtained by lifting the splitting of $\mathfrak{R}(G^o)$ implied by Theorem 4.2.2 and Theorem 3.2.5.

**Definition 4.3.1.** A representation $V$ of $G$ is said to be **cuspidal** if $V|_{G^o}$ is finite.

*Remark* 4.3.2. Refer §2.2 of [Rum] or [BZ] for a proof of the equivalence of the above definition to the more usual one mentioned at the beginning of the report.

Let $\mathfrak{R}^c(G)$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are all cuspidal representations of $G$ and write $\mathfrak{R}^{nc}(G)$ for the full subcategory of $\mathfrak{R}(G)$ whose objects are all representations of $G$ with no non-zero cuspidal subquotients. The splitting 3.8 for $G^o$ gives us

$$\mathfrak{R}(G) = \mathfrak{R}^c(G) \times \mathfrak{R}^{nc}(G)$$

Finally, we want to further split $\mathfrak{R}^c(G)$ along the lines of Lemma 3.2.4.

Let $\pi$ and $\pi'$ be irreducible $G$-representations. Define $\pi \sim \pi'$ if $\pi|_{G^o} \cong \pi'|_{G^o}$. Recall the equivalent conditions discussed in Proposition 4.1.3. Let $[\pi]$ denote the equivalence class of $\pi$ under $\sim$.

Further, let $\overline{[\pi]}$ be the collection of isomorphism classes of all irreducible $G^o$-representations $\tilde{\pi}$, such that $\tilde{\pi}$ is a subquotient of $\tau|_{G^o}$ for some $\tau \in [\pi]$. Property (**ii**) of Proposition 4.1.3 tells us that if $[\pi] \neq [\pi']$, then $\overline{[\pi]}$ and $\overline{[\pi']}$ are disjoint.

**Definition 4.3.3.** Let $\mathfrak{R}^{[\pi]}(G)$ be the category of those $G$-representations, each of whose irreducible subquotients belongs to $[\pi]$.

**Theorem 4.3.4.** *Let $\mathcal{B}^c(G)$ be the set of all equivalence classes $[\pi]$. We have the following decomposition:*

$$\mathfrak{R}^c(G) = \prod_{[\pi]\in\mathcal{B}^c(G)} \mathfrak{R}^{[\pi]}(G). \tag{4.2}$$

*Proof.* If $\pi \in \mathfrak{R}^c(G)$, then $\pi|_{G^o}$ is finite. Observe that the objects of $\mathfrak{R}^{[\pi]}(G)$ are exactly those $G$-representations, which when restricted to $G^o$, belong to $\prod_{\omega\in\overline{[\pi]}} \mathfrak{R}^{\omega}(G^o)$. The splitting of Lemma 3.2.4 applied to $G^o$ then implies that $\mathfrak{R}^{[\pi]}(G)$ is a direct factor of $\mathfrak{R}^c(G)$ and that $\mathfrak{R}^c(G)$ is the product of the various subcategories $\mathfrak{R}^{[\pi]}(G)$. □

*Remark* 4.3.5. The various subcategories $\mathfrak{R}^{[\pi]}(G)$ are called the **cuspidal components** of $\mathfrak{R}(G)$. These subcategories are indecomposable. The reader can find a proof in section 1.6.2 of [Roc].

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